For Thought

1. True, for the sum of the measurements of the three angles is 180°.

2. False, $a \sin 17^\circ = 88 \sin 9^\circ$ and $a = \dfrac{88 \sin 9^\circ}{\sin 17^\circ}$.

3. False, since $\alpha = \sin^{-1}\left(\dfrac{5 \sin 44^\circ}{18}\right) \approx 11^\circ$ and $\alpha = 180 - 11^\circ = 169^\circ$.

4. True

5. True, since $\dfrac{\sin 60^\circ}{\sqrt{3}} = \dfrac{\sqrt{3}/2}{\sqrt{3}} = \dfrac{1}{2}$ and $\dfrac{\sin 30^\circ}{1} = \sin 30^\circ = \dfrac{1}{2}$.

6. False, a triangle exists since $a = 500$ is bigger than $h = 10 \sin 60^\circ \approx 8.7$.

7. True, since the triangle that exists is a right triangle.

8. False, there exists only one triangle and it is an obtuse triangle.

9. True **10.** True

7.1 Exercises

1. oblique

2. law of sines

3. Note $\gamma = 180^\circ - (64^\circ + 72^\circ) = 44^\circ$.

By the sine law $\dfrac{b}{\sin 72^\circ} = \dfrac{13.6}{\sin 64^\circ}$ and $\dfrac{c}{\sin 44^\circ} = \dfrac{13.6}{\sin 64^\circ}$. So $b = \dfrac{13.6}{\sin 64^\circ} \cdot \sin 72^\circ \approx 14.4$ and $c = \dfrac{13.6}{\sin 64^\circ} \cdot \sin 44^\circ \approx 10.5$.

4. Note $\alpha = 180^\circ - (16^\circ + 121^\circ) = 43^\circ$.

By the sine law $\dfrac{c}{\sin 16^\circ} = \dfrac{4.2}{\sin 121^\circ}$ and $\dfrac{a}{\sin 43^\circ} = \dfrac{4.2}{\sin 121^\circ}$. So $c = \dfrac{4.2}{\sin 121^\circ} \cdot \sin 16^\circ \approx 1.4$ and $a = \dfrac{4.2}{\sin 121^\circ} \cdot \sin 43^\circ \approx 3.3$.

5. Note $\beta = 180^\circ - (12.2^\circ + 33.6^\circ) = 134.2^\circ$.

By the sine law $\dfrac{a}{\sin 12.2^\circ} = \dfrac{17.6}{\sin 134.2^\circ}$ and $\dfrac{c}{\sin 33.6^\circ} = \dfrac{17.6}{\sin 134.2^\circ}$.

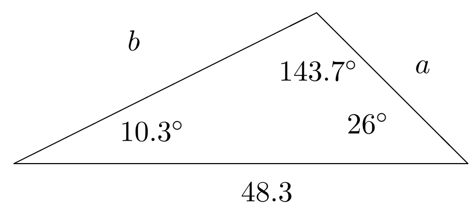
So $a = \dfrac{17.6}{\sin 134.2^\circ} \cdot \sin 12.2^\circ \approx 5.2$ and $c = \dfrac{17.6}{\sin 134.2^\circ} \cdot \sin 33.6^\circ \approx 13.6$.

6. Note $\alpha = 180^\circ - (39.5^\circ + 66.7^\circ) = 73.8^\circ$.

By the sine law $\dfrac{b}{\sin 66.7^\circ} = \dfrac{6.4}{\sin 73.8^\circ}$ and $\dfrac{c}{\sin 39.5^\circ} = \dfrac{6.4}{\sin 73.8^\circ}$.

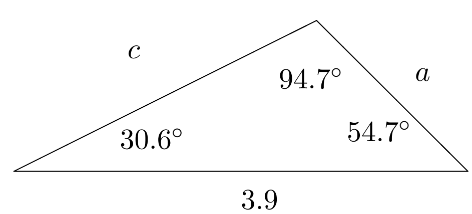
So $b = \dfrac{6.4}{\sin 73.8^\circ} \cdot \sin 66.7^\circ \approx 6.1$ and $c = \dfrac{6.4}{\sin 73.8^\circ} \cdot \sin 39.5^\circ \approx 4.2$.

7. Note $\beta = 180^\circ - (10.3^\circ + 143.7^\circ) = 26^\circ$.



Since $\dfrac{a}{\sin 10.3^\circ} = \dfrac{48.3}{\sin 143.7^\circ}$ and $\dfrac{b}{\sin 26^\circ} = \dfrac{48.3}{\sin 143.7^\circ}$, we have $a = \dfrac{48.3}{\sin 143.7^\circ} \cdot \sin 10.3^\circ \approx 14.6$ and $b = \dfrac{48.3}{\sin 143.7^\circ} \cdot \sin 26^\circ \approx 35.8$

8. Note $\gamma = 180^\circ - (94.7^\circ + 30.6^\circ) = 54.7^\circ$.



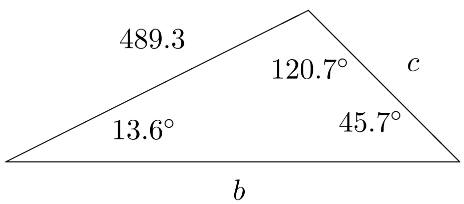
Since $\dfrac{a}{\sin 30.6^\circ} = \dfrac{3.9}{\sin 94.7^\circ}$ and

$\dfrac{c}{\sin 54.7^\circ} = \dfrac{3.9}{\sin 94.7^\circ}$, we have

$a = \dfrac{3.9}{\sin 94.7^\circ} \cdot \sin 30.6^\circ \approx 2.0$ and

$c = \dfrac{3.9}{\sin 94.7^\circ} \cdot \sin 54.7^\circ \approx 3.2$.

9. Note $\alpha = 180^\circ - (120.7^\circ + 13.6^\circ) = 45.7^\circ$.



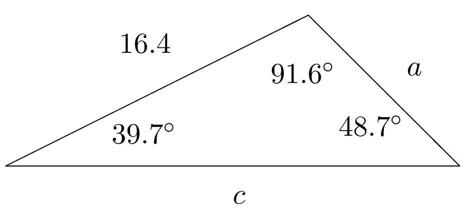
Since $\dfrac{c}{\sin 13.6^\circ} = \dfrac{489.3}{\sin 45.7^\circ}$ and

$\dfrac{b}{\sin 120.7^\circ} = \dfrac{489.3}{\sin 45.7^\circ}$, we have

$c = \dfrac{489.3}{\sin 45.7^\circ} \cdot \sin 13.6^\circ \approx 160.8$ and

$b = \dfrac{489.3}{\sin 45.7^\circ} \cdot \sin 120.7^\circ \approx 587.9$

10. Note $\beta = 180^\circ - (39.7^\circ + 91.6^\circ) = 48.7^\circ$.



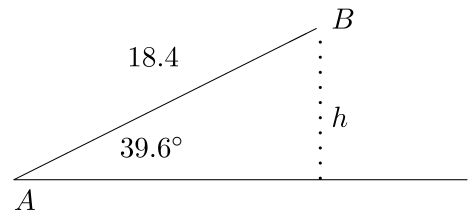
Since $\dfrac{a}{\sin 39.7^\circ} = \dfrac{16.4}{\sin 48.7^\circ}$ and

$\dfrac{c}{\sin 91.6^\circ} = \dfrac{16.4}{\sin 48.7^\circ}$, we have

$a = \dfrac{16.4}{\sin 48.7^\circ} \cdot \sin 39.7^\circ \approx 13.9$ and

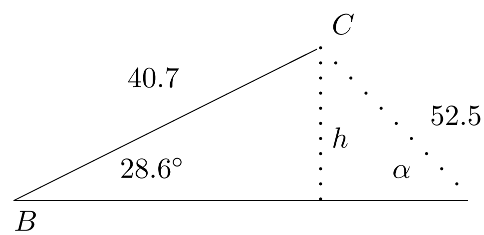
$c = \dfrac{16.4}{\sin 48.7^\circ} \cdot \sin 91.6^\circ \approx 21.8$

11. Draw angle $\alpha = 39.6^\circ$ and let h be the height.



Since $\sin 39.6^\circ = \dfrac{h}{18.4^\circ}$, we have $h = 18.4 \sin 39.6^\circ \approx 11.7$. There is no triangle since $a = 3.7$ is smaller than $h \approx 11.7$.

12. Draw angle $\beta = 28.6^\circ$ and let h be the height.



Since $h = 40.7 \sin 28.6^\circ \approx 19.5$ and $b > h$ and $b > 40.7$, there is exactly one triangle. By the sine law, we obtain

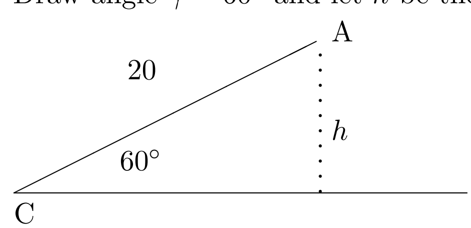
$$\begin{aligned} \frac{40.7}{\sin\alpha} &= \frac{52.5}{\sin 28.6^\circ} \\ \sin\alpha &= \frac{40.7 \sin 28.6^\circ}{52.5} \\ \sin\alpha &\approx 0.3711 \\ \alpha = \sin^{-1}(0.3711) &\approx 21.8^\circ. \end{aligned}$$

Then $\gamma = 180^\circ - (28.6^\circ + 21.8^\circ) = 129.6^\circ$.

Since $\dfrac{c}{\sin 129.6^\circ} = \dfrac{52.5}{\sin 28.6^\circ}$, we get

$$c = \sin 129.6^\circ \cdot \frac{52.5}{\sin 28.6^\circ} \approx 84.5.$$

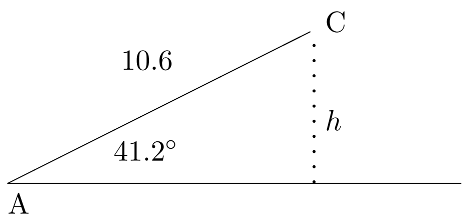
13. Draw angle $\gamma = 60^\circ$ and let h be the height.



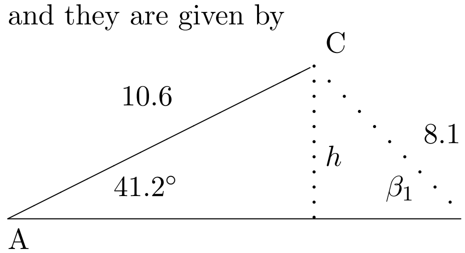
Since $h = 20\sin 60° = 10\sqrt{3}$ and $c = h$, there is exactly one triangle and it is a right triangle. So $\beta = 90°$ and $\alpha = 30°$.
By the Pythagorean Theorem,

$$a = \sqrt{20^2 - (10\sqrt{3})^2} = \sqrt{400 - 300} = 10.$$

14. Draw angle $\alpha = 41.2°$ and let h be the height.



Since $h = 10.6 \sin 41.2° \approx 7.0$ and $7.0 < a < 10.6$, there are two triangles and they are given by



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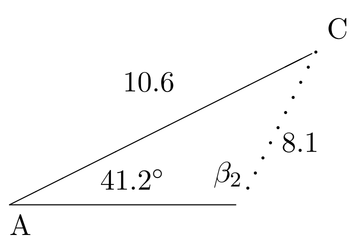


Apply the sine law to the acute triangle.

$$\begin{aligned}
\frac{8.1}{\sin 41.2°} &= \frac{10.6}{\sin \beta_1} \\
\sin \beta_1 &= \frac{10.6 \sin 41.2°}{8.1} \\
\sin \beta_1 &\approx 0.862 \\
\beta_1 = \sin^{-1}(0.862) &\approx 59.5°
\end{aligned}$$

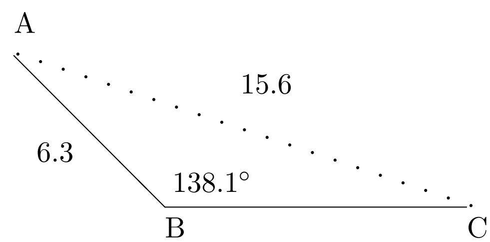
So $\gamma_1 = 180° - (59.5° + 41.2°) = 79.3°$. By the sine law, $c_1 = \dfrac{8.1}{\sin 41.2°} \sin 79.3° \approx 12.1$.

On the obtuse triangle, $\beta_2 = 180° - \beta_1 = 120.5°$ and

$$\gamma_2 = 180° - (120.5° + 41.2°) = 18.3°.$$

By the sine law, $c_2 = \dfrac{8.1}{\sin 41.2°} \sin 18.3° \approx 3.9$.

15. Since β is an obtuse angle and $b > c$, there is exactly one triangle.

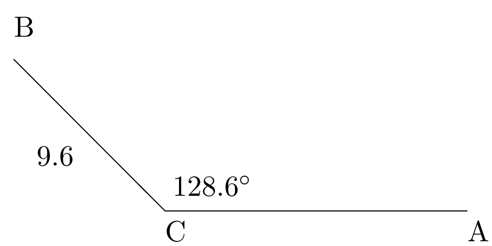


Apply the sine law.

$$\begin{aligned}
\frac{15.6}{\sin 138.1°} &= \frac{6.3}{\sin \gamma} \\
\sin \gamma &= \frac{6.3 \sin 138.1°}{15.6} \\
\sin \gamma &\approx 0.2697 \\
\gamma = \sin^{-1}(0.2697) &\approx 15.6°
\end{aligned}$$

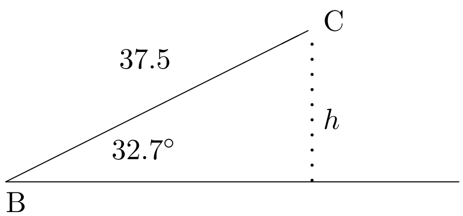
So $\alpha = 180° - (15.6° + 138.1°) = 26.3°$. By the sine law, $a = \dfrac{15.6}{\sin 138.1°} \sin 26.3° \approx 10.3$.

16. Draw angle $\gamma = 128.6°$.

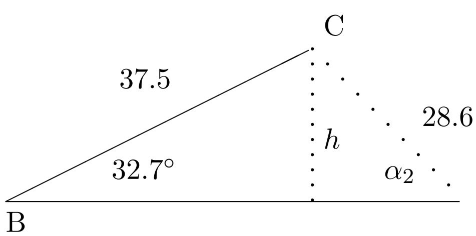


Since $c < 9.6$, no triangle exists.

17. Draw angle $\beta = 32.7°$ and let h be the height.



Since $h = 37.5 \sin 32.7° \approx 20.3$ and $20.3 < b < 37.5$, there are two triangles and they are given by



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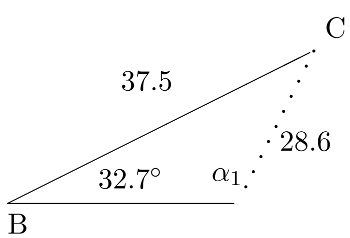
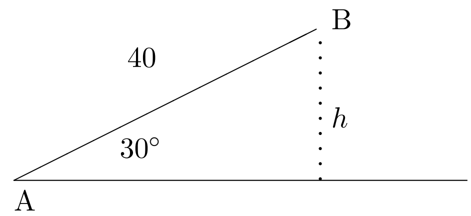


Apply the sine law to the acute triangle.

$$\begin{aligned}
\frac{28.6}{\sin 32.7°} &= \frac{37.5}{\sin \alpha_2} \\
\sin \alpha_2 &= \frac{37.5 \sin 32.7°}{28.6} \\
\sin \alpha_2 &\approx 0.708 \\
\alpha_2 = \sin^{-1}(0.708) &\approx 45.1°
\end{aligned}$$

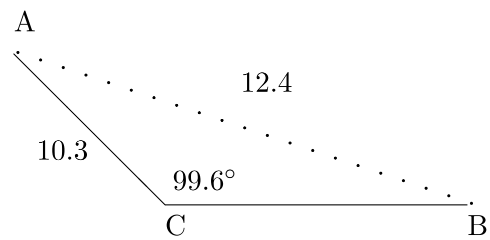
So $\gamma_2 = 180° - (45.1° + 32.7°) = 102.2°$. By the sine law, $c_2 = \dfrac{28.6}{\sin 32.7°} \sin 102.2° \approx 51.7$.
On the obtuse triangle, we find $\alpha_1 = 180° - \alpha_2 = 134.9°$ and $\gamma_1 = 180° - (134.9° + 32.7°) = 12.4°$. By the sine law, $c_1 = \dfrac{28.6}{\sin 32.7°} \sin 12.4° \approx 11.4$.

18. Draw angle $\alpha = 30°$ and let h be the height.



Since $h = 40 \sin 30° = 20$ and $a = h$, there is exactly one triangle and it is a right triangle. So $\beta = 60°$, $\gamma = 90°$, and by the Pythagorean Theorem we get $b = \sqrt{40^2 - 20^2} = 20\sqrt{3}$.

19. Draw angle $\gamma = 99.6°$. Note, there is exactly one triangle since $12.4 > 10.3$.



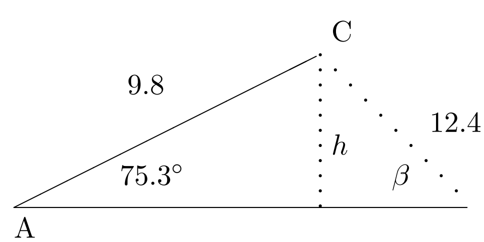
By the sine law, we obtain

$$\begin{aligned}
\frac{12.4}{\sin 99.6°} &= \frac{10.3}{\sin \beta} \\
\sin \beta &= \frac{10.3 \sin 99.6°}{12.4} \\
\sin \beta &\approx 0.819 \\
\beta = \sin^{-1}(0.819) &\approx 55.0°.
\end{aligned}$$

So $\alpha = 180° - (55.0° + 99.6°) = 25.4°$.

By the sine law, $a = \dfrac{12.4}{\sin 99.6°} \sin 25.4° \approx 5.4$.

20. Draw angle $\alpha = 75.3°$ and let h be the height.



So $h = 9.8 \sin 75.3° \approx 9.5$. Since $a > h$ and $a > 9.8$, there is exactly one triangle. By the sine law, we find

$$\begin{aligned} \frac{9.8}{\sin\beta} &= \frac{12.4}{\sin 75.3°} \\ \sin\beta &= \frac{9.8 \sin 75.3°}{12.4} \\ \sin\beta &\approx 0.7645 \\ \beta = \sin^{-1}(0.7645) &\approx 49.9°. \end{aligned}$$

So $\gamma = 180° - (49.9° + 75.3°) = 54.8°$.

Since $\dfrac{c}{\sin 54.8°} = \dfrac{12.4}{\sin 75.3°}$, we have

$c = \sin 54.8° \cdot \dfrac{12.4}{\sin 75.3°} \approx 10.5.$

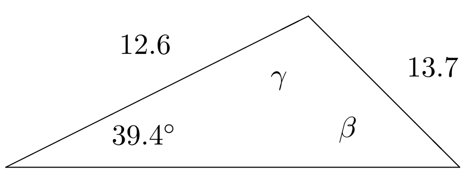
21. Since two sides and an included angle are given, the area is

$A = \dfrac{1}{2}(12.9)(6.4)\sin 13.7° \approx 9.8.$

22. Since two sides and an included angle are given, the area is

$A = \dfrac{1}{2}(42.7)(64.1)\sin 74.2° \approx 1316.8.$

23. Draw angle $\alpha = 39.4°$.



By the sine law, we obtain

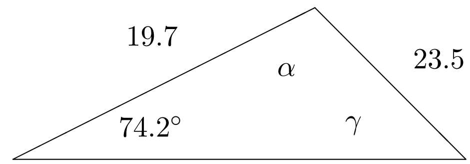
$$\begin{aligned} \frac{12.6}{\sin\beta} &= \frac{13.7}{\sin 39.4°} \\ \sin\beta &= \frac{12.6 \sin 39.4°}{13.7} \\ \sin\beta &\approx 0.5838 \\ \beta = \sin^{-1}(0.5838) &\approx 35.7°. \end{aligned}$$

Then $\gamma = 180° - (35.7° + 39.4°) = 104.9°$.

The area is $A = \dfrac{1}{2} \cdot ab \sin\gamma =$

$\dfrac{1}{2} \cdot (13.7)(12.6)\sin 104.9° \approx 83.4.$

24. Draw angle $\beta = 74.2°$.



By the sine law, we find

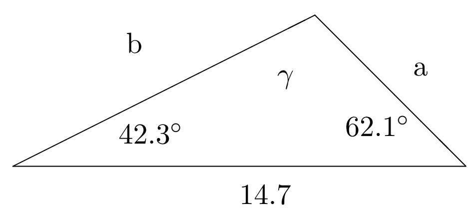
$$\begin{aligned} \frac{19.7}{\sin\gamma} &= \frac{23.5}{\sin 74.2°} \\ \sin\gamma &= \frac{19.7 \sin 74.2°}{23.5} \\ \sin\gamma &\approx 0.8066 \\ \gamma = \sin^{-1}(0.8066) &\approx 53.77°. \end{aligned}$$

Then $\alpha = 180° - (53.77° + 74.2°) = 52.03°$.

The area is $A = \dfrac{1}{2}bc\sin\alpha =$

$\dfrac{1}{2}(23.5)(19.7)\sin 52.03° \approx 182.5.$

25. Draw angle $\alpha = 42.3°$.



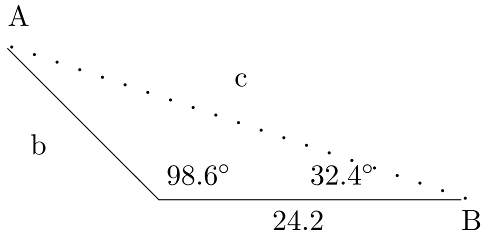
Note $\gamma = 180° - (42.3° + 62.1°) = 75.6°$.
By the sine law,

$$\begin{aligned} \frac{b}{\sin 62.1°} &= \frac{14.7}{\sin 75.6°} \\ b &= \frac{14.7}{\sin 75.6°} \cdot \sin 62.1° \\ b &\approx 13.41. \end{aligned}$$

The area is $A = \dfrac{1}{2}bc\sin\alpha =$

$\dfrac{1}{2}(13.41)(14.7)\sin 42.3° \approx 66.3.$

26. Draw angle $\gamma = 98.6°$.



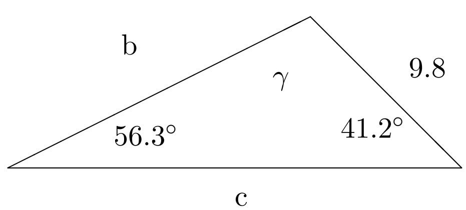
Note $\alpha = 180° - (98.6° + 32.4°) = 49°$.
By the sine law,

$$\begin{aligned} \frac{b}{\sin 32.4°} &= \frac{24.2}{\sin 49°} \\ b &= \frac{24.2}{\sin 49°} \sin 32.4° \\ b &\approx 17.181. \end{aligned}$$

The area is $A = \frac{1}{2} ab \sin \gamma =$

$\frac{1}{2}(24.2)(17.18) \sin 98.6° \approx 205.6.$

27. Draw angle $\alpha = 56.3°$.



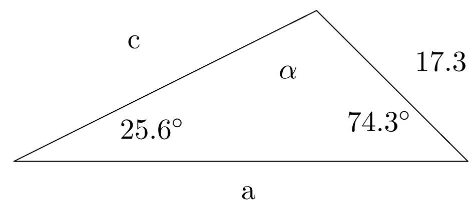
Note $\gamma = 180° - (56.3° + 41.2°) = 82.5°$.
By the sine law, we obtain

$$\begin{aligned} \frac{c}{\sin 82.5°} &= \frac{9.8}{\sin 56.3°} \\ c &= \frac{9.8}{\sin 56.3°} \sin 82.5° \\ c &\approx 11.679. \end{aligned}$$

The area is $A = \frac{1}{2} ac \sin \beta =$

$\frac{1}{2}(9.8)(11.679) \sin 41.2° \approx 37.7.$

28. Draw angle $\beta = 25.6°$.



Note $\alpha = 180° - (25.6° + 74.3°) = 80.1°$.
By the sine law, we get

$$\begin{aligned} \frac{a}{\sin 80.1°} &= \frac{17.3}{\sin 25.6°} \\ a &= \frac{17.3}{\sin 25.6°} \sin 80.1° \\ a &\approx 39.44. \end{aligned}$$

The area is $A = \frac{1}{2} ab \sin \gamma =$

$\frac{1}{2}(39.44)(17.3) \sin 74.3° \approx 328.4.$

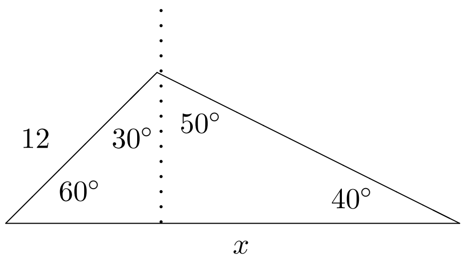
29. Divide the given 4-sided polygon into two triangles by drawing the diagonal that connects the 60° angle to the 135° angle. On each triangle two sides and an included angle are given. The area of the polygon is equal to the sum of the areas of the two triangles. Namely,

$$\frac{1}{2}(4)(10) \sin 120° + \frac{1}{2}(12+2\sqrt{3})(2\sqrt{6}) \sin 45° =$$
$$20(\sqrt{3}/2) + \frac{1}{2}(24\sqrt{6} + 4\sqrt{18})(\sqrt{2}/2) =$$
$$10\sqrt{3} + \frac{1}{2}(12\sqrt{12} + 2\sqrt{36}) =$$
$$10\sqrt{3} + 6\sqrt{12} + \sqrt{36} = 10\sqrt{3} + 12\sqrt{3} + 6 =$$
$$22\sqrt{3} + 6.$$

30. Divide the given 4-sided polygon into two triangles by drawing the diagonal that connects the 89° angle to the 109° angle. On each triangle two sides and an included angle are given. The area of the polygon is equal to the sum of the areas of the two triangles. Namely,

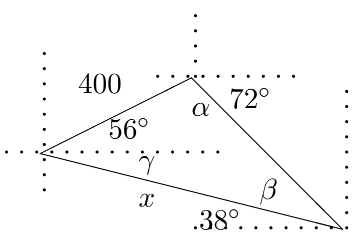
$$\frac{1}{2}\cdot(140)(129.44)\sin 70^\circ + \frac{1}{2}\cdot(120)(93.67)\sin 92^\circ \approx 14,131 \text{ ft}^2.$$

31. Let x be the number of miles flown along I-20.



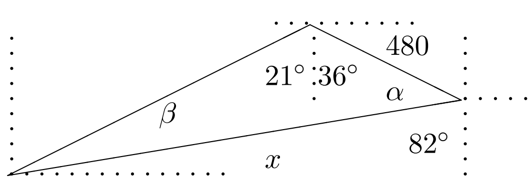
By the sine law, $\dfrac{x}{\sin 80^\circ} = \dfrac{12}{\sin 40^\circ}$.
Then $x \approx 18.4$ miles.

32. Let x be the distance of the final leg.



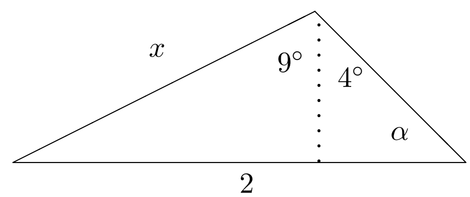
There is a 72° angle because of the 162° bearing. There is a 38° angle because of the 308° bearing. Since $\beta + 38^\circ = 72^\circ$, $\beta = 34^\circ$. Since opposite angles are equal, $\gamma = 38^\circ$. So $\alpha = 52^\circ$. From the sine law, $\dfrac{x}{\sin 52^\circ} = \dfrac{400}{\sin 34^\circ}$. So $x \approx 563.7$ miles.

33. Let x be the length of the third side.



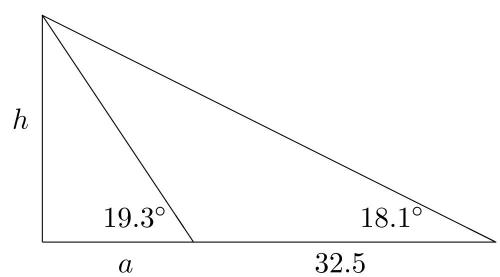
There is a 21° angle because of the $S21^\circ W$ direction. There are 36° and 82° angles because opposite angles are equal and because of the directions $N36^\circ W$ and $N82^\circ E$.
Note $\alpha = 180^\circ - (82^\circ + 36^\circ) = 62^\circ$ and $\beta = 180 - (21^\circ + 36^\circ + 62^\circ) = 61^\circ$. By the sine law, $x = \dfrac{480}{\sin 61^\circ}\sin 57^\circ \approx 460.27$. The area is $\dfrac{1}{2}(460.27)(480)\sin 62^\circ \approx 97,535$ sq ft.

34. Let x be the distance Jill sailed.



Note that $\alpha = 86^\circ$. By the sine law, $\dfrac{2}{\sin 13^\circ} = \dfrac{x}{\sin 86^\circ}$. Then $x \approx 8.9$ miles.

35. Let h be the height of the tower.

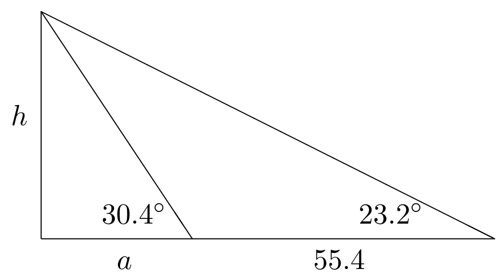


By using right triangle trigonometry, we get $\tan 19.3^\circ = \dfrac{h}{a}$ or $a = \dfrac{h}{\tan 19.3^\circ}$. Similarly, we have $\tan 18.1^\circ = \dfrac{h}{a + 32.5}$. Then

$$\begin{aligned}
\tan 18.1^\circ(a + 32.5) &= h \\
a\tan 18.1^\circ + 32.5\tan 18.1^\circ &= h \\
\frac{h}{\tan 19.3^\circ}\cdot\tan 18.1^\circ + 32.5\tan 18.1^\circ &= h \\
h\cdot\frac{\tan 18.1^\circ}{\tan 19.3^\circ} + 32.5\tan 18.1^\circ &= h.
\end{aligned}$$

Solving for h, we find that the height of the tower is $h \approx 159.4$ ft.

36. Let h be the height of the building.

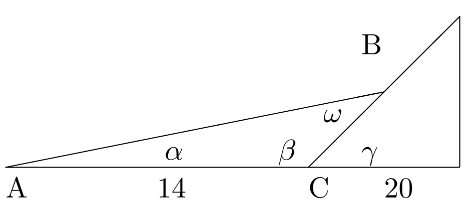


By using right triangle trigonometry, we get $\tan 30.4° = \dfrac{h}{a}$ or $a = \dfrac{h}{\tan 30.4°}$. Similarly, we have $\tan 23.2° = \dfrac{h}{a + 55.4}$. Then

$$\begin{aligned}
\tan 23.2°(a + 55.4) &= h \\
a \tan 23.2° + 55.4 \tan 23.2° &= h \\
\frac{h}{\tan 30.4°} \cdot \tan 23.2° + 55.4 \tan 23.2° &= h \\
h \cdot \frac{\tan 23.2°}{\tan 30.4°} + 55.4 \tan 23.2° &= h.
\end{aligned}$$

Solving for h, we find that the height of the building is $h \approx 88.1$ ft.

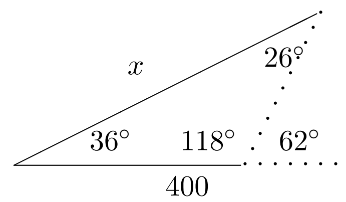
37. Note, $\tan \gamma = 6/12$ and $\gamma = \tan^{-1}(0.5) \approx 26.565°$. Also, $\tan \alpha = 3/12$ and $\alpha = \tan^{-1}(0.25) \approx 14.036°$.



The remaining angles are $\beta = 153.435°$ and $\omega = 12.529°$.

By the sine law, $\dfrac{AB}{\sin 153.435°} = \dfrac{14}{\sin 12.529°}$ and $\dfrac{BC}{\sin 14.036°} = \dfrac{14}{\sin 12.529°}$.
Then $AB \approx 28.9$ ft and $BC \approx 15.7$ ft.

38. Let x be the distance up the hill.



By the sine law, we obtain $\dfrac{x}{\sin 118°} = \dfrac{400}{\sin 26°}$.
So $x \approx 805.7$ ft, and yes the tree will have to be excavated.

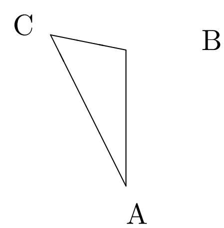
39. The kite consists of two equal triangles. The area of the kite is twice the area of the triangle.

It is $2\dfrac{1}{2}(24)(18)\sin 40° \approx 277.7$ in^2.

40. Since two sides and an included angle are given, the area of one wing, which is a two-sided triangle, is

$\dfrac{1}{2}(37.6)(19.2)\sin 68° \approx 334.7$ ft^2.

41. **a)** Consider the triangle where A is the center of the earth, B is a point on the surface of the earth, and C is a point on the atmosphere.



The angle at B is 120°. Let γ be the angle at C. Using the Sine Law, we obtain

$$\begin{aligned}
\frac{3960}{\sin 120°} &= \frac{3950}{\sin \gamma} \\
\sin \gamma &= \frac{3950 \sin 120°}{3960} \\
\gamma &\approx 59.75°.
\end{aligned}$$

The angle at A is $\alpha \approx 0.25°$.
Then d is given by

$$\frac{3960}{\sin 120°} \approx \frac{d}{\sin 0.25°}$$

$$\begin{aligned} d &\approx \frac{3960 \sin 0.25^\circ}{\sin 120^\circ} \\ d &\approx 19.9 \text{ miles.} \end{aligned}$$

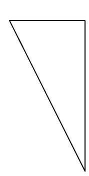
b) In the picture given in part a), let θ be the angle at A. Using sine law, one finds

$$\begin{aligned} \frac{\sin\theta}{93,000,000} &= \frac{\sin(120^\circ)}{93,003,950} \\ \theta &= \sin^{-1}\left(\frac{93,000,000\sin(120^\circ)}{93,003,950}\right) \\ \theta &\approx 59.99578^\circ. \end{aligned}$$

Suppose the sun is overhead at noon and the earth rotates 15° every hour. Since $\frac{59.99578}{15} \approx 3.999719$ (the number of hours since noon), when the angle of elevation is 30° the time is 1 second before 4:00 p.m.

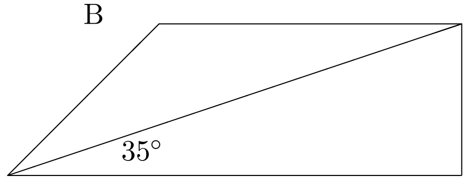
c) In the triangle in part a), at sunset the angle at B is 90°. If d_s is the distance through the atmosphere at sunset, then $d_s^2 + 3950^2 = 3960^2$ or $d = \sqrt{3960^2 - 3950^2} \approx 281$ miles.

42. Consider the right triangle where A is a point on the surface of the earth.



The distance AC that the sunlight passes is given by $AC = \frac{10}{\cos 60^\circ} = 20$ miles.

43. Let t be the number of seconds since the cruise missile was spotted.

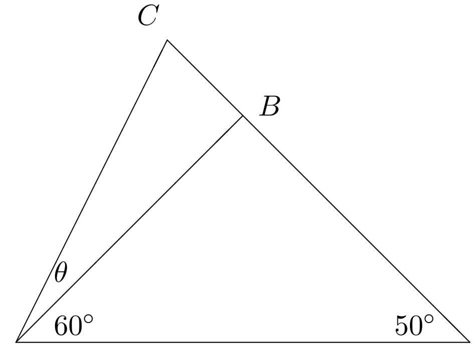


Let β be the angle at B. The angle formed by BAC is $180^\circ - 35^\circ - \beta$. After t seconds, the cruise missile would have traveled $548\frac{t}{3600}$ miles and the projectile $688\frac{t}{3600}$ miles. Using the law of sines, we have

$$\begin{aligned} \frac{\frac{548t}{3600}}{\sin(145^\circ - \beta)} &= \frac{\frac{688t}{3600}}{\sin 35^\circ} \\ \frac{548}{\sin(145^\circ - \beta)} &= \frac{688}{\sin 35^\circ} \\ \beta &= 145^\circ - \sin^{-1}\left(\frac{548\sin 35^\circ}{688}\right) \\ \beta &\approx 117.8^\circ. \end{aligned}$$

Then angle BAC is 27.2°. The angle of elevation of the projectile must be the angle DAB which is 62.2° ($= 35^\circ + 27.2^\circ$).

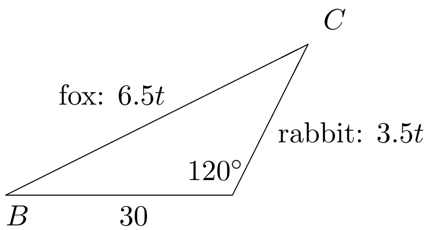
44. Let t be the number of seconds since Jones threw the ball.



After t seconds, Smith would have ran $17t$ feet and the ball would have covered $60t$ feet. Note, the angle $\angle ABC$ is 110°. Using the law of sines, the angle θ is given by

$$\begin{aligned} \frac{17t}{\sin\theta} &= \frac{60t}{\sin 110^\circ} \\ \frac{17}{\sin\theta} &= \frac{60}{\sin 110^\circ} \\ \theta &= \sin^{-1}\left(\frac{17\sin 110^\circ}{60}\right) \\ \theta &\approx 15.4^\circ. \end{aligned}$$

45. Let t be the number of seconds it takes the fox to catch the rabbit. The distances travelled by the fox and rabbit are indicated below.



Apply the sine law as follows:

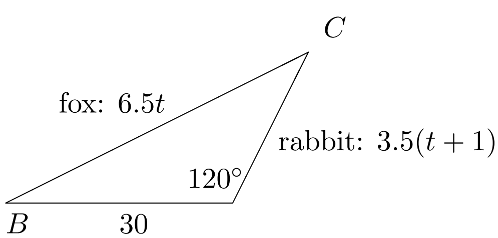
$$\begin{aligned} \frac{6.5t}{\sin 120°} &= \frac{3.5t}{\sin B} \\ \sin B &= \frac{3.5\sqrt{3}}{13} \end{aligned}$$

Note, $C = 60° - \arcsin\left(\frac{3.5\sqrt{3}}{13}\right)$. Then

$$\begin{aligned} \frac{30}{\sin C} &= \frac{3.5t}{\sin B} \\ t &= \frac{30 \sin B}{3.5 \sin C} = 7.5 \end{aligned}$$

It will take 7.5 sec to catch the rabbit.

46. Let t be the number of seconds it takes the fox to catch the rabbit. The distances travelled by the fox and rabbit are indicated below.



Apply the sine law as follows:

$$\begin{aligned} \frac{3.5(t+1)}{\sin B} &= \frac{6.5t}{\sin 120°} \\ \sin B &= \frac{7\sqrt{3}}{26}\left(1 + \frac{1}{t}\right) \end{aligned}$$

Note, $C = 60° - \arcsin\left(\frac{7\sqrt{3}}{26}\left(1 + \frac{1}{t}\right)\right)$.

Then

$$\begin{aligned} \frac{30}{\sin C} &= \frac{6.5t}{\sin 120°} \\ \frac{30}{\sin\left(60° - \arcsin\left(\frac{7\sqrt{3}}{26}\left(1 + \frac{1}{t}\right)\right)\right)} &= \frac{13t}{\sqrt{3}} \end{aligned}$$

Using a solver from a calculator, the solution to the above equation is

$$t \approx 8.37 \text{ sec}$$

which is the time it will take the fox to catch the rabbit.

49. The shortest side is 7 cm. The angle opposite the shortest side is $90° - 64° = 26°$. If h is the hypotenuse, then

$$h = \frac{7}{\sin 26°} \approx 16.0 \text{ cm.}$$

If x is the longer leg, then

$$x = h \sin 64° \approx 14.4 \text{ cm.}$$

50. $\cos\alpha = -\sqrt{1 - (-5/6)^2} = -\dfrac{\sqrt{11}}{6}$

$\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{-5/6}{-\sqrt{11}/6} = \dfrac{5\sqrt{11}}{11}$

51. Rewrite as $y = 5\sin\left(4\left(x - \frac{\pi}{4}\right)\right) - 3$.

The amplitude is 5, period is $\dfrac{2\pi}{B} = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$, phase shift is $\dfrac{\pi}{4}$, and the range is $[-5-3, 5-3]$ or $[-8, 2]$.

52. Since $B = \pi$, the period is $\dfrac{2\pi}{B} = \dfrac{2\pi}{\pi} = 2$.

Solve $\pi x - \pi = j\pi$ where j is an integer. Then

$$\begin{aligned} x - 1 &= j \\ x &= j + 1 \end{aligned}$$

Since $k = j + 1$ is an integer, the asymptotes are the vertical lines $x = k$.

The range is $(-\infty, -3 + 2] \cup [3 + 2, \infty)$ or $(-\infty, -1] \cup [5, \infty)$.

53. $3 - 3\sin^2 x = 3(1 - \sin^2 x) = 3\cos^2 x$

54. Note, $(\sin x + \cos x)^2 = \sin^2 x + 2\sin x \cos x + \cos^2 x = 1 + 2\sin x \cos x = 1 + \sin 2x$.

Then $(\sin x + \cos x)^2 - \sin 2x = 1$

Thinking Outside the Box

LX. Place the box on the floor so that it stands 24 in. up the floor, and the dimensions of the box that faces the floor is 4 in.-by-24 in. Then place 8 cans on the bottom row of the box, then 7 cans on top of the row of 8 cans, then 8 cans on the top of the row of 7 cans, and so on. All the cans in the box are tangent to each other.

Note, the top most part of any can in the second row is

$$3 + \sqrt{7.75} \text{ inches}$$

from the floor by the Pythagorean Theorem. Also, the distance from the floor to the top most part of any can on the top row is

$$3 + 7\sqrt{7.75} \approx 22.5 \text{ inches.}$$

Thus, you will stack nine rows of cans with five of the rows having 8 cans each, and four of the rows have 7 cans each. Hence, there are a total of 68 cans $(= 5(8) + 4(7))$.

LXI. Let x be the number of bricks. Working together for six hours, the number of bricks that Red lays is $6x/10 - 60$, and Slim lays $6x/12 - 60$ bricks. All together, we find

$$\frac{6}{10}x - 60 + \frac{6}{12}x - 60 = x.$$

The solution is $x = 1200$ bricks.

7.1 Pop Quiz

1. $\gamma = 180° - 8° - 121° = 51°$

2. Using the sine law, we obtain

$$\frac{a}{\sin 20.4°} = \frac{38.5}{\sin 132.3°}.$$

Solving for a, we find

$$a = \frac{38.5 \sin 20.4°}{\sin 132.3°} \approx 18.1.$$

3. Using the sine law, we obtain

$$\frac{7.4}{\sin 33.5°} = \frac{10.6}{\sin \sin \beta}.$$

Solving for $\sin\beta$, we find

$$\sin\beta = \frac{10.6 \sin 33.5°}{7.4}.$$

Solving for β, we obtain

$$\beta = \sin^{-1}\left(\frac{10.6 \sin 33.5°}{7.4}\right)$$

or

$$\beta = 180° - \sin^{-1}\left(\frac{10.6 \sin 33.5°}{7.4}\right).$$

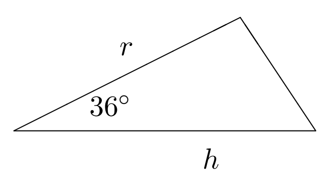
Thus, $\beta \approx 52.2°$ or $\beta \approx 127.8°$.

4. The area is

$$A = \frac{1}{2}(6)(15)\sin 66.7° \approx 41.3 \text{ ft}^2.$$

7.1 Linking Concepts

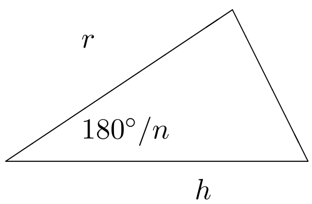
a) Consider the right triangle where the length of the side opposite the 36° angle is one-half the length of a side of the pentagon.



Note, $h = \dfrac{r}{\cos 36°}$. The area of an isosceles triangle with two sides of equal length h and an included angle of 72° is $\frac{1}{2}(h)^2 \sin 72°$. The area, A, of the pentagon is given by

$$\begin{aligned} A &= 5 \cdot \frac{1}{2}\left(\frac{r}{\cos 36°}\right)^2 \sin 72° \\ &= \frac{5}{2}\frac{r^2}{\cos^2 36°} \sin 72° \\ &= \frac{5}{2}\frac{r^2}{\cos^2 36°}(2\sin(36°)\cos(36°)) \\ A &= 5r^2 \tan 36°. \end{aligned}$$

b) Consider the right triangle where the length of the side opposite the $\dfrac{180°}{n}$ angle is one-half the length of a side of the regular polygon.



Note, $h = \dfrac{r}{\cos(180°/n)}$. The area A_t of an isosceles triangle with two sides of equal length h and an included angle of $\dfrac{360°}{n}$ is

$$A_t = \frac{1}{2}(h)^2 \sin\left(\frac{360°}{n}\right).$$

The area, A, of the regular polygon is given by

$$\begin{aligned} A &= n \cdot \frac{1}{2}\left(\frac{r}{\cos(180°/n)}\right)^2 \sin(360°/n) \\ &= \frac{n}{2}\frac{r^2}{\cos^2(180°/n)} \sin(360°/n) \\ &= \frac{nr^2}{2\cos^2(180°/n)}(2\sin(180°/n)\cos(180°/n)) \\ A &= nr^2 \tan\left(\frac{180°}{n}\right). \end{aligned}$$

c) If A is the area of a regular polygon and r is the radius of the circle inscribed in the polygon, then A varies directly with r^2.

d) The proportion constants are
$10\tan 18° \approx 3.249$,
$1000\tan(0.18°) \approx 3.141603$, and
$10^6 \tan\left(\dfrac{180°}{10^6}\right) \approx 3.141592654$.

e) When n is a large number, the shape of a regular n-gon approximates the shape of a circle. The area A of a circle with radius r can be approximated by $nr^2\tan\left(\dfrac{180°}{n}\right)$ where n is a large number. That is, if $n = 10^6$ then $A \approx 3.141592654r^2$; or better, $A = \pi r^2$.

f) Based on the triangle in part b), if L is the length of one side of a regular n-gon then $\sin\left(\dfrac{180°}{n}\right) = \dfrac{L/2}{h}$. Moreover,

$$\begin{aligned} L &= 2h\sin\left(\frac{180°}{n}\right) \\ &= 2\frac{r}{\cos\left(\frac{180°}{n}\right)}\sin\left(\frac{180°}{n}\right) \\ &= 2r\tan\left(\frac{180°}{n}\right). \end{aligned}$$

Multiplying the last quantity by n, we get the perimeter of an n-gon, namely,

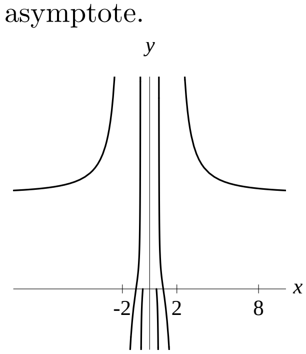
$$P = 2nr\tan\left(\frac{180°}{n}\right).$$

g) The circumference C of a circle with radius r could be approximated by $2nr\tan\left(\dfrac{180°}{n}\right)$ where n is a large number, or

$$C \approx 2r(3.141592654)$$

or exactly by $C = 2\pi r$.

h) The vertical asymptotes of $y = x\tan\left(\dfrac{\pi}{x}\right)$ are $x = w$ where $\dfrac{\pi}{w} = \dfrac{\pi}{2} + k\pi$. Solving for w, one finds $w = \dfrac{2}{2k+1}$. Thus, the vertical asymptotes are $x = \dfrac{2}{2k+1}$ where k is an integer. Note, $y = \pi$ is the horizontal asymptote.



For Thought

1. True

2. False, $a = \sqrt{c^2 + b^2 - 2bc\cos\alpha}$.

3. False, $c^2 = a^2 + b^2 - 2ab\cos\gamma$.

4. True

5. False, it has only one solution.

6. True

7. True

8. True

9. True, since $\cos\gamma = \dfrac{3.4^2 + 4.2^2 - 8.1^2}{2(3.4)(4.2)} \approx -1.27$ has no real solution γ.

10. False, an obtuse triangle exists.

7.2 Exercises

1. law of cosines

2. triangle inequality

3. By the cosine law, we obtain $c = \sqrt{3.1^2 + 2.9^2 - 2(3.1)(2.9)\cos 121.3°}$ $\approx 5.23 \approx 5.2$. By the sine law, we find

$$\begin{aligned} \frac{3.1}{\sin\alpha} &= \frac{5.23}{\sin 121.3°} \\ \sin\alpha &= \frac{3.1\sin 121.3°}{5.23} \\ \sin\alpha &\approx 0.50647 \\ \alpha \approx \sin^{-1}(0.50647) &\approx 30.4°. \end{aligned}$$

Then $\beta = 180° - (30.4° + 121.3°) = 28.3°$.

4. By the cosine law, we get $a = \sqrt{11.4^2 + 10.3^2 - 2(11.4)(10.3)\cos 40.2°}$ $\approx 7.53 \approx 7.5$. By the sine law,

$$\begin{aligned} \frac{10.3}{\sin\beta} &= \frac{7.53}{\sin 40.2°} \\ \sin\beta &= \frac{10.3\sin 40.2°}{7.53} \\ \sin\beta &\approx 0.8829 \\ \beta \approx \sin^{-1}(0.8829) &\approx 62.0°. \end{aligned}$$

Then $\gamma = 180° - (62° + 40.2°) = 77.8°$.

5. By the cosine law, we find $\cos\beta = \dfrac{6.1^2 + 5.2^2 - 10.3^2}{2(6.1)(5.2)} \approx -0.6595$ and so $\beta \approx \cos^{-1}(-0.6595) \approx 131.3°$. By the sine law,

$$\begin{aligned} \frac{6.1}{\sin\alpha} &= \frac{10.3}{\sin 131.3°} \\ \sin\alpha &= \frac{6.1\sin 131.3°}{10.3} \\ \sin\alpha &\approx 0.4449 \\ \alpha \approx \sin^{-1}(0.4449) &\approx 26.4°. \end{aligned}$$

So $\gamma = 180° - (26.4° + 131.3°) = 22.3°$.

6. By the cosine law, we obtain $\cos\gamma = \dfrac{7.9^2 + 6.5^2 - 13.6^2}{2(7.9)(6.5)} \approx -0.7819$ and so $\gamma \approx \cos^{-1}(-0.7819) \approx 141.4°$. By the sine law, we have

$$\begin{aligned} \frac{6.5}{\sin\alpha} &= \frac{13.6}{\sin 141.4°} \\ \sin\alpha &= \frac{6.5\sin 141.4°}{13.6} \\ \sin\alpha &\approx 0.29818 \\ \alpha \approx \sin^{-1}(0.29818) &\approx 17.3° \end{aligned}$$

Also, $\beta = 180° - (17.3° + 141.4°) = 21.3°$.

7. By the cosine law, $b = \sqrt{2.4^2 + 6.8^2 - 2(2.4)(6.8)\cos 10.5°}$ $\approx 4.46167 \approx 4.5$ and $\cos\alpha = \dfrac{2.4^2 + 4.46167^2 - 6.8^2}{2(2.4)(4.46167)} \approx -0.96066$. So $\alpha = \cos^{-1}(-0.96066) \approx 163.9°$ and $\gamma = 180° - (163.9° + 10.5°) = 5.6°$

8. By the cosine law, $c = \sqrt{1.3^2 + 14.9^2 - 2(1.3)(14.9)\cos 9.8°}$ $\approx 13.62 \approx 13.6$ and $\cos\alpha = \dfrac{14.9^2 + 13.62^2 - 1.3^2}{2(14.9)(13.62)} \approx 0.99987$. So $\alpha = \cos^{-1}(0.99987) \approx 0.9°$ and $\beta = 180° - (0.9° + 9.8°) = 169.3°$.

9. By the cosine law,

$$\cos\alpha = \frac{12.2^2 + 8.1^2 - 18.5^2}{2(12.2)(8.1)} \approx -0.6466.$$

Then $\alpha = \cos^{-1}(-0.6466) \approx 130.3°$.
By the sine law,

$$\begin{aligned} \frac{12.2}{\sin\beta} &= \frac{18.5}{\sin 130.3°} \\ \sin\beta &= \frac{12.2\sin 130.3°}{18.5} \\ \sin\beta &\approx 0.5029 \\ \beta \approx \sin^{-1}(0.5029) &\approx 30.2° \end{aligned}$$

So $\gamma = 180° - (30.2° + 130.3°) = 19.5°$

10. By the cosine law,

$$\cos\gamma = \frac{30.4^2 + 28.9^2 - 31.6^2}{2(30.4)(28.9)} \approx 0.433.$$

So $\gamma = \cos^{-1}(0.433) \approx 64.3°$.
By the sine law,

$$\begin{aligned} \frac{28.9}{\sin\beta} &= \frac{31.6}{\sin 64.3°} \\ \sin\beta &= \frac{28.9\sin 64.3°}{31.6} \\ \sin\beta &\approx 0.824 \\ \beta \approx \sin^{-1}(0.824) &\approx 55.5° \end{aligned}$$

So $\alpha = 180° - (55.5° + 64.3°) = 60.2°$

11. By the cosine law, we obtain
$a = \sqrt{9.3^2 + 12.2^2 - 2(9.3)(12.2)\cos 30°}$
$\approx 6.23 \approx 6.2$ and

$$\cos\gamma = \frac{6.23^2 + 9.3^2 - 12.2^2}{2(6.23)(9.3)} \approx -0.203.$$

So $\gamma = \cos^{-1}(-0.203) \approx 101.7°$ and
$\beta = 180° - (101.7° + 30°) = 48.3°$.

12. By the cosine law, we find
$b = \sqrt{10.3^2 + 8.4^2 - 2(10.3)(8.4)\cos 88°}$
$\approx 13.062 \approx 13.1$, and using $b \approx 13.062$ we find

$$\cos\alpha = \frac{13.062^2 + 8.4^2 - 10.3^2}{2(13.062)(8.4)} \approx 0.616.$$

So $\alpha = \cos^{-1}(0.616) \approx 52.0°$ and
$\gamma \approx 180° - (52.0° + 88°) \approx 40.0°$.

13. By the cosine law,

$$\cos\beta = \frac{6.3^2 + 6.8^2 - 7.1^2}{2(6.3)(6.8)} \approx 0.4146.$$

So $\beta = \cos^{-1}(0.4146) \approx 65.5°$.
By the sine law, we have

$$\begin{aligned} \frac{6.8}{\sin\gamma} &= \frac{7.1}{\sin 65.5°} \\ \sin\gamma &= \frac{6.8\sin 65.5°}{7.1} \\ \sin\gamma &\approx 0.8715 \\ \gamma \approx \sin^{-1}(0.8715) &\approx 60.6°. \end{aligned}$$

So $\alpha = 180° - (60.6° + 65.5°) = 53.9°$.

14. By the cosine law,

$$\cos\beta = \frac{4.1^2 + 6.2^2 - 9.8^2}{2(4.1)(6.2)} \approx -0.8023.$$

So $\beta = \cos^{-1}(-0.8023) \approx 143.4°$.
By the sine law, we find

$$\begin{aligned} \frac{6.2}{\sin\gamma} &= \frac{9.8}{\sin 143.4°} \\ \sin\gamma &= \frac{6.2\sin 143.4°}{9.8} \\ \sin\gamma &\approx 0.3372 \\ \gamma \approx \sin^{-1}(0.3372) &\approx 22.2°. \end{aligned}$$

Then $\alpha = 180° - (22.2° + 143.4°) = 14.4°$.

15. Note, $\alpha = 180° - 25° - 35° = 120°$.
Then by the sine law, we obtain

$$\frac{7.2}{\sin 120°} = \frac{b}{\sin 25°} = \frac{c}{\sin 35°}$$

from which we have

$$b = \frac{7.2\sin 25°}{\sin 120°} \approx 3.5$$

and

$$c = \frac{7.2\sin 35°}{\sin 120°} \approx 4.8.$$

16. Note, $\beta = 180° - 120° - 20° = 40°$.
Then by the sine law, we obtain

$$\frac{12.3}{\sin 40°} = \frac{a}{\sin 20°} = \frac{c}{\sin 120°}$$

from which we have

$$a = \frac{12.3 \sin 20°}{\sin 40°} \approx 6.5$$

and

$$c = \frac{12.3 \sin 120°}{\sin 40°} \approx 16.6.$$

17. There is no such triangle. Note, $a + b = c$ and in a triangle the sum of the lengths of two sides is greater than the length of the third side.

18. There is no such triangle. Note, $a + c < b$ and in a triangle the sum of the lengths of two sides is greater than the length of the third side.

19. One triangle exists. The angles are uniquely determined by the law of cosines.

20. One triangle exists. The angles are uniquely determined by the law of cosines.

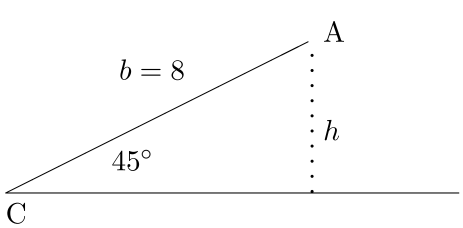
21. There is no such triangle since the sum of the angles in a triangle is 180°.

22. There is no such triangle. By drawing angles $\gamma = 120°$ and $\alpha = 62°$, one will find that sides a and c do not intersect. And, so a third vertex does not exist.

23. Exactly one triangle exists. This is seen by constructing a 179°-angle with two sides that have lengths 1 and 10. The third side is constructed by joining the ndpoints of the first two sides.

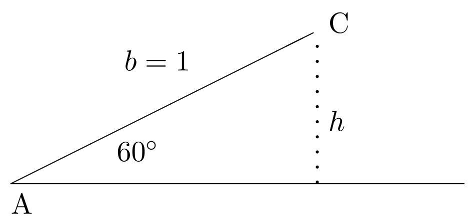
24. Exactly one triangle exists. This is seen by constructing a 2°-angle with two sides that have lengths 10 and 4. The third side is constructed by joining the ndpoints of the first two sides.

25. Consider the figure below.



Note, $h = 8 \sin 45° = 2\sqrt{2}$. Then the minimum value of c so that we will be able to make a triangle is $2\sqrt{2}$. Since $c = 2$, no such triangle is possible.

26. Consider the figure below.



Note, $h = \sin 60° = \dfrac{\sqrt{3}}{2}$. So the minimum value of a so that we will be able to make a triangle is $\dfrac{\sqrt{3}}{2}$. Since $a = \dfrac{\sqrt{3}}{2}$, exactly one triangle exists and it is a right triangle.

27. Note $S = \dfrac{16 + 9 + 10}{2} = 17.5$. The area is

$$A = \sqrt{17.5(17.5 - 16)(17.5 - 9)(17.5 - 10)}$$
$$= \sqrt{17.5(1.5)(8.5)(7.5)} \approx 40.9 \ .$$

28. Note $S = \dfrac{12 + 8 + 17}{2} = 18.5$. The area is

$$A = \sqrt{18.5(18.5 - 12)(18.5 - 8)(18.5 - 17)}$$
$$= \sqrt{18.5(6.5)(10.5)(1.5)} \approx 43.5 \ .$$

29. Note $S = \dfrac{3.6 + 9.8 + 8.1}{2} = 10.75$. The area is

$$\sqrt{10.75(10.75 - 3.6)(10.75 - 9.8)(10.75 - 8.1)}$$
$$= \sqrt{10.75(7.15)(0.95)(2.65)} \approx 13.9 \ .$$

30. Note $S = \dfrac{5.4 + 8.2 + 12}{2} = 12.8$. The area is

$$\sqrt{12.8(12.8 - 5.4)(12.8 - 8.2)(12.8 - 12)} =$$
$$\sqrt{12.8(7.4)(4.6)(0.8)} \approx 18.7 \ .$$

31. Note $S = \dfrac{346 + 234 + 422}{2} = 501$. The area is

$$\sqrt{501(501 - 346)(501 - 234)(501 - 422)} =$$
$$\sqrt{501(155)(267)(79)} \approx 40,471.9 \ .$$

32. Note $S = \dfrac{124.8 + 86.4 + 154.2}{2} = 182.7$.

The area is

$$\sqrt{182.7(182.7 - 124.8)} \times \sqrt{(182.7 - 86.4)(182.7 - 154.2)} =$$

$\sqrt{182.7(57.9)(96.3)(28.5)} \approx 5388.2$.

33. Since the base is 20 and the height is 10, the area is $\dfrac{1}{2}bh = \dfrac{1}{2}(20)(10) = 100$.

34. Note $S = \dfrac{7+8+5}{2} = 10$.

Area is $\sqrt{10(10-7)(10-8)(10-5)} = \sqrt{10(3)(2)(5)} \approx 17.3$.

35. Since two sides and an included angle are given, the area is $\dfrac{1}{2}(6)(8)\sin 60^\circ \approx 20.8$.

36. Since the base is 12 and the height is 9, the area is $\dfrac{1}{2}bh = \dfrac{1}{2}(12)(9) = 54$.

37. Note $S = \dfrac{9+5+12}{2} = 13$.

The area is $\sqrt{13(13-9)(13-5)(13-12)} = \sqrt{13(4)(8)(1)} \approx 20.4$

38. Since two sides and an included angle are given, the area is $\dfrac{1}{2}(9)(15)\sin 14^\circ \approx 16.3$.

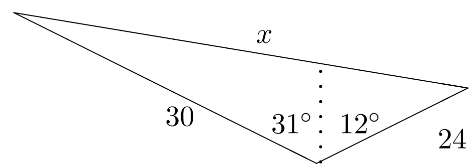
39. Recall, a central angle α in a circle of radius r intercepts a chord of length $r\sqrt{2 - 2\cos\alpha}$. Since $r = 30$ and $\alpha = 19^\circ$, the length is $30\sqrt{2 - 2\cos 19^\circ} \approx 9.90$ ft.

40. Recall, a central angle α in a circle of radius r intercepts a chord of length $r\sqrt{2 - 2\cos\alpha}$. Since $r = 3$ and $\alpha = 20^\circ$, the length is $3\sqrt{2 - 2\cos 20^\circ} \approx 1.04$ miles.

41. Note, a central angle α in a circle of radius r intercepts a chord of length $r\sqrt{2 - 2\cos\alpha}$. Since $921 = r\sqrt{2 - 2\cos 72^\circ}$ (where $360 \div 5 = 72$), we get $r = \dfrac{921}{\sqrt{2 - 2\cos 72^\circ}} \approx 783.45$ ft.

42. Note, a central angle α in a circle of radius r intercepts a chord of length $r\sqrt{2 - 2\cos\alpha}$. Since $10 = r\sqrt{2 - 2\cos 60^\circ}$ (where $360 \div 6 = 60$), we get $r = \dfrac{10}{\sqrt{2 - 2\cos 60^\circ}} = 10$ ft.

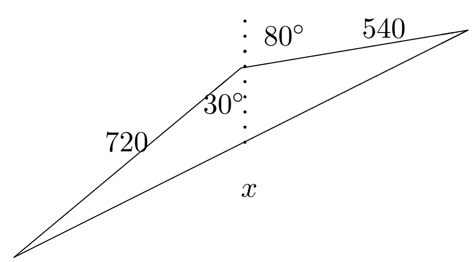
43. After 6 hours, Jan hiked a distance of 24 miles and Dean hiked 30 miles. Let x be the distance between them after 6 hrs.



By the cosine law,

$x = \sqrt{30^2 + 24^2 - 2(30)(24)\cos 43^\circ} = \sqrt{1476 - 1440\cos 43^\circ} \approx 20.6$ miles.

44. After 3 hours, Andrea flew a distance of 540 miles and Carlos flew 720 miles. Let x be the distance between them after 3 hrs.



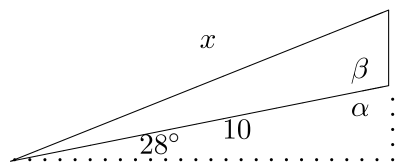
The obtuse angle in the triangle is 130°.
By the cosine law,

$x = \sqrt{720^2 + 540^2 - 2(720)(540)\cos 130^\circ} = \sqrt{810,000 - 777,600\cos 130^\circ} \approx 1144.5$ miles.

45. By the cosine law, we find

$$\begin{aligned}
\cos\alpha &= \frac{1.2^2 + 1.2^2 - 0.4^2}{2(1.2)(1.2)} \\
\cos\alpha &\approx 0.9444 \\
\alpha &\approx \cos^{-1}(0.9444) \\
\alpha &\approx 19.2^\circ.
\end{aligned}$$

46. Let x be the length of the guy wire.



Note $\alpha = 62^\circ$ and $\beta = 118^\circ$.
By the cosine law,
$x = \sqrt{10^2 + 6^2 - 2(10)(6)\cos 118^\circ} = \sqrt{136 - 120\cos 118^\circ} \approx 13.9$ ft.

47. Let α, β, and γ be the angles at gears A, B, and C. The length of the sides of the triangle are 5, 6, and 7. By the cosine law,

$$\begin{aligned} \cos\alpha &= \frac{5^2+6^2-7^2}{2(5)(6)} \\ \cos\alpha &= 0.2 \\ \alpha &= \cos^{-1}(0.2) \\ \alpha &\approx 78.5^\circ. \end{aligned}$$

By the sine law,

$$\begin{aligned} \frac{6}{\sin\beta} &= \frac{7}{\sin 78.5^\circ} \\ \sin\beta &\approx 0.8399 \\ \beta &\approx \sin^{-1}(0.8399) \\ \beta &\approx 57.1^\circ. \end{aligned}$$

Then $\gamma = 180^\circ - (57.1^\circ + 78.5^\circ) = 44.4^\circ$.

48. Let x be the distance the target has moved from the time it was fired to the time it was hit. By the cosine law,
$x = \sqrt{924^2 + 820^2 - 2(924)(820)\cos 9^\circ} = \sqrt{1,526,176 - 1,515,360\cos 9^\circ} \approx 171.7$ m.

49. By the cosine law,
$AB = \sqrt{5.3^2 + 7.6^2 - 2(5.3)(7.6)\cos 28^\circ} = \sqrt{85.85 - 80.56\cos 28^\circ} \approx 3.837 \approx 3.8$ miles.
Likewise,

$$\begin{aligned} \cos(\angle CBA) &= \frac{3.837^2+5.3^2-7.6^2}{2(3.837)(5.3)} \\ \cos(\angle CBA) &\approx -0.3675 \\ \angle CBA &\approx \cos^{-1}(-0.3675) \\ \angle CBA &\approx 111.6^\circ \end{aligned}$$

and $\angle CAB = 180^\circ - (111.6^\circ + 28^\circ) = 40.4^\circ$.

50. By the cosine law, one finds

$$\begin{aligned} 1.017^2 &= .133^2 + .894^2 - 2(.133)(.894)\cos\alpha \\ \cos\alpha &= \frac{.133^2+.894^2-1.017^2}{2(.133)(.894)} \\ \alpha &= \cos^{-1}\left(\frac{.133^2+.894^2-1.017^2}{2(.133)(.894)}\right) \\ \alpha &\approx 156^\circ \end{aligned}$$

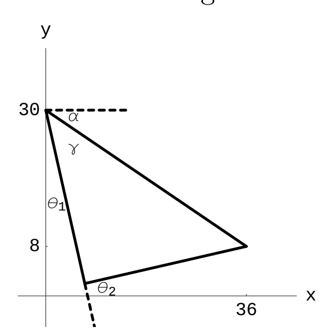
and $\theta \approx 180^\circ - 156^\circ = 24^\circ$.

51. The pentagon consists of 5 chords each of which intercepts a $\dfrac{360^\circ}{5} = 72^\circ$ angle.
By the cosine law, the length of a chord is given by $\sqrt{10^2 + 10^2 - 2(10)(10)\cos 72^\circ} = \sqrt{200 - 200\cos 72^\circ} \approx 11.76$ m.

52. By the cosine law, we obtain

$$\begin{aligned} \cos\alpha &= \frac{5^2+5^2-1^2}{2(5)(5)} \\ \cos\alpha &= \frac{49}{50} \\ \alpha &= \cos^{-1}(49/50) \\ \alpha &\approx 11.5^\circ. \end{aligned}$$

53. Consider the figure below.



Note $\tan\alpha = 22/36$ and $\alpha = \tan^{-1}(22/36) \approx 31.4^\circ$. The distance between $(36, 8)$ and $(0, 30)$ is approximately 42.19. By the cosine law,

$$\begin{aligned}\cos\beta &= \frac{30^2+30^2-42.19^2}{2(30)(30)}\\ \cos\beta &\approx 0.011\\ \beta\approx\cos^{-1}(0.011) &\approx 89.4^\circ.\end{aligned}$$

So $\theta_2 = 180^\circ - 89.4^\circ = 90.6^\circ$.

By the sine law, we find

$$\begin{aligned}\frac{30}{\sin\gamma} &= \frac{42.19}{\sin 89.4^\circ}\\ \sin\gamma &\approx 0.711\\ \gamma\approx\sin^{-1}(0.711) &\approx 45.3^\circ.\end{aligned}$$

Then $\theta_1 = 90^\circ - (45.3^\circ + 31.4^\circ) = 13.3^\circ$.

54. Since the arc length $s = 2.5$ mm intercepts an arc α, we have $2.5 = 10\alpha$ or $\alpha = 0.25$. Then a flat side of the shaft intercepts a $\dfrac{2\pi - 3(0.25)}{3} \approx 1.844$ radian angle. By the cosine law, the length of a flat side is $\sqrt{10^2+10^2-2(10)(10)\cos(1.844)} \approx$ 15.94 mm.

55. **a)** Let α_m and α_M be the minimum and maximum values of α, respectively. By the law of cosines, we get

$$865,000^2 = 2(91,400,000)^2 - 2(91,400,000)^2\cos\alpha_M.$$

Then

$$\begin{aligned}\alpha_M &= \cos^{-1}\left(\frac{2(91400000)^2-865000^2}{2(91400000)^2}\right)\\ \alpha_M &\approx 0.54^\circ.\end{aligned}$$

Likewise,

$$\begin{aligned}\alpha_m &= \cos^{-1}\left(\frac{2(94500000)^2-865000^2}{2(94500000)^2}\right)\\ \alpha_m &\approx 0.52^\circ.\end{aligned}$$

b) Let β_m and β_M be the minimum and maximum values of β, respectively. By the law of cosines, one obtains

$$2163^2 = 2(225,800)^2 - 2(225,800)^2\cos\beta_M.$$

Then

$$\begin{aligned}\beta_M &= \cos^{-1}\left(\frac{2(225800)^2-2163^2}{2(225800)^2}\right)\\ \beta_M &\approx 0.55^\circ.\end{aligned}$$

Likewise,

$$\begin{aligned}\beta_m &= \cos^{-1}\left(\frac{2(252000)^2-2163^2}{2(252000)^2}\right)\\ \beta_m &\approx 0.49^\circ.\end{aligned}$$

c) Yes, even in perfect alignment a total eclipse may not occur, for instance when $\beta = 0.49^\circ$ and $\alpha = 0.52^\circ$.

56. **a)** Let α_m and α_M be the minimum and maximum values of α (diameters of Jupiter), respectively. By the law of cosines, one obtains

$$\alpha_M = \cos^{-1}\left(\frac{2(7.406\times10^8)^2-(1.39\times10^6)^2}{2(7.406\times10^8)^2}\right)$$

or $\alpha_M \approx 0.11^\circ$, and

$$\alpha_m = \cos^{-1}\left(\frac{2(8.160\times10^8)^2-(1.39\times10^6)^2}{2(8.160\times10^8)^2}\right)$$

or $\alpha_m \approx 0.10^\circ$.

b) Let β be the diameter of Callisto. By the law of cosines, one obtains

$$\beta = \cos^{-1}\left(\frac{2(1.884\times10^6)^2-(2420)^2}{2(1.884\times10^6)^2}\right)$$

$\approx 0.07^\circ$.

c) No, a total eclipse is not possible since Callisto is too small.

57. Let d_b and d_h be the distance from the bear and hiker, respectively, to the base of the tower. Then $d_b = 150\tan 80^\circ$ and $d_h = 150\tan 75^\circ$.

Since the line segments joining the base of the tower to the bear and hiker form a 45° angle, by the cosine law the distance, d, between the bear and the hiker is

$$\begin{aligned}d &= \sqrt{d_b^2+d_h^2-2(d_b)(d_h)\cos 45^\circ}\\ &\approx \big((850.69)^2+(559.81)^2-\\ &\qquad 2(850.69)(559.81)\cos 45^\circ\big)^{1/2}\\ &\approx 603 \text{ feet}.\end{aligned}$$

58. Let t be the number of hours since midnight. Since the smuggler's have been riding for t hours and the DEA boat for $t-1$ hours, then $(20(t-1))^2 = (20t)^2 + 80^2 - 2(20t)(80)\cos 40^\circ$. Subtracting $400t^2$ from both sides, one obtains

$$\begin{aligned} -800t + 400 &= 6400 - 3200t\cos 40^\circ \\ (3200\cos 40^\circ - 800)t &= 6000 \\ t &= \frac{6000}{3200\cos 40^\circ - 800} \\ t &\approx 3.63 \quad \text{hours.} \end{aligned}$$

The interception occured at 3:38 a.m. since $(0.63)60 \approx 38$. The distances covered by the DEA's and smuggler's boats are 20(2.63) miles and 20(3.63) miles, respectively. To find θ, we use the sine law. Then

$$\begin{aligned} \frac{20(2.63)}{\sin 40^\circ} &= \frac{20(3.63)}{\sin\theta} \\ \frac{2.63}{\sin 40^\circ} &= \frac{3.63}{\sin\theta} \\ \theta &= \sin^{-1}\left(\frac{3.63\sin 40^\circ}{2.63}\right) \\ \theta &\approx 62.5^\circ. \end{aligned}$$

59. By using the cosine law, we obtain

$a = \sqrt{2r^2 - 2r^2\cos(\theta)} = \sqrt{4r^2\frac{1-\cos\theta}{2}} =$

$2r\sin(\theta/2)$.

60. If the second largest side were opposite an obtuse angle, then the triangle would have two obtuse sides and the angles would add up to more than 180°.

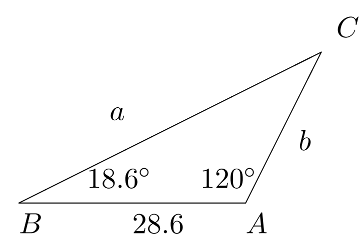
61. If $\alpha = 90^\circ$ in $a^2 = b^2 + c^2 - 2bc\cos\alpha$, then $a^2 = b^2 + c^2$ since $\cos 90^\circ = 0^\circ$. Thus the Pythagorean Theorem is a special case i.e., when the angle is 90°) of the law of cosines.

62. Note $S = \dfrac{37+48+86}{2} = 85.5$. By Heron's formula, the area of the triangle is suppose to be $\sqrt{85.5(85.5-37)(85.5-48)(85.5-86)}$. But this area is undefined since the area is a complex imaginary number. Thus, no triangle exists with sides 37, 48, and 86.

63. Note $S = \dfrac{31+87+56}{2} = 87$. By Heron's formula, the area of the triangle is suppose to be $\sqrt{87(87-31)(87-87)(87-56)}$. But this area is zero. Thus, no triangle exists with sides 31, 87, and 56.

64. Let $a = 6$, $b = 9$,and $c = 13$. Then $4b^2c^2 = 54,756$ and $(b^2+c^2-a^2)^2 = 45,796$. The area is given by $\frac{1}{4}\sqrt{4b^2c^2 - (b^2+c^2-a^2)^2} =$ $\frac{1}{4}\sqrt{54,756 - 45,796} = \frac{1}{4}\sqrt{8960} = 4\sqrt{35}$.

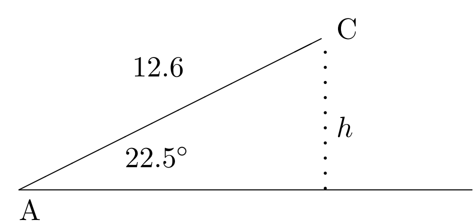
65. Using the triangle below,



we find $C = 180^\circ - A - B = 53.3^\circ$. Using the sine law, we obtain

$$\begin{aligned} a &= \frac{28.6\sin 108.1^\circ}{\sin 53.3^\circ} \approx 33.9 \\ b &= \frac{28.6\sin 18.6^\circ}{\sin 53.3^\circ} \approx 11.4 \end{aligned}$$

66. Draw angle $A = 22.5^\circ$ and let h be the height.



Then $h = 12.6\sin 22.5^\circ \approx 4.8$. Since side a opposite angle A satisfies $h < a < 12.6$, there are two triangles.

67. Since $\sin x = \pm 1$, we find $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

68. Since $\sin 2x = 0$ or $\cos 2x = 0$, we obtain $2x = k\pi$ or $2x = \frac{(2k+1)\pi}{2}$ where k is an integer. Then $x = \frac{k\pi}{2}$ or $x = \frac{(2k+1)\pi}{4}$. If $k = 0, 1, 2, 3$, we find

$$x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}.$$

69. **a)** Domain $[-1, 1]$, range $[-\pi 2, \pi/2]$

b) Domain $[-1, 1]$, range $[0, \pi]$

c) Domain $(-\infty, \infty)$, range $(-\pi/2, \pi/2)$

70. If $\frac{2\pi}{B} = 16$, we find $B = \frac{\pi}{8}$. The amplitude is $A = 10$, and an equation is

$$y = 10 \sin\left(\frac{\pi t}{8}\right).$$

Thinking Outside the Box LXII

Using Heron's formula, the area of the triangle is

$$A = \sqrt{15(6)4)(5)}.$$

Let α, β, and γ be the angles included by sides 9 & 10, 9 & 11, and 10 & 11, respectively. By the cosine law, we find

$$\alpha = \cos^{-1}\left(\frac{9^2 + 10^2 - 11^2}{2(9)(10)}\right)$$

$$\beta = \cos^{-1}\left(\frac{9^2 + 11^2 - 10^2}{2(9)(11)}\right)$$

$$\gamma = \cos^{-1}\left(\frac{10^2 + 11^2 - 9^2}{2(10)(11)}\right)$$

Draw a sector with central angle α and radius 4, and the area of this sector is

$$S_1 = \frac{1}{2}\left(4^2\alpha\right) \approx 9.847675339.$$

Similarly, let S_2 and S_3 be the areas of the sectors with central angle β and radius 5, and central angle γ and radius 6, respectively.

Thus, the area that is not sprayed by any of the three sprinklers is

$$A - (S_1 + S_2 + S_3) \approx 3.850 \text{ meters}^2.$$

7.2 Pop Quiz

1. By the law of cosines, we find

$$a = \sqrt{10.4^2 + 8.1^2 - 2(10.4)(8.1)\cos 12.3^\circ} \approx 3.0$$

2. By the law of cosines, we obtain

$$\gamma = \cos^{-1}\left(\frac{6^2 + 7^2 - 12^2}{2(6)(7)}\right) \approx 134.6^\circ.$$

3. Note, $S = \dfrac{7 + 8 + 9}{2} = 12$. Using Heron's formula, the area of the triangle is

$$\begin{aligned} \text{Area} &= \sqrt{12(12-7)(12-8)(12-9)} \\ &= \sqrt{720} = 12\sqrt{5}. \end{aligned}$$

7.2 Linking Concepts

a) From the point $(30, 10)$, the distance to the top of the screen and the bottom of the screen are

$$\sqrt{(10-60)^2 + (30-0)^2} = \sqrt{3400} = 10\sqrt{34}$$

and

$$\sqrt{(10-0)^2 + (30-0)^2} = \sqrt{1000} = 10\sqrt{10},$$

respectively.

b) Using the law of cosines, one finds that the viewing angle α is given by

$$\begin{aligned} \alpha &= \cos^{-1}\left(\frac{3400 + 1000 - 60^2}{2\sqrt{3400}\sqrt{1000}}\right) \\ \alpha &\approx 77.5^\circ. \end{aligned}$$

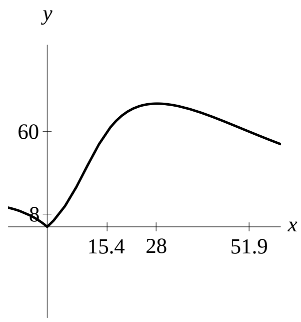
c) Note, the coordinates of any seat is of the form $(x, x - 20)$ for some real number $x \geq 20$. The distances of this seat from the top of the screen and bottom of the screen are $\sqrt{x^2 + (x-80)^2}$ and $\sqrt{x^2 + (x-20)^2}$, respectively. By the cosine law, the angle α is given by

$$\cos^{-1}\left(\frac{x^2 + (x-80)^2 + x^2 + (x-20)^2 - 60^2}{2\sqrt{x^2 + (x-80)^2}\sqrt{x^2 + (x-20)^2}}\right)$$

or equivalently

$$\cos^{-1}\left(\frac{x^2 - 50x + 800}{\sqrt{x^2 - 80x + 3200}\sqrt{x^2 - 20x + 200}}\right)$$

and a sketch of its graph is shown.



d) Since $\alpha = 60°$ when $x \approx 15.4, 51.9$, then the viewing angle is greater than $60°$ when $15.4 < x < 51.9$.

e) The largest viewing angle α is seen from the seat with coordinates $(28, 8)$.

For Thought

1. True

2. False, if $\boldsymbol{A} = \langle 1, 0\rangle$ and $\boldsymbol{B} = \langle 0, 1\rangle$ then $|\boldsymbol{A} + \boldsymbol{B}| = |\langle 1, 1\rangle| = \sqrt{2}$ and $|\boldsymbol{A}| + |\boldsymbol{B}| = 2$.

3. True

4. True

5. False, rather the parallelogram law says that the magnitude of $\boldsymbol{A} + \boldsymbol{B}$ is the length of a diagonal of the parallelogram formed by $\boldsymbol{A}$ and $\boldsymbol{B}$.

6. False, the direction angle is formed with the positive x-axis.

7. True

8. True

9. True, the direction angle of a vector is unchanged when it is multipied by a positive scalar.

10. True

7.3 Exercises

1. vector

2. equal

3. magnitude

4. sum, resultant

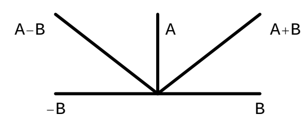
5. parallelogram law

6. direction

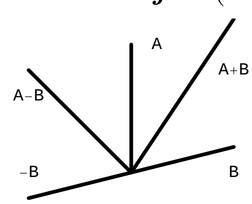
7. component

8. perpendicular, orthogonal

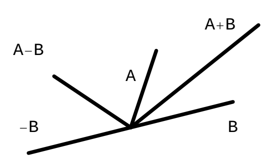
9. $\boldsymbol{A} + \boldsymbol{B} = 5\boldsymbol{j} + 4\boldsymbol{i} = 4\boldsymbol{i} + 5\boldsymbol{j}$ and $\boldsymbol{A} - \boldsymbol{B} = 5\boldsymbol{j} - 4\boldsymbol{i} = -4\boldsymbol{i} + 5\boldsymbol{j}$



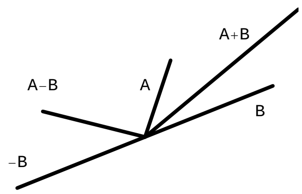
10. $\boldsymbol{A} + \boldsymbol{B} = 5\boldsymbol{j} + (4\boldsymbol{i} + \boldsymbol{j}) = 4\boldsymbol{i} + 6\boldsymbol{j}$ and $\boldsymbol{A} - \boldsymbol{B} = 5\boldsymbol{j} - (4\boldsymbol{i} + \boldsymbol{j}) = -4\boldsymbol{i} + 4\boldsymbol{j}$



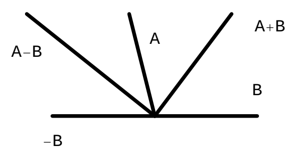
11. $\boldsymbol{A} + \boldsymbol{B} = (\boldsymbol{i} + 3\boldsymbol{j}) + (4\boldsymbol{i} + \boldsymbol{j}) = 5\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{A} - \boldsymbol{B} = (\boldsymbol{i} + 3\boldsymbol{j}) - (4\boldsymbol{i} + \boldsymbol{j}) = -3\boldsymbol{i} + 2\boldsymbol{j}$



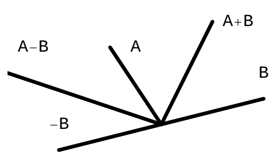
12. $\boldsymbol{A}+\boldsymbol{B} = (\boldsymbol{i} + 3\boldsymbol{j}) + (5\boldsymbol{i} + 2\boldsymbol{j}) = 6\boldsymbol{i} + 5\boldsymbol{j}$ and $\boldsymbol{A}-\boldsymbol{B} = (\boldsymbol{i} + 3\boldsymbol{j}) - (5\boldsymbol{i} + 2\boldsymbol{j}) = -4\boldsymbol{i} + \boldsymbol{j}$



13. $\boldsymbol{A}+\boldsymbol{B} = (-\boldsymbol{i} + 4\boldsymbol{j}) + (4\boldsymbol{i}) = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{A}-\boldsymbol{B} = (-\boldsymbol{i} + 4\boldsymbol{j}) - (4\boldsymbol{i}) = -5\boldsymbol{i} + 4\boldsymbol{j}$



14. $\boldsymbol{A}+\boldsymbol{B} = (-2\boldsymbol{i} + 3\boldsymbol{j}) + (4\boldsymbol{i} + \boldsymbol{j}) = 2\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{A}-\boldsymbol{B} = (-2\boldsymbol{i} + 3\boldsymbol{j}) - (4\boldsymbol{i} + \boldsymbol{j}) = -6\boldsymbol{i} + 2\boldsymbol{j}$



15. D **16.** A **17.** E **18.** F

19. B **20.** C

21. $|\boldsymbol{v_x}| = |4.5\cos 65.2°| = 1.9$,
$|\boldsymbol{v_y}| = |4.5\sin 65.2°| = 4.1$

22. $|\boldsymbol{v_x}| = |6000\cos 13.1°| \approx 5843.9$,
$|\boldsymbol{v_y}| = |6000\sin 13.1°| \approx 1359.9$

23. $|\boldsymbol{v_x}| = |8000\cos 155.1°| \approx 7256.4$,
$|\boldsymbol{v_y}| = |8000\sin 155.1°| \approx 3368.3$

24. $|\boldsymbol{v_x}| = |445\cos 211.1°| \approx 381.0$,
$|\boldsymbol{v_y}| = |445\sin 211.1°| \approx 229.9$

25. $|\boldsymbol{v_x}| = |234\cos 248°| \approx 87.7$,
$|\boldsymbol{v_y}| = |234\sin 248°| \approx 217.0$

26. $|\boldsymbol{v_x}| = |48.3\cos 349°| \approx 47.4$,
$|\boldsymbol{v_y}| = |48.3\sin 349°| \approx 9.2$

27. The magnitude is $\sqrt{\sqrt{3}^2 + 1^2} = 2$.
Since $\tan\alpha = 1/\sqrt{3}$, the direction angle is $\alpha = 30°$.

28. The magnitude is $\sqrt{(-1)^2 + \sqrt{3}^2} = 2$.
Since $\tan\alpha = -\sqrt{3}$, the direction angle is $\alpha = 120°$.

29. The magnitude is $\sqrt{(-\sqrt{2})^2 + \sqrt{2}^2} = 2$.
Since $\tan\alpha = -\sqrt{2}/\sqrt{2} = -1$, the direction angle is $\alpha = 135°$.

30. The magnitude is $\sqrt{\sqrt{2}^2 + (-\sqrt{2})^2} = 2$.
Since $\tan\alpha = -\sqrt{2}/\sqrt{2} = -1$, the direction angle is $\alpha = 315°$.

31. The magnitude is $\sqrt{8^2 + (-8\sqrt{3})^2} = 16$.
Since $\tan\alpha = -8\sqrt{3}/8 = -\sqrt{3}$, the direction angle is $\alpha = 300°$.

32. The magnitude is $\sqrt{(-1/2)^2 + (-\sqrt{3}/2)^2} = 1$.
Since $\tan\alpha = \dfrac{-\sqrt{3}/2}{-1/2} = \sqrt{3}$, the direction angle is $\alpha = 240°$.

33. The magnitude is $\sqrt{5^2 + 0^2} = 5$.
Since the terminal point is on the positive x-axis, the direction angle is $0°$.

34. The magnitude is $\sqrt{0^2 + (-6)^2} = 6$.
Since the terminal point is on the negative y-axis, the direction angle is $270°$.

35. The magnitude is $\sqrt{(-3)^2 + 2^2} = \sqrt{13}$.
Since $\tan^{-1}(-2/3) \approx -33.7°$, the direction angle is $180° - 33.7° = 146.3°$.

36. The magnitude is $\sqrt{(-4)^2+(-2)^2}=2\sqrt{5}$.
Since $\tan^{-1}(2/4)\approx 26.6°$, the direction angle is $180°+26.6°=206.6°$.

37. The magnitude is $\sqrt{3^2+(-1)^2}=\sqrt{10}$.
Since $\tan^{-1}(-1/3)\approx -18.4°$, the direction angle is $360°-18.4°=341.6°$.

38. The magnitude is $\sqrt{2^2+(-6)^2}=2\sqrt{10}$.
Since $\tan^{-1}(-6/2)\approx -71.6°$, the direction angle is $360°-71.6°=288.4°$.

39. $\langle 8\cos 45°, 8\sin 45°\rangle=\langle 8(\sqrt{2}/2), 8(\sqrt{2}/2)\rangle$
$=\langle 4\sqrt{2}, 4\sqrt{2}\rangle$

40. $\langle 12\cos 120°, 12\sin 120°\rangle=$
$\langle 12(-1/2), 12(\sqrt{3}/2)\rangle=\langle -6, 6\sqrt{3}\rangle$

41. $\langle 290\cos 145°, 290\sin 145°\rangle\approx\langle -237.6, 166.3\rangle$

42. $\langle 5.3\cos 321°, 5.3\sin 321°\rangle\approx\langle 4.1, -3.3\rangle$

43. $\langle 18\cos 347°, 18\sin 347°\rangle\approx\langle 17.5, -4.0\rangle$

44. $\langle 3000\cos 209.1°, 3000\sin 209.1°\rangle\approx$
$\langle -2621.3, -1459.0\rangle$

45. $\langle 15, -10\rangle$ **46.** $\langle 4, -20\rangle$

47. $\langle 6, -4\rangle+\langle 12, -18\rangle=\langle 18, -22\rangle$

48. $\langle -1, 4\rangle$ **49.** $\langle -1, 5\rangle+\langle 12, -18\rangle=\langle 11, -13\rangle$

50. $\dfrac{\langle 2, 3\rangle}{2}=\langle 1, 1.5\rangle$

51. $\langle 3, -2\rangle-\langle 3, -1\rangle=\langle 0, -1\rangle$

52. $\langle 3, -2\rangle-\langle -1, 5\rangle-\langle 4, -6\rangle=\langle 0, -1\rangle$

53. $(3)(-1)+(-2)(5)=-13$

54. $(-1)(4)+(5)(-6)=-34$

55. If $\boldsymbol{A}=\langle 2, 1\rangle$ and $\boldsymbol{B}=\langle 3, 5\rangle$, then the angle between these vectors is given by

$$\cos^{-1}\left(\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{A}|\cdot|\boldsymbol{B}|}\right)=\cos^{-1}\left(\frac{11}{\sqrt{5}\sqrt{34}}\right)\approx 32.5°$$

56. If $\boldsymbol{A}=\langle 2, 3\rangle$ and $\boldsymbol{B}=\langle 1, 5\rangle$, then the angle between these vectors is given by

$$\cos^{-1}\left(\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{A}|\cdot|\boldsymbol{B}|}\right)=\cos^{-1}\left(\frac{17}{\sqrt{13}\sqrt{26}}\right)\approx 22.4°$$

57. If $\boldsymbol{A}=\langle -1, 5\rangle$ and $\boldsymbol{B}=\langle 2, 7\rangle$, then the angle between these vectors is given by

$$\cos^{-1}\left(\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{A}|\cdot|\boldsymbol{B}|}\right)=\cos^{-1}\left(\frac{33}{\sqrt{26}\sqrt{53}}\right)\approx 27.3°$$

58. If $\boldsymbol{A}=\langle -2, -5\rangle$ and $\boldsymbol{B}=\langle 1, -9\rangle$, then the angle between these vectors is given by

$$\cos^{-1}\left(\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{A}|\cdot|\boldsymbol{B}|}\right)=\cos^{-1}\left(\frac{43}{\sqrt{29}\sqrt{82}}\right)\approx 28.1°$$

59. Since $\langle -6, 5\rangle\cdot\langle 5, 6\rangle=0$, angle between them is $90°$.

60. Since $\langle 2, 7\rangle\cdot\langle 7, -2\rangle=0$, angle between them is $90°$.

61. Perpendicular since their dot product is zero

62. Parallel since $4\langle 2, 3\rangle=\langle 8, 12\rangle$

63. Parallel since $-2\langle 1, 7\rangle=\langle -2, -14\rangle$

64. Perpendicular since their dot product is zero

65. Neither

66. Neither

67. $\boldsymbol{2i+j}$ **68.** $\boldsymbol{i+5j}$ **69.** $\boldsymbol{-3i+\sqrt{2}j}$

70. $\boldsymbol{\sqrt{2}i-5j}$ **71.** $-9\,\boldsymbol{j}$ **72.** $-\dfrac{1}{2}\,\boldsymbol{i}$

73. $-7\,\boldsymbol{i}-\boldsymbol{j}$ **74.** $\boldsymbol{i}+\boldsymbol{j}$

75. The magnitude of $\boldsymbol{A+B}=\langle 1, 4\rangle$ is $\sqrt{1^2+4^2}=\sqrt{17}$ and the direction angle is $\tan^{-1}(4/1)\approx 76.0°$

76. The magnitude of $\boldsymbol{A-B}=\langle 5, -2\rangle$ is $\sqrt{5^2+(-2)^2}=\sqrt{29}$.
Since $\tan^{-1}(-2/5)\approx -21.8°$, the direction angle is $360°-21.8°=338.2°$

77. The magnitude of $\boldsymbol{-3A}=\langle -9, -3\rangle$ is $\sqrt{(-9)^2+(-3)^2}=\sqrt{90}=3\sqrt{10}$.
Since $\tan^{-1}(3/9)\approx 18.4°$, the direction angle is $180°+18.4°=198.4°$

78. The magnitude of $\boldsymbol{5B}=\langle -10, 15\rangle$ is $\sqrt{(-10)^2+(15)^2}=\sqrt{325}=5\sqrt{13}$.
Since $\tan^{-1}(-15/10)\approx -56.3°$, the direction angle is $180°-56.3°=123.7°$

79. The magnitude of $\boldsymbol{B}-\boldsymbol{A} = \langle -5, 2\rangle$ is $\sqrt{(-5)^2+2^2}=\sqrt{29}$.
Since $\tan^{-1}(-2/5)\approx -21.8°$, the direction angle is $180°-21.8°=158.2°$

80. The magnitude of $\boldsymbol{B}+\boldsymbol{A} = \langle 1, 4\rangle$ is $=\sqrt{1^2+4^2}=\sqrt{17}$ and the direction angle is $\tan^{-1}(4/1)\approx 76.0°$

81. Note $-\boldsymbol{A}+\frac{\boldsymbol{1}}{\boldsymbol{2}}\boldsymbol{B} = \langle -3-1, -1+3/2\rangle$ $=\langle -4, 1/2\rangle$. The magnitude is $\sqrt{(-4)^2+(1/2)^2}=\sqrt{65}/2$.
Since $\tan^{-1}\left(\frac{1/2}{-4}\right)\approx -7.1°$, the direction angle is $180°-7.1°=172.9°$

82. Note $\frac{\boldsymbol{1}}{\boldsymbol{2}}\boldsymbol{A}-\boldsymbol{2B} = \langle 3/2+4, 1/2-6\rangle =$ $\langle 11/2, -11/2\rangle$. The magnitude is $\sqrt{(11/2)^2+(-11/2)^2}=\sqrt{242}/2\approx 7.8$.
Since $\tan^{-1}\left(-\frac{11/2}{11/2}\right)=-45°$, the direction angle is $360°-45°=315°$

83. The resultant is $\langle 2+6, 3+2\rangle=\langle 8,5\rangle$. So the magnitude is $\sqrt{8^2+5^2}=\sqrt{89}$ and direction angle is $\tan^{-1}(5/8)=32.0°$.

84. The resultant is $\langle 4+4, 2+6\rangle=\langle 8,8\rangle$. So the magnitude is $\sqrt{8^2+8^2}=\sqrt{128}=8\sqrt{2}$ and direction angle is $\tan^{-1}(8/8)=45°$.

85. The resultant is $\langle -6+4, 4+2\rangle=\langle -2,6\rangle$ and its magnitude is

$$\sqrt{(-2)^2+6^2}=2\sqrt{10}.$$

Since $\tan^{-1}(-6/2)\approx -71.6°$, the direction angle is $180°-71.6°=108.4°$.

86. The resultant is $\langle -6+3, 2+6\rangle=\langle -3,8\rangle$ and its magnitude is $\sqrt{(-3)^2+8^2}=\sqrt{73}$.
Since $\tan^{-1}(-8/3)\approx -69.4°$, the direction angle is $180°-69.4°=110.6°$.

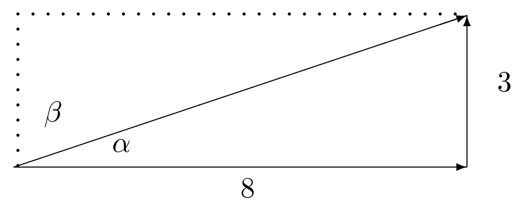
87. The resultant is $\langle -4+3, 4-6\rangle=\langle -1,-2\rangle$ and its magnitude is

$$\sqrt{(-1)^2+(-2)^2}=\sqrt{5}.$$

Since $\tan^{-1}(2/1)\approx 63.4°$, the direction angle is $180°+63.4°=243.4°$.

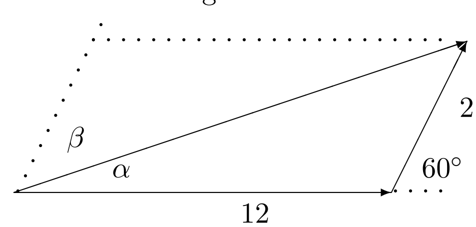
88. The resultant is $\langle -5+6, -4-2\rangle=\langle 1,-6\rangle$ and its magnitude is $\sqrt{1^2+(-6)^2}=\sqrt{37}$.
Since $\tan^{-1}(-6/1)\approx -80.5°$, the direction angle is $360°-80.5°=279.5°$.

89. Draw two perpendicular vectors whose magnitudes are 3 and 8.



The magnitude of the resultant force is $\sqrt{8^2+3^2}=\sqrt{73}$ pounds by the Pythagorean Theorem. The angles between the resultant and each force are $\tan^{-1}(3/8)\approx$ $20.6°$ and $\beta=90°-20.6°=69.4°$.

90. Draw two vectors with magnitudes 12 and 2 that act at an angle of 60° with each other as shown in the figure.



By the cosine law, the magnitude of the resultant force, in pounds, is

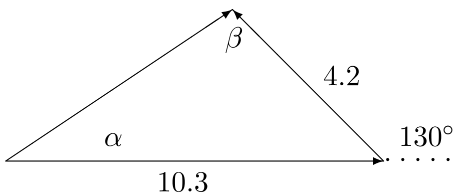
$$\sqrt{12^2+2^2-2(12)(2)\cos 120°}\approx 13.11.$$

By the sine law,

$$\begin{aligned}\frac{2}{\sin\alpha} &= \frac{13.11}{\sin 120°}\\ \sin\alpha &= \frac{2\sin 120°}{13.11}\\ \sin\alpha &\approx 0.1321\\ \alpha\approx\sin^{-1}(0.1321) &\approx 7.6°.\end{aligned}$$

The angles between the resultant and each force are $7.6°$ and $\beta = 180° - 7.6° - 120° = 52.4°$.

91. Draw two vectors with magnitudes 10.3 and 4.2 that act at an angle of $130°$ with each other.



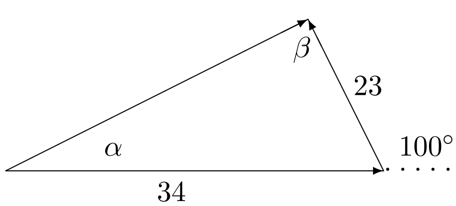
By using the cosine law, the magnitude of the resultant force is
$\sqrt{10.3^2 + 4.2^2 - 2(10.3)(4.2)\cos 50°} \approx 8.253 \approx 8.25$ newtons.

By the sine law,

$$\begin{aligned} \frac{4.2}{\sin\alpha} &= \frac{8.253}{\sin 50°} \\ \sin\alpha &= \frac{4.2\sin 50°}{8.253} \\ \sin\alpha &\approx 0.3898 \\ \alpha \approx \sin^{-1}(0.3898) &\approx 22.9°. \end{aligned}$$

The angles between the resultant and each force are $22.9°$ and $\beta = 180° - 22.9° - 50° = 107.1°$.

92. Draw two vectors with magnitudes 34 and 23 that act at an angle of $100°$ with each other.



By the cosine law, the magnitude of the resultant force is

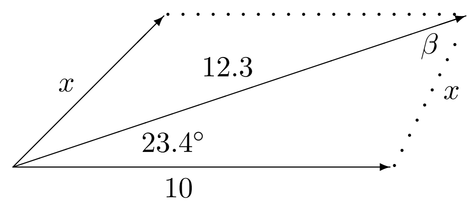
$\sqrt{34^2 + 23^2 - 2(34)(23)\cos 80°} \approx 37.6$ newtons.

By the sine law, we find

$$\begin{aligned} \frac{23}{\sin\alpha} &= \frac{37.6}{\sin 80°} \\ \sin\alpha &= \frac{23\sin 80°}{37.6} \\ \sin\alpha &\approx 0.6024 \\ \alpha \approx \sin^{-1}(0.6024) &\approx 37°. \end{aligned}$$

The angles between the resultant and each force are $37°$ and $\beta = 180° - 37° - 80° = 63°$.

93. Draw two vectors with magnitudes 10 & 12.3 and whose angle between them is $23.4°$.

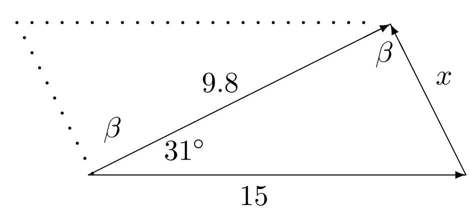


By the cosine law, the magnitude of the other force is
$x = \sqrt{10^2 + 12.3^2 - 2(10)(12.3)\cos 23.4°}$
$\approx 5.051 \approx 5.1$ pounds. By the sine law,

$$\begin{aligned} \frac{10}{\sin\beta} &= \frac{5.051}{\sin 23.4°} \\ \sin\beta &= \frac{10\sin 23.4°}{5.051} \\ \sin\beta &\approx 0.7863 \\ \beta \approx \sin^{-1}(0.7863) &\approx 51.8°. \end{aligned}$$

The angle between the two forces is $51.8° + 23.4° = 75.2°$.

94. Draw two vectors with magnitudes 15 and 9.8 and whose angle between them is $31°$.

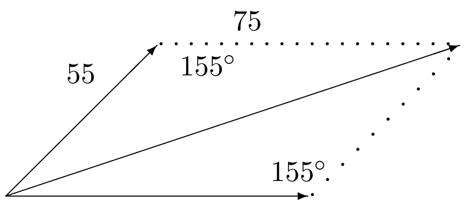


By the cosine law, the magnitude of the other force is
$x = \sqrt{15^2 + 9.8^2 - 2(15)(9.8)\cos 31°} \approx 8.31 \approx 8.3$ pounds. Once again by the cosine law,

$$\begin{aligned}\cos\beta &= \frac{9.8^2+8.31^2-15^2}{2(9.8)(8.31)}\\ \cos\beta &\approx -0.3678\\ \beta &\approx \cos^{-1}(-0.3678)\approx 111.6^\circ.\end{aligned}$$

The angle between the resultant and the other force is $\beta = 111.6^\circ$.

95. Since the angles in a parallelogram must add up to 360°, the angle formed by the two forces is $\dfrac{360^\circ - 2(25^\circ)}{2} = 155^\circ$.

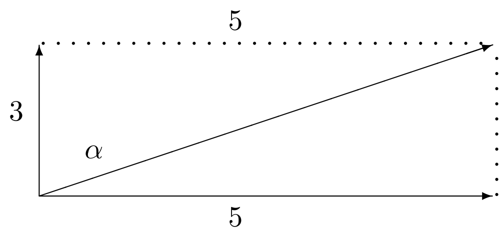


By the cosine law, the magnitude of the resultant force is

$\sqrt{55^2+75^2-2(55)(75)\cos 155^\circ} \approx 127.0$ pounds.

So the donkey must pull a force of 127 pounds in the direction opposite that of the resultant's.

96. Draw two perpendicular vectors one with magnitude 5 and the other with magnitude 3.

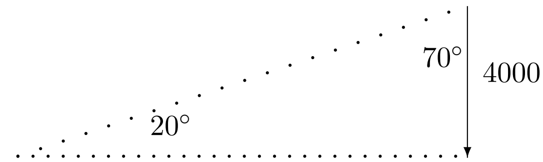


By the Pythagorean Theorem, Phyllis' speed is $\sqrt{3^2+5^2} \approx 5.83$ mph. The direction from the north is $\alpha = \tan^{-1}(5/3) \approx 59.0^\circ$.

97. The magnitudes of the horizontal and vertical components are $|520\cos 30^\circ| \approx 450.3$ mph and $|520\sin 30^\circ| = 260$ mph, respectively.

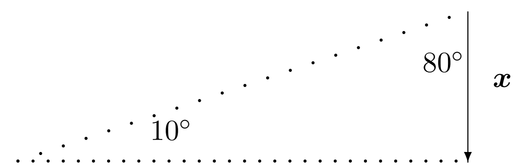
98. The magnitudes of the horizontal and vertical components are $|30\cos 22^\circ| \approx 27.8$ m/sec ^{2} and $|30\sin 22^\circ| \approx 11.2$ m/sec^2, respectively.

99. Draw a vector pointing vertically down with magnitude 4000.



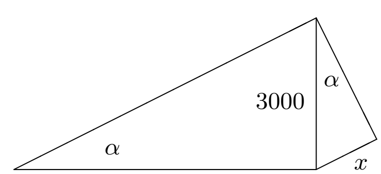
The magnitude of the component along the hill of the given vector is $4000\cos 70^\circ \approx 1368.1$ pounds and this is the amount of force needed to prevent the rock from falling.

100. Draw a vector $\boldsymbol{x}$ representing the force exerted vertically down by the steel ball.



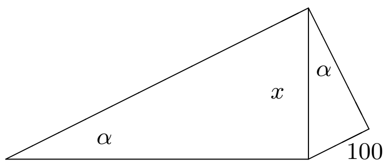
Since 3.2 is the magnitude of the component of $\boldsymbol{x}$ along the incline, we have $3.2 = |\,\boldsymbol{x}\,| \cdot \cos 80^\circ$. The weight of the ball is $|\,\boldsymbol{x}\,| = 3.2/\cos 80^\circ \approx 18.4$ pounds.

101. In the diagram, $\alpha = 20^\circ$ and x is the required force.



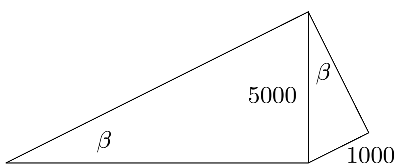
We find $x = 3000\sin 20^\circ \approx 1026.1$ lb.

102. In the figure, $\alpha = 25°$ and x is the weight of the block.



We find $x = 100/\sin 25° \approx 236.6$ lb.

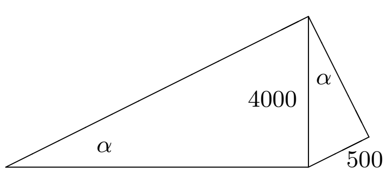
103. Let β be the angle of inclination.



Since $\sin\beta = \dfrac{1000}{5000}$, we find

$$\beta = \sin^{-1}\frac{1000}{5000} \approx 11.5°.$$

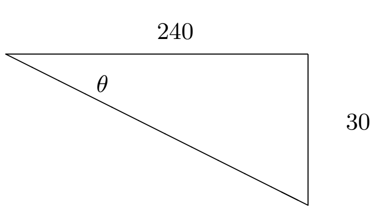
104. Let α be the angle of inclination.



Since $\sin\alpha = \dfrac{500}{4000}$, we find

$$\alpha = \sin^{-1}\frac{500}{4000} \approx 7.2°.$$

105. Consider the figure below



Since $\tan\theta = \dfrac{30}{240}$, we find

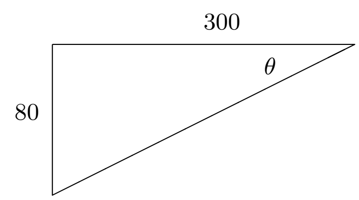
$$\theta = \tan^{-1}\frac{30}{240} \approx 7.1°.$$

Thus, the bearing of the plane's course is about 97.1° ($\approx 90° + \theta$).

Using the Pythagorean theorem, the ground speed is

$$\sqrt{240^2 + 30^2} \approx 241.9 \text{ mph}.$$

106. Consider the vectors with their indicated magnitudes.



Since $\tan\theta = \dfrac{80}{300}$, we find

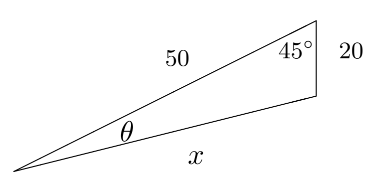
$$\theta = \tan^{-1}\frac{80}{300} \approx 14.9°.$$

Thus, the bearing of the plane's course is about 255.1° ($\approx 270° - \theta$).

Using the Pythagorean theorem, the ground speed is

$$\sqrt{300^2 + 80^2} \approx 310.5 \text{ mph}.$$

107. Let x be the ground speed in the figure below.



Using the cosine law, we find that the ground speed is

$$x = \sqrt{50^2 + 20^2 - 2(50)(20)\cos 45^\circ} \approx 38.5 \text{ mph.}$$

Using the sine law, we obtain

$$\frac{x}{\sin 45^\circ} = \frac{20}{\sin\theta}.$$

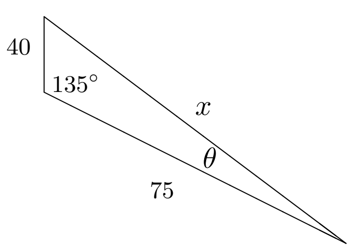
Solving for θ, we find

$$\theta = \sin^{-1}\left(\frac{20\sin 45^\circ}{x}\right) \approx 21.5^\circ.$$

Thus, the bearing for the course is

$$45^\circ + \theta \approx 66.5^\circ.$$

108. Let x be the ground speed.



Using the cosine law, the ground speed is

$$\begin{aligned} x &= \sqrt{75^2 + 40^2 - 2(75)(40)\cos 135^\circ} \\ &\approx 107.1 \text{ mph.} \end{aligned}$$

Using the sine law, we find

$$\frac{x}{\sin 135^\circ} = \frac{40}{\sin\theta}.$$

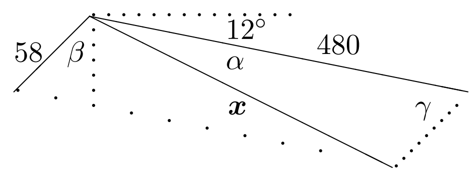
Solving for θ, we find

$$\theta = \sin^{-1}\left(\frac{40\sin 135^\circ}{x}\right) \approx 15.3^\circ.$$

Thus, the bearing for the course is

$$315^\circ + \theta \approx 330.3^\circ.$$

109. Draw a vector $\boldsymbol{x}$ representing the airplane's course and ground speed. There is a 12° angle in the picture because of the plane's 102° heading.



Note, $\beta = 45^\circ$. Since the four angles in a parallelogram adds up to 360° and $45^\circ + 78^\circ = 123^\circ$, we have

$$\gamma = \frac{360 - 2(123^\circ)}{2} = 57^\circ.$$

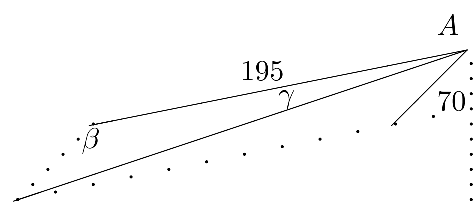
By the cosine law, the ground speed is $|\,\boldsymbol{x}\,| = \sqrt{480^2 + 58^2 - 2(480)(58)\cos 57^\circ}$ ≈ 451.0 mph. By the sine law, we get

$$\begin{aligned} \frac{58}{\sin\alpha} &= \frac{451}{\sin 57^\circ} \\ \sin\alpha &= \frac{58\sin 57^\circ}{451} \\ \sin\alpha &\approx 0.107856 \\ \alpha &\approx \sin^{-1}(0.107856) \approx 6.2^\circ. \end{aligned}$$

The bearing of the airplane is

$$6.2^\circ + 102^\circ = 108.2^\circ.$$

110. Draw two vectors representing the wind and the helicopter.



Note, the angle of the parallelogram at A is 55°. Since the four angles in a parallelogram add up to 360°, we get

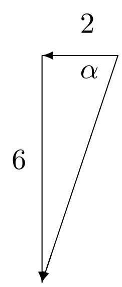
$$\beta = \frac{360 - 2(55^\circ)}{2} = 125^\circ.$$

By the cosine law, the ground speed is
$\sqrt{195^2 + 70^2 - 2(195)(70)\cos 125°} \approx 242.0$ mph.
By the sine law, we find

$$\begin{aligned} \frac{70}{\sin\gamma} &= \frac{242}{\sin 125°} \\ \sin\gamma &= \frac{70\sin 125°}{242} \\ \sin\gamma &\approx 0.2369 \\ \gamma &\approx \sin^{-1}(0.2369) \\ \gamma &\approx 13.7°. \end{aligned}$$

The bearing of the course of the helicopter is $240° - 13.7° = 226.3°$.

111. Draw two vectors representing the canoe and river current; the magnitudes of these vectors are 2 and 6, respectively.



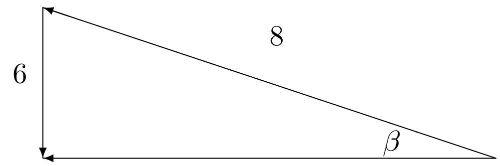
Since $\alpha = \tan^{-1}(6/2) \approx 71.6°$, the direction measured from the north is

$$270° - 71.6° = 198.4°.$$

Also, if d is the distance downstream from a point directly across the river to the point where she will land, then $\tan\alpha = d/2000$. Since $\tan\alpha = 6/2 = 3$, we get

$$d = 2000 \cdot 3 = 6000 \text{ ft}.$$

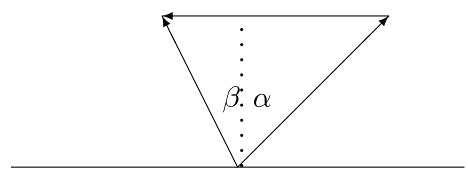
112. Draw two vectors representing the canoe and the river current; the magnitudes of these vectors are 8 and 6, respectively.



Since $\beta = \sin^{-1}(6/8) \approx 48.6°$, she must paddle in the direction $270° + 48.6° = 318.6°$ from the north direction measured clockwise if she want to go directly across.

Also, if d is the distance in feet she will paddle as she crosses the river then $\cos 48.6° = 2000/d$ and $d = 2000/\cos 48.6° \approx 3024.3$ ft.
Equivalently, $d = 3024.3/5280 \approx 0.5728$ miles.
So the time it takes to cross the river is
$t = 0.5728/8 \approx 0.0716$ hr or
$t = 0.0716(60) \approx 4.3$ minutes.

113. a) Assume we have a coordinate system where the origin is the point where the boat will start.



The intended direction and speed of the boat that goes 3 mph in still water is defined by the vector $3\sin\alpha\ \boldsymbol{i} + 3\cos\alpha\ \boldsymbol{j}$ and its actual direction and speed is determined by the vector

$$\boldsymbol{v} = (3\sin\alpha - 1)\ \boldsymbol{i} + 3\cos\alpha\ \boldsymbol{j}.$$

The number t of hours it takes the boat to cross the river is given by

$$t = \frac{0.2}{3\cos\alpha},$$

the solution to $3t\cos\alpha = 0.2$.
Suppose $\beta > 0$ if $3\sin\alpha - 1 < 0$ and $\beta < 0$ if $3\sin\alpha - 1 > 0$. Using right triangle trigonometry, we find

$$\tan\beta = \frac{|3\sin\alpha - 1|}{3\cos\alpha}.$$

The distance d the boat travels as a function of β is given by

$$\begin{aligned} d &= |t\, \boldsymbol{v}\,| \\ &= \frac{0.2}{3\cos\alpha}|\,\boldsymbol{v}\,| \\ &= \frac{0.2}{3\cos\alpha}\sqrt{(3\sin\alpha-1)^2+(3\cos\alpha)^2} \\ &= 0.2\sqrt{\tan^2\beta+1} \\ d &= 0.2|\sec\beta|. \end{aligned}$$

b) Since speed is distance divided by time, then by using the answer from part a) the speed s as a function of α and β is

$$\begin{aligned} s &= \frac{d}{t} \\ &= \frac{0.2|\sec\beta|}{0.2/(3\cos\alpha)} \\ &= 3\cos(\alpha)|\sec\beta|. \end{aligned}$$

c) As seen in the previous exercise, the number t of hours the trip will take as a function of α is

$$t = \frac{0.2}{3\cos\alpha} = \frac{1}{15}\sec\alpha.$$

The minimum value of t is attained when $\sec\alpha$ is the least, i.e., when $\alpha = 0°$.

114. Let the forces exerted by the papa, mama, and baby elephant be represented by the vectors
$\boldsymbol{v_p} = 800\cos 30°\ \boldsymbol{i}\ + 800\sin 30°\ \boldsymbol{j}$,
$\boldsymbol{v_m} = 500\ \boldsymbol{i}$, and
$\boldsymbol{v_b} = 200\cos 20°\ \boldsymbol{i}\ - 200\sin 20°\ \boldsymbol{j}$,
respectively. With a calculator, we find

$$\begin{aligned} \boldsymbol{F} &= \boldsymbol{v_p} + \boldsymbol{v_m} + \boldsymbol{v_b} \\ &\approx 1380.76\ \boldsymbol{i}\ + 331.60\ \boldsymbol{j}\ . \end{aligned}$$

The magnitude of the resultant of the three forces is

$$|\ \boldsymbol{F}\ | \approx \sqrt{1380.76^2+331.60^2} \approx 1420 \text{ lbs}$$

and the direction is

$$\tan^{-1}\left(\frac{331.60}{1380.76}\right) \approx 13.5°$$

or E13.5°N.

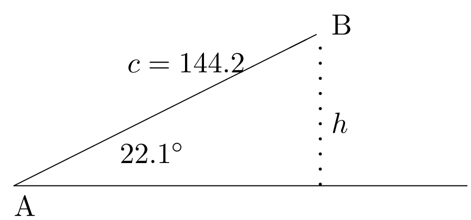
117. Note, $\alpha < 90°$. By the sine law, we find

$$\begin{aligned} \sin\alpha &= \frac{19.4\sin 122.1°}{22.6} \\ \alpha &\approx 11.2° \end{aligned}$$

Then $\gamma = 180° - 122.1° - \alpha \approx 11.2°$.
By the sine law, we obtain

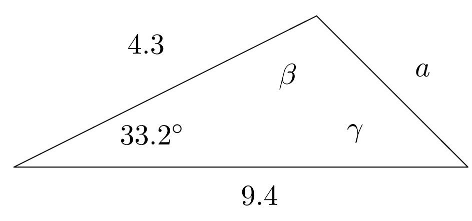
$$\begin{aligned} c &= \frac{22.6\sin\gamma}{\sin 122.1°} \\ c &\approx 5.2 \end{aligned}$$

118. Draw angle $\alpha = 22.1°$ and let h be the height.



Then $h = 144.2\sin 22.1° \approx 54.3$. Since side a opposite angle A satisfies $a = 19.4 < h$, no triangles exists.

119. Apply the cosine law to the given triangle.



$$\begin{aligned} a^2 &= 4.3^2+9.4^2-2(4.3)(9.4)\cos 33.2° \\ a &\approx 6.3 \end{aligned}$$

Since $\gamma < \beta$, we find γ by the sine law.

$$\begin{aligned} \sin\gamma &= \frac{4.3\sin 33.2°}{a} \\ \gamma &\approx 22.1° \end{aligned}$$

Then $\beta = 180° - 33.2° - \gamma \approx 124.7°$.

120. Since $\cos x = \pm\frac{1}{2}$, we find

$$x = \frac{2\pi}{3} + 2k\pi \quad \text{or} \quad x = \frac{4\pi}{3} + 2k\pi$$

where k is an integer.

121. Since $\sin 3x = \pm\frac{\sqrt{3}}{2}$, we find

$$3x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \ldots$$

Since x lies in $(0, \pi)$, we obtain

$$x = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}$$

122. The period is $\dfrac{\pi}{B} = \dfrac{\pi}{\pi} = 1$.

If we solve $\pi x + \pi = \frac{\pi}{2} + m\pi$. where m is an integer, we find

$$x = \frac{1}{2} + (m - 1)$$

Thus, the asymptotes are the vertical lines

$$x = \frac{1}{2} + k$$

where k is an integer.

Thinking Outside the Box LXIII

Let $\beta = 2\alpha$. By the cosine law, we find

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

and

$$b^2 = a^2 + c^2 - 2ac\cos 2\alpha.$$

Since $\cos 2\alpha = 2\cos^2\alpha - 1$, we obtain

$$b^2 = a^2 + c^2 - 2ac\left[2\left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2 - 1\right].$$

We may rewrite the above equation as

$$\frac{(a - b - c)(a + b - c)(a + c)(a^2 - b^2 + ac)}{b^2 c} = 0.$$

Since the sum of any two sides of a triangle must be larger than the remaining side, we obtain $a^2 - b^2 + ac = 0$ or equivalently

$$c = \frac{b^2 - a^2}{a}.$$

a) Using the above equation, such a triangle with the smallest perimeter has sides

$$a = 4, b = 6, c = 5.$$

b) As seen in the above discussion, we have $\beta = 2\alpha$ and

$$c = \frac{b^2 - a^2}{a}.$$

c) The next two larger such triangles have sides

$$a = 9, b = 12, c = 7$$

and

$$a = 9, b = 15, c = 16$$

7.3 Pop Quiz

1. $\boldsymbol{v_x} = 5.6\cos 33.9° \approx 4.6,$

$\boldsymbol{v_y} = 5.6\sin 33.9° \approx 3.1$

2. $|\,\boldsymbol{v}\,| = \sqrt{(-2)^2 + 6^2} = \sqrt{40} = 2\sqrt{10},$

$\theta = 180° - \tan^{-1}\dfrac{6}{2} = 108.4°$

3. $\boldsymbol{v} - \boldsymbol{w} = \langle -1, 3\rangle - \langle 2, 6\rangle = \langle -3, -3\rangle$

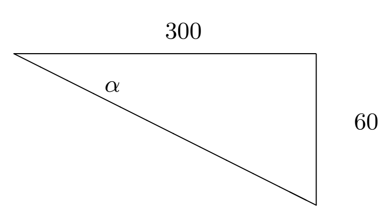
$3\,\boldsymbol{v} = 3\langle -1, 3\rangle = \langle -3, 9\rangle$

$\boldsymbol{v} \cdot \boldsymbol{w} = \langle -1, 3\rangle \cdot \langle 2, 6\rangle = -2 + 18 = 16$

4. If θ is the smallest positive angle, then

$$\begin{aligned}
\boldsymbol{v} \cdot \boldsymbol{w} &= |\,\boldsymbol{v}\,||\,\boldsymbol{w}\,|\cos\theta \\
\boldsymbol{v} \cdot \boldsymbol{w} &= |\,\boldsymbol{v}\,||\,\boldsymbol{w}\,|\cos\theta \\
26 &= \sqrt{17}\sqrt{40}\cos\theta \\
\theta &= \cos^{-1}\left(\frac{2}{\sqrt{17}\sqrt{40}}\right) \\
\theta &\approx 4.4°.
\end{aligned}$$

5. Consider the figure below



Since $\tan\alpha = \dfrac{60}{300}$, we find

$$\alpha = \tan^{-1}\frac{60}{300} \approx 11.3^\circ.$$

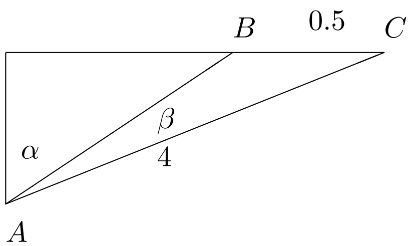
Thus, the bearing of the plane's course is about 101.3° ($\approx 90^\circ + \alpha$).

Using the Pythagorean theorem, the ground speed is

$$\sqrt{300^2 + 60^2} \approx 305.9 \text{ mph}.$$

7.3 Linking Concepts

a) The angle on the right of vertex B is $90^\circ + \alpha$.



By applying the sine law, we have

$$\begin{aligned}\frac{4}{\sin(90^\circ + \alpha)} &= \frac{0.5}{\sin\beta}\\ \frac{4}{\cos\alpha} &= \frac{0.5}{\sin\beta}\\ \beta &= \sin^{-1}\left(\frac{\cos\alpha}{8}\right).\end{aligned}$$

Likewise, the length AB (which is the actual speed) is found to be

$$\begin{aligned}\frac{AB}{\sin(90^\circ - \alpha - \beta)} &= \frac{4}{\cos\alpha}\\ \frac{AB}{\cos(\alpha+\beta)} &= \frac{4}{\cos\alpha}\\ AB &= \frac{4\cos(\alpha+\beta)}{\cos\alpha}.\end{aligned}$$

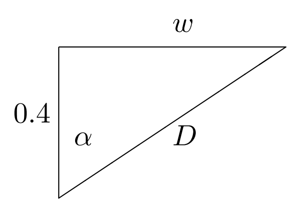
Thus, if $\alpha = 12^\circ$ then

$$\beta = \sin^{-1}\left(\frac{\cos 12^\circ}{8}\right) \approx 7.02^\circ$$

and the actual speed is

$$AB \approx \frac{4\cos(12^\circ + 7.02^\circ)}{\cos 12^\circ} \approx 3.87 \text{ mph}.$$

b) Let D be the distance across the river.



Note, $D = \dfrac{0.4}{\cos\alpha}$ and $w = 0.4\tan\alpha$.

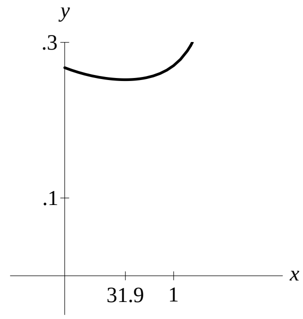
If $\alpha = 12^\circ$, then the time spent on crossing the river plus the time spent on biking is given by

$$\begin{aligned}\frac{0.4/\cos\alpha}{4\cos(\alpha+\beta)/\cos\alpha} + \frac{1 - .4\tan\alpha}{6} &=\\ \frac{0.1}{\cos(\alpha+\beta)} + \frac{1 - .4\tan\alpha}{6} &=\\ \frac{0.1}{\cos(19.02^\circ)} + \frac{1 - .4\tan 12^\circ}{6} &\approx\\ 0.258 \text{ hour} &\approx\\ 15.5 \text{ minutes}.\end{aligned}$$

c) As seen in part b), the total time $T(\alpha)$ spent for the trip is

$$\begin{aligned}T(\alpha) &= \frac{0.1}{\cos(\alpha+\beta)} + \frac{1 - 0.4\tan\alpha}{6}\\ &= \frac{0.1}{\cos\left(\alpha + \sin^{-1}\left(\frac{\cos\alpha}{8}\right)\right)} +\\ &\qquad \frac{1 - 0.4\tan\alpha}{6}\end{aligned}$$

and its graph is given in the next column.



d) From the graph in part c), the total time $T(\alpha)$ is minimized when

$$\alpha \approx 31.9^\circ.$$

Correspondingly, we find

$$\beta \approx \sin^{-1}\left(\frac{0.5\cos 31.9^\circ}{4}\right) \approx 6.1^\circ.$$

For Thought

1. True **2.** False, the absolute value is $\sqrt{(-2)^2+(-5)^2} = \sqrt{29}$.

3. True **4.** False, $\tan\theta = 3/2$.

5. False, $i = 1(\cos 90^\circ + i\sin 90^\circ)$.

6. True **7.** True **8.** True

9. True, since $\dfrac{\pi}{4} + \dfrac{\pi}{2} = \dfrac{3\pi}{4}$.

10. False, since $\dfrac{3(\cos\pi/4 + i\sin\pi/4)}{3(\cos\pi/2 + i\sin\pi/2)} =$ $1.5(\cos(-\pi/4) + i\sin(-\pi/4)) =$ $1.5(\cos\pi/4 - i\sin\pi/4)$.

7.4 Exercises

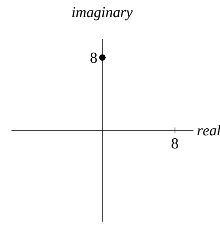
1. real

2. imaginary

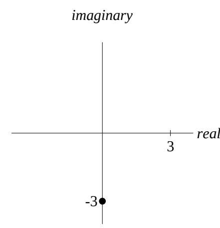
3. absolute value

4. multiplying, adding

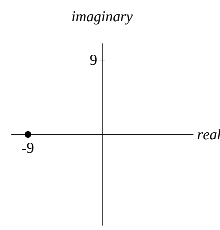
5. $\sqrt{0^2+8^2} = 8$



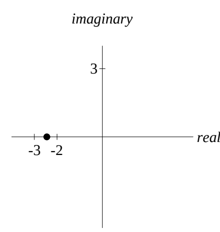
6. $\sqrt{0^2+(-3)^2} = 3$



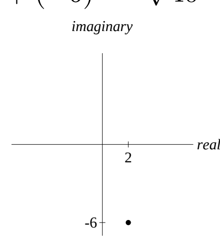
7. $\sqrt{(-9)^2+0^2} = 9$



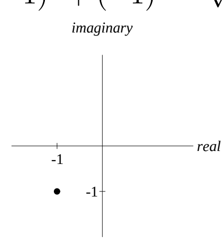
8. $\sqrt{(-\sqrt{6})^2+0^2} = \sqrt{6}$



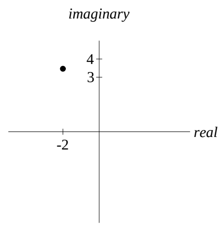
9. $\sqrt{2^2+(-6)^2} = \sqrt{40} = 2\sqrt{10}$



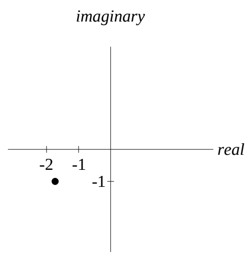
10. $\sqrt{(-1)^2+(-1)^2} = \sqrt{2}$



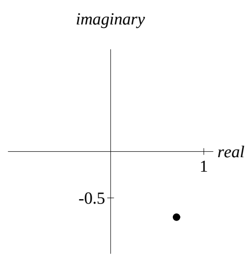
11. $\sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4$



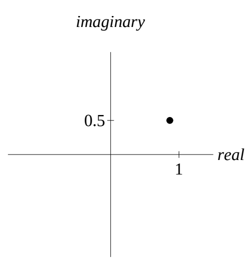
12. $\sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$



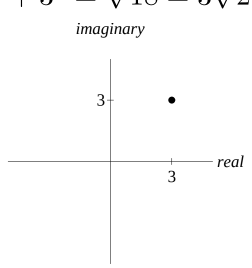
13. $\sqrt{(1/\sqrt{2})^2 + (-1/\sqrt{2})^2} = \sqrt{1/2+1/2} = 1$



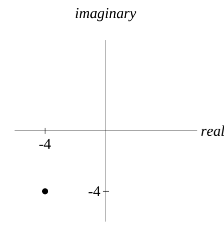
14. $\sqrt{(\sqrt{3}/2)^2 + (1/2)^2} = \sqrt{3/4+1/4} = 1$



15. $\sqrt{3^2+3^2} = \sqrt{18} = 3\sqrt{2}$



16. $\sqrt{(-4)^2 + (-4)^2} = \sqrt{32} = 4\sqrt{2}$



17. Since terminal side of $0°$ goes through $(8, 0)$, the trigonomeric form is $8(\cos 0° + i \sin 0°)$

18. Since terminal side of $180°$ goes through $(-7, 0)$, the trigonomeric form is $7(\cos 180° + i \sin 180°)$

19. Since terminal side of $90°$ goes through $(0, \sqrt{3})$, the trigonomeric form is $\sqrt{3}(\cos 90° + i \sin 90°)$

20. Since terminal side of $270°$ goes through $(0, -5)$, the trigonomeric form is $5(\cos 270° + i \sin 270°)$

21. Since $|-3+3i| = \sqrt{(-3)^2+3^2} = 3\sqrt{2}$ and if the terminal side of θ goes through $(-3, 3)$ then $\cos\theta = -3/(3\sqrt{2}) = -1/\sqrt{2}$. One can choose $\theta = 135°$ and the trigonometric form is $3\sqrt{2}(\cos 135° + i \sin 135°)$.

22. Since $|4-4i| = \sqrt{4^2+(-4)^2} = 4\sqrt{2}$ and if the terminal side of θ goes through $(4, -4)$ then $\cos\theta = 4/(4\sqrt{2}) = 1/\sqrt{2}$. One can choose $\theta = 315°$. Trigonometric form is $4\sqrt{2}(\cos 315° + i \sin 315°)$.

23. Since $\sqrt{(-3/\sqrt{2})^2 + (3/\sqrt{2})^2} = 3$ and if the terminal side of θ goes through $(-3/\sqrt{2}, 3/\sqrt{2})$ then $\cos\theta = (-3/\sqrt{2})/3 = -1/\sqrt{2}$. One can choose $\theta = 135°$. The trigonometric form is $3(\cos 135° + i \sin 135°)$.

24. Since $\sqrt{(\sqrt{3}/6)^2 + (1/6)^2} = \sqrt{4/36} = 1/3$ and if the terminal side of θ goes through $(\sqrt{3}/6, 1/6)$ then $\cos\theta = \dfrac{\sqrt{3}/6}{1/3} = \sqrt{3}/2$. One can choose $\theta = 30°$. The trigonometric form is $\dfrac{1}{3}(\cos 30° + i \sin 30°)$.

25. Since $|-\sqrt{3}+i| = \sqrt{(-\sqrt{3})^2+1^2} = 2$ and if the terminal side of θ goes through $(-\sqrt{3},1)$ then $\cos\theta = -\sqrt{3}/2$. One can choose $\theta = 150°$. Trigonometric form is $2(\cos 150° + i\sin 150°)$.

26. Since $|-2-2i\sqrt{3}| = \sqrt{(-2)^2+(-2\sqrt{3})^2} = 4$ and if the terminal side of θ goes through $(-2,-2\sqrt{3})$ then $\cos\theta = -2/4 = -1/2$. One can choose $\theta = 240°$. Trigonometric form is $4(\cos 240° + i\sin 240°)$.

27. Since $|3+4i| = \sqrt{3^2+4^2} = 5$ and if the terminal side of θ goes through $(3,4)$ then $\cos\theta = 3/5$. One can choose $\theta = \cos^{-1}(3/5) \approx 53.1°$. The trigonometric form is $5(\cos 53.1° + i\sin 53.1°)$.

28. Since $|-2+i| = \sqrt{(-2)^2+1^2} = \sqrt{5}$ and if the terminal side of θ goes through $(-2,1)$ then $\cos\theta = -2/\sqrt{5}$. One can choose $\theta = \cos^{-1}(-2/\sqrt{5}) \approx 153.4°$. Trigonometric form is $\sqrt{5}(\cos 153.4° + i\sin 153.4°)$.

29. Since $|-3+5i| = \sqrt{(-3)^2+5^2} = \sqrt{34}$ and if the terminal side of θ goes through $(-3,5)$ then $\cos\theta = -3/\sqrt{34}$. One can choose $\theta = \cos^{-1}(-3/\sqrt{34}) \approx 121.0°$. Trigonometric form is $\sqrt{34}(\cos 121.0° + i\sin 121.0°)$.

30. Note $|-2-4i| = \sqrt{(-2)^2+(-4)^2} = \sqrt{20} = 2\sqrt{5}$. If the terminal side of θ goes through $(-2,-4)$ then $\tan\theta = (-4)/(-2) = 2$. Since $\tan^{-1}(2) \approx 63.4°$, one can choose $\theta = 180° + 63.4° = 243.4°$. The trigonometric form is $2\sqrt{5}(\cos 243.4° + i\sin 243.4°)$.

31. Note $|3-6i| = \sqrt{3^2+(-6)^2} = \sqrt{45} = 3\sqrt{5}$. If the terminal side of θ goes through $(3,-6)$ then $\tan\theta = -6/3 = -2$. Since $\tan^{-1}(-2) \approx -63.4°$, one can choose $\theta = 360° - 63.4° = 296.6°$. The trigonometric form is $3\sqrt{5}(\cos 296.6° + i\sin 296.6°)$.

32. Note $|5-10i| = \sqrt{5^2+(-10)^2} = \sqrt{125} = 5\sqrt{5}$. If the terminal side of θ goes through $(5,-10)$ then $\tan\theta = -10/5 = -2$. Since $\tan^{-1}(-2) \approx -63.4°$, one can choose $\theta = 360° - 63.4° = 296.6°$. The trigonometric form is $5\sqrt{5}(\cos 296.6° + i\sin 296.6°)$.

33. $\sqrt{2}\left(\dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}\right) = 1+i$

34. $6\left(\dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}\right) = 3\sqrt{3}+3i$

35. $\dfrac{\sqrt{3}}{2}\left(-\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) = -\dfrac{3}{4} + \dfrac{\sqrt{3}}{4}i$

36. $12(0.95 + i\cdot 0.31) \approx 11.4 + 3.7i$

37. $\dfrac{1}{2}(-0.848 - 0.53i) \approx -0.42 - 0.26i$

38. $4.3(0.94 + i\cdot 0.342) \approx 4.04 + 1.47i$

39. $3(0+i) = 3i$ **40.** $4(-1+0i) = -4$

41. $\sqrt{3}(0-i) = -i\sqrt{3}$ **42.** $8.1(-1+0i) = -8.1$

43. $\sqrt{6}\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right) =$

$\dfrac{\sqrt{6}}{2} + \dfrac{\sqrt{18}}{2}i = \dfrac{\sqrt{6}}{2} + \dfrac{3\sqrt{2}}{2}i$

44. $0.5\left(-\dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}\right) = -\dfrac{\sqrt{3}}{4} + i\dfrac{1}{4}$

45. $6(\cos 450° + i\sin 450°) =$
$6(\cos 90° + i\sin 90°) = 6(0+i) = 6i$

46. $\sqrt{6}(\cos 360° + i\sin 360°) =$
$\sqrt{6}(1 + i\cdot 0) = \sqrt{6}$

47. $\sqrt{6}(\cos 30° + i\sin 30°) =$

$\sqrt{6}\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) =$

$\dfrac{\sqrt{18}}{2} + \dfrac{\sqrt{6}}{2}i = \dfrac{3\sqrt{2}}{2} + \dfrac{\sqrt{6}}{2}i$

48. $24(\cos 135° + i\sin 135°) =$

$24\left(-\dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}\right) = -12\sqrt{2} + 12i\sqrt{2}$

49. $9(\cos 90° + i\sin 90°) =$
$9(0+i) = 9i$

50. $5(\cos(\pi/6) + i\sin(\pi/6)) =$

$5\left(\dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}\right) = \dfrac{5\sqrt{3}}{2} + i\dfrac{5}{2}$

51. $2(\cos(\pi/6) + i\sin(\pi/6)) =$
$2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i$

52. $3(\cos(-\pi) + i\sin(-\pi)) =$
$3(-1 + i \cdot 0) = -3$

53. $0.5(\cos(-47.5°) + i\sin(-47.5°)) \approx$
$0.5(0.6756 - i \cdot 0.7373) \approx 0.34 - 0.37i$

54. $9(\cos(-203.7°) + i\sin(-203.7°)) \approx$
$9(-0.9157 - i \cdot 0.4019) \approx -8.24 + 3.62i$

55. Since $z_1 = 4\sqrt{2}(\cos 45° + i\sin 45°)$ and $z_2 = 5\sqrt{2}(\cos 225° + i\sin 225°)$, we have
$z_1 z_2 = 40(\cos 270° + i\sin 270°) =$
$40(0 - i) = -40i$ and
$\frac{z_1}{z_2} = 0.8(\cos(-180°) + i\sin(-180°)) =$
$0.8(-1 + i \cdot 0) = -0.8$

56. Since $z_1 = 3\sqrt{2}(\cos 135° + i\sin 135°)$ and $z_2 = 2\sqrt{2}(\cos 225° + i\sin 225°)$, we have
$z_1 z_2 = 12(\cos 360° + i\sin 360°) =$
$12(1 + i \cdot 0) = 12$ and
$\frac{z_1}{z_2} = 1.5(\cos(-90°) + i\sin(-90°)) =$
$1.5(0 - i) = -1.5i$

57. Since $z_1 = 2(\cos 30° + i\sin 30°)$ and $z_2 = 4(\cos 60° + i\sin 60°)$, we have
$z_1 z_2 = 8(\cos 90° + i\sin 90°) =$
$8(0 + i) = 8i$ and
$\frac{z_1}{z_2} = \frac{1}{2}(\cos(-30°) + i\sin(-30°)) =$
$\frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \frac{\sqrt{3}}{4} - \frac{1}{4}i.$

58. Since $z_1 = 2(\cos 150° + i\sin 150°)$ and $z_2 = 8(\cos 330° + i\sin 330°)$, we have
$z_1 z_2 = 16(\cos 480° + i\sin 480°)$
$z_1 z_2 = 16(\cos 120° + i\sin 120°)$
$16\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -8 + 8i\sqrt{3}$ and
$\frac{z_1}{z_2} = \frac{1}{4}(\cos(-180°) + i\sin(-180°)) =$
$\frac{1}{4}(-1 + i \cdot 0) = -\frac{1}{4}.$

59. Since $z_1 = 2\sqrt{2}(\cos 45° + i\sin 45°)$ and $z_2 = 2(\cos 315° + i\sin 315°)$, we have
$z_1 z_2 = 4\sqrt{2}(\cos 360° + i\sin 360°) =$
$4\sqrt{2}(1 + i \cdot 0) = 4\sqrt{2}$ and
$\frac{z_1}{z_2} = \sqrt{2}(\cos(-270°) + i\sin(-270°)) =$
$\sqrt{2}(0 + i) = \sqrt{2}i.$

60. Since $z_1 = \sqrt{10}(\cos 45° + i\sin 45°)$ and $z_2 = 2\sqrt{3}(\cos 225° + i\sin 225°)$, we have
$z_1 z_2 = 2\sqrt{30}(\cos 270° + i\sin 270°) =$
$2\sqrt{30}(0 - i) = -2i\sqrt{30}$ and
$\frac{z_1}{z_2} = \frac{\sqrt{10}}{2\sqrt{3}}(\cos(-180°) + i\sin(-180°)) =$
$\frac{\sqrt{30}}{6}(-1 + i \cdot 0) = -\frac{\sqrt{30}}{6}.$

61. Let α and β be angles whose terminal side goes through $(3, 4)$ and $(-5, -2)$, respectively. Since $|3 + 4i| = 5$ and $|-5 - 2i| = \sqrt{29}$, we have $\cos\alpha = 3/5$, $\sin\alpha = 4/5$, $\cos\beta = -5/\sqrt{29}$, and $\sin\beta = -2/\sqrt{29}$. From the sum and difference identities, we find
$\cos(\alpha + \beta) = -\frac{7}{5\sqrt{29}}$, $\sin(\alpha + \beta) = -\frac{26}{5\sqrt{29}}$,
$\cos(\alpha - \beta) = -\frac{23}{5\sqrt{29}}$, $\sin(\alpha - \beta) = -\frac{14}{5\sqrt{29}}$.
Note $z_1 = 5(\cos\alpha + i\sin\alpha)$ and $z_2 = \sqrt{29}(\cos\beta + i\sin\beta)$. Then

$$\begin{aligned} z_1 z_2 &= 5\sqrt{29}(\cos(\alpha + \beta) + i\sin(\alpha + \beta)) \\ &= 5\sqrt{29}\left(-\frac{7}{5\sqrt{29}} - i\frac{26}{5\sqrt{29}}\right) \\ z_1 z_2 &= -7 - 26i \end{aligned}$$

and

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{5}{\sqrt{29}}(\cos(\alpha - \beta) + i\sin(\alpha - \beta)) \\ &= \frac{5}{\sqrt{29}}\left(-\frac{23}{5\sqrt{29}} - i\frac{14}{5\sqrt{29}}\right) \\ \frac{z_1}{z_2} &= -\frac{23}{29} - \frac{14}{29}i \end{aligned}$$

62. Let α and β be angles whose terminal sides go through $(3, -4)$ and $(-1, 3)$, respectively. Since $|3 - 4i| = 5$ and $|-1 + 3i| = \sqrt{10}$,

$\cos\alpha = 3/5$, $\sin\alpha = -4/5$, $\cos\beta = -1/\sqrt{10}$, and $\sin\beta = 3/\sqrt{10}$. From the sum and difference identities,

$$\cos(\alpha+\beta) = \frac{9}{5\sqrt{10}},\ \sin(\alpha+\beta) = \frac{13}{5\sqrt{10}},$$

$$\cos(\alpha-\beta) = -\frac{3}{\sqrt{10}},\ \sin(\alpha-\beta) = -\frac{1}{\sqrt{10}}.$$

Note $z_1 = 5(\cos\alpha + i\sin\alpha)$ and $z_2 = \sqrt{10}(\cos\beta + i\sin\beta)$. Then

$$\begin{aligned} z_1 z_2 &= 5\sqrt{10}\left(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right) \\ &= 5\sqrt{10}\left(\frac{9}{5\sqrt{10}} + i\frac{13}{5\sqrt{10}}\right) \\ z_1 z_2 &= 9 + 13i \end{aligned}$$

and

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{5}{\sqrt{10}}\left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right) \\ &= \frac{5}{\sqrt{10}}\left(-\frac{3}{\sqrt{10}} - i\frac{1}{\sqrt{10}}\right) \\ \frac{z_1}{z_2} &= -\frac{3}{2} - \frac{1}{2}i \end{aligned}$$

63. Let α and β be angles whose terminal sides goes through $(2,-6)$ and $(-3,-2)$, respectively. Since $|2-6i| = 2\sqrt{10}$ and $|-3-2i| = \sqrt{13}$, we have $\cos\alpha = 1/\sqrt{10}$, $\sin\alpha = -3/\sqrt{10}$, $\cos\beta = -3/\sqrt{13}$, and $\sin\beta = -2/\sqrt{13}$. From the sum and difference identities,

$$\cos(\alpha+\beta) = -\frac{9}{\sqrt{130}},\ \sin(\alpha+\beta) = \frac{7}{\sqrt{130}},$$

$$\cos(\alpha-\beta) = \frac{3}{\sqrt{130}},\ \sin(\alpha-\beta) = \frac{11}{\sqrt{130}}.$$

Note $z_1 = 2\sqrt{10}(\cos\alpha + i\sin\alpha)$ and $z_2 = \sqrt{13}(\cos\beta + i\sin\beta)$. Thus,

$$\begin{aligned} z_1 z_2 &= 2\sqrt{130}\left(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right) \\ &= 2\sqrt{130}\left(-\frac{9}{\sqrt{130}} + i\frac{7}{\sqrt{130}}\right) \\ z_1 z_2 &= -18 + 14i \end{aligned}$$

and

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2\sqrt{10}}{\sqrt{13}}\left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right) \\ &= \frac{2\sqrt{10}}{\sqrt{13}}\left(\frac{3}{\sqrt{130}} + i\frac{11}{\sqrt{130}}\right) \\ \frac{z_1}{z_2} &= \frac{6}{13} + \frac{22}{13}i \end{aligned}$$

64. Let α and β be angles whose terminal side goes through $(1,4)$ and $(-4,-2)$, respectively. Since $|1+4i| = \sqrt{17}$ and $|-4-2i| = 2\sqrt{5}$, we have $\cos\alpha = 1/\sqrt{17}$, $\sin\alpha = 4/\sqrt{17}$, $\cos\beta = -2/\sqrt{5}$, and $\sin\beta = -1/\sqrt{5}$. From the sum and difference identities,

$$\cos(\alpha+\beta) = \frac{2}{\sqrt{85}},\ \sin(\alpha+\beta) = -\frac{9}{\sqrt{85}},$$

$$\cos(\alpha-\beta) = -\frac{6}{\sqrt{85}},\ \sin(\alpha-\beta) = -\frac{7}{\sqrt{85}}.$$

Note $z_1 = \sqrt{17}(\cos\alpha + i\sin\alpha)$ and $z_2 = 2\sqrt{5}(\cos\beta + i\sin\beta)$. So

$$\begin{aligned} z_1 z_2 &= 2\sqrt{85}\left(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right) \\ &= 2\sqrt{85}\left(\frac{2}{\sqrt{85}} - i\frac{9}{\sqrt{85}}\right) \\ z_1 z_2 &= 4 - 18i \end{aligned}$$

and

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{\sqrt{17}}{2\sqrt{5}}\left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right) \\ &= \frac{\sqrt{17}}{2\sqrt{5}}\left(-\frac{6}{\sqrt{85}} - i\frac{7}{\sqrt{85}}\right) \\ \frac{z_1}{z_2} &= -\frac{3}{5} - \frac{7}{10}i \end{aligned}$$

65. Note $3i = 3(\cos 90^\circ + i\sin 90^\circ)$ and $1 + i = \sqrt{2}(\cos 45^\circ + i\sin 45^\circ)$. Then we get

$$\begin{aligned} (3i)(1+i) &= 3\sqrt{2}\left(\cos 135^\circ + i\sin 135^\circ\right) \\ &= 3\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\cdot\frac{\sqrt{2}}{2}\right) \\ (3i)(1+i) &= -3 + 3i \end{aligned}$$

and

$$\begin{aligned}\frac{3i}{1+i} &= \frac{3}{\sqrt{2}}(\cos 45^\circ + i\sin 45^\circ)\\ &= \frac{3}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)\\ (3i)(1+i) &= \frac{3}{2}+\frac{3}{2}i.\end{aligned}$$

66. Let α be an angle whose terminal side goes through $(-3,1)$. Since $|-3+i| = \sqrt{10}$, we have $\cos\alpha = -3/\sqrt{10}$ and $\sin\alpha = 1/\sqrt{10}$. So

$$4(-3+i) = 4\sqrt{10}\,(\cos(0^\circ+\alpha) + i\sin(0^\circ+\alpha))$$

$$\begin{aligned}4(-3+i) &= 4\sqrt{10}\left(-\frac{3}{\sqrt{10}}+i\frac{1}{\sqrt{10}}\right)\\ 4(-3+i) &= -12+4i\end{aligned}$$

and

$$\begin{aligned}\frac{4}{-3+i} &= \frac{4}{\sqrt{10}}(\cos(0^\circ-\alpha)+i\sin(0^\circ-\alpha))\\ &= \frac{4}{\sqrt{10}}(\cos\alpha - i\sin\alpha)\\ &= \frac{4}{\sqrt{10}}\left(-\frac{3}{\sqrt{10}}-i\frac{1}{\sqrt{10}}\right)\\ \frac{4}{-3+i} &= -\frac{6}{5}-\frac{2}{5}i.\end{aligned}$$

67. $3\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$

68. $\sqrt{2}\left(\cos\left(-\frac{\pi}{8}\right)+i\sin\left(-\frac{\pi}{8}\right)\right)$

69. $2\sqrt{3}\,(\cos(20^\circ)+i\sin(20^\circ))$

70. $9\,(\cos(-14^\circ)+i\sin(-14^\circ))$

71. $\left[3\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]\left[3\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)\right]$
$= 9\left(\cos^2\frac{\pi}{6}+\sin^2\frac{\pi}{6}\right) = 9(1) = 9$

72. $\left[5\left(\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}\right)\right]\left[5\left(\cos\frac{\pi}{7}-i\sin\frac{\pi}{7}\right)\right]$
$= 25\left(\cos^2\frac{\pi}{7}+\sin^2\frac{\pi}{7}\right) = 25(1) = 25$

73. $[2(\cos 7^\circ + i\sin 7^\circ)]\,[2(\cos 7^\circ - i\sin 7^\circ)]$
$= 4(\cos^2 7^\circ + \sin^2 7^\circ) = 4(1) = 4$

74. $[6(\cos 5^\circ + i\sin 5^\circ)]\,[6(\cos 5^\circ - i\sin 5^\circ)]$
$= 36(\cos^2 5^\circ + \sin^2 5^\circ) = 36(1) = 36$

75. Since $4+4i = 4\sqrt{2}\,(\cos 45^\circ + i\sin 45^\circ)$,
$(4+4i)^2 = (4\sqrt{2})^2\,(\cos(2\cdot 45^\circ)+i\sin(2\cdot 45^\circ))$
$32\,(\cos 90^\circ + i\sin 90^\circ) =$
$32\,(0+i\cdot 1) = 32i.$

76. Since $-3+3i = 3\sqrt{2}\,(\cos 135^\circ + i\sin 135^\circ)$,
we obtain $(-3+3i)^2 =$
$(3\sqrt{2})^2\,(\cos(2\cdot 135^\circ)+i\sin(2\cdot 135^\circ)) =$
$18\,(\cos 270^\circ + i\sin 270^\circ) =$
$18\,(0+i\cdot(-1)) = -18i.$

77. Since $3+3i = 3\sqrt{2}\,(\cos 45^\circ + i\sin 45^\circ)$,
$(3+3i)^3 = (3\sqrt{2})^3\,(\cos(3\cdot 45^\circ)+i\sin(3\cdot 45^\circ))$
$54\sqrt{2}\,(\cos 135^\circ + i\sin 135^\circ) =$
$54\sqrt{2}\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right) = -54+54i.$

78. Since $\sqrt{3}+i = 2\,(\cos 30^\circ + i\sin 30^\circ)$,
$(\sqrt{3}+i)^4 = (2)^4\,(\cos(4\cdot 30^\circ)+i\sin(4\cdot 30^\circ))$
$16\,(\cos 120^\circ + i\sin 120^\circ) =$
$16\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = -8+8i\sqrt{3}.$

79. $\overline{z} = \overline{r(\cos\theta + i\sin\theta)} =$
$r(\cos\theta - i\sin\theta) = r(\cos(-\theta)+i\sin(-\theta))$

80. By Exercise 79, one derives
$z\overline{z} = [r(\cos\theta+i\sin\theta)]\,[r(\cos(-\theta)+i\sin(-\theta))]$
$r^2\,[\cos(\theta-\theta)+i\sin(\theta-\theta)] = r^2(1+i0) = r^2$

81. The reciprocal of z is $\frac{1}{z} = \frac{\cos 0 + i\sin 0}{r\,[\cos\theta + i\sin\theta]} =$
$r^{-1}\,[\cos(0-\theta)+i\sin(0-\theta)] =$
$r^{-1}\,[\cos\theta - i\sin\theta]$ provided $r \neq 0$.

82. $z^2 = r^2\,(\cos 2\theta + i\sin 2\theta)$ and
$z^{-2} = \frac{1}{r^2}\,(\cos 2\theta - i\sin 2\theta)$

83. The first sum is
$6(\cos 9^\circ + i\sin 9^\circ) + 3(\cos 5^\circ + i\sin 5^\circ) =$
$6\cos 9^\circ + 3\cos 5^\circ + i(6\sin 9^\circ + 3\sin 5^\circ)$.
Also, $(1+3i)+(5-7i) = 6-4i$. It is easier to add complex numbers in standard form.

84. Using the trigonometric form, we obtain

$$\begin{aligned} \frac{6(\cos 9^\circ + i\sin 9^\circ)}{3(\cos 5^\circ + i\sin 5^\circ)} &= \\ \frac{6}{3}(\cos(9^\circ - 5^\circ) + i\sin(9^\circ - 5^\circ)) &= \\ 2(\cos 4^\circ + i\sin 4^\circ). \end{aligned}$$

While using the standard form, we have

$$\begin{aligned} \frac{1+3i}{5-7i} &= \frac{(1+3i)(5+7i)}{(5-7i)(5+7i)} \\ &= \frac{-16+22i}{25+49} \\ &= \frac{-16+22i}{74} \\ &= \frac{-8}{37} + \frac{11}{37}i. \end{aligned}$$

It is easier to divide complex numbers in trigonometric form.

85. $-2(3) + 6(5) = 24$

86. Let $v_1 = \langle -3, 5\rangle$ and $v_2 = \langle 1, 6\rangle$. Then $v_1 \cdot v_2 = 27$, $|v_1| = \sqrt{34}$ and $|v_2| = \sqrt{37}$. If α is the angle between v_1 and v_2, then

$$\begin{aligned} v_1 \cdot v_2 &= |v_1||v_2|\cos\alpha \\ \alpha &= \cos^{-1}\left(\frac{27}{\sqrt{34}\sqrt{37}}\right) \\ \alpha &\approx 40.4^\circ \end{aligned}$$

87. The largest angle is γ and is opposite the longest side. By the cosine law,

$$\begin{aligned} 10^2 &= 7^2 + 5^2 - 2(7)(5)\cos\gamma \\ 100 &= 74 - 70\cos\gamma \\ \gamma &= \cos^{-1}\left(-\frac{26}{70}\right) \\ \gamma &\approx 111.8^\circ \end{aligned}$$

The remaining angles are less than 90°.

By the sine law,

$$\begin{aligned} \sin\beta &= \frac{7\sin\gamma}{10} \\ \beta &= \sin^{-1}\left(\frac{7\sin\gamma}{10}\right) \\ \beta &\approx 40.5^\circ \end{aligned}$$

Finally, $\alpha = 180^\circ - \gamma - \beta = 27.7^\circ$.

88. The area is

$$\begin{aligned} A &= \frac{1}{2}bc\sin\alpha \\ &= \frac{1}{2}(5.7)(12.2)\sin 10.6^\circ \\ &\approx 6.4 \text{ ft}^2 \end{aligned}$$

89. If x is the height of the building, then

$$x = 230\tan 48^\circ \approx 255.4 \text{ ft}$$

90. $\dfrac{1-\sin^2 x\csc^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1 - 1 + \sin^2 x}{\cos^2 x} = \dfrac{\sin^2 x}{\cos^2 x} = \tan^2 x$

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Note, $CT = CA = 5$ and $CS = CB = 6$. Then ΔABC is congruent to ΔTSC. We use Heron's formula, and let

$$s = \frac{5+6+7}{2} = 9.$$

The area of ΔTSC is

$$\text{Area } = \sqrt{9(9-5)(9-6)(9-7)} = 6\sqrt{6}.$$

7.4 Pop Quiz

1. $|3-i| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$

2. Since $|5+5i| = \sqrt{5^2+5^2} = 5\sqrt{2}$ and $\tan^{-1}\left(\dfrac{5}{5}\right) = \pi/4$, we obtain

$$5 + 5i = 5\sqrt{2}(\cos\pi/4 + i\sin\pi/4).$$

3. $2\sqrt{3}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-3+i\sqrt{3}$

4. The product zw is equal to
$$3\sqrt{2}\cdot\sqrt{2}\left[\cos\left(\frac{\pi}{12}+\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}+\frac{\pi}{12}\right)\right]$$
or equivalently to
$$6\left[\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right]=6\left[\frac{\sqrt{3}}{2}+\frac{1}{2}i\right]=3\sqrt{3}+3i.$$

5. The product is
$$\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]\left[\sqrt{2}\left(\cos\frac{-\pi}{4}+i\sin\frac{-\pi}{4}\right)\right]$$
or equivalently to
$$\sqrt{2}\sqrt{2}\left[\cos\left(\frac{\pi}{4}-\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}-\frac{\pi}{4}\right)\right]$$
and to
$$2\left[\cos 0+i\sin 0\right]=2.$$

For Thought

1. False, $(2+3i)^2=4+12i+9i^2$.

2. False, $z^3=8(\cos 360^\circ+i\sin 360^\circ)=8$.

3. True **4.** False, the argument is 4θ.

5. False, since $\left[\frac{1}{2}+i\cdot\frac{1}{2}\right]^2=\frac{i}{2}$ and
$\cos 2\pi/3+i\sin\pi/3=-\frac{1}{2}+i\cdot\frac{\sqrt{3}}{2}$.

6. False, since $\cos 5\pi/6=\cos 7\pi/6$ and $5\pi/6\neq 7\pi/6+2k\pi$ for any integer k. It is possible that $\alpha=2k\pi-\beta$.

7. True **8.** True

9. True, $x=i$ is a solution not on $y=\pm x$.

10. False, it has four imaginary solutions.

7.5 Exercises

1. $3^3(\cos 90^\circ+i\sin 90^\circ)=27(0+i)=27i$

2. $2^5(\cos 225^\circ+i\sin 225^\circ)=32\left(-\frac{\sqrt{2}}{2}-i\cdot\frac{\sqrt{2}}{2}\right)$
$=-16\sqrt{2}-16i\sqrt{2}$

3. $(\sqrt{2})^4(\cos 480^\circ+i\sin 480^\circ)=$
$4(\cos 120^\circ+i\sin 120^\circ)=$
$4\left(-\frac{1}{2}+i\cdot\frac{\sqrt{3}}{2}\right)=-2+2i\sqrt{3}$

4. $(\sqrt{3})^3(\cos 450^\circ+i\sin 450^\circ)=$
$3\sqrt{3}(\cos 90^\circ+i\sin 90^\circ)=$
$3\sqrt{3}\,(0+i)=3i\sqrt{3}$

5. $\cos(8\pi/12)+i\sin(8\pi/12)=$
$\cos(2\pi/3)+i\sin(2\pi/3)=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

6. $\cos(9\pi/6)+i\sin(9\pi/6)=$
$\cos(3\pi/2)+i\sin(3\pi/2)=0-i=-i$

7. $(\sqrt{6})^4\left[\cos(8\pi/3)+i\sin(8\pi/3)\right]=$
$36\left[\cos(2\pi/3)+i\sin(2\pi/3)\right]=$
$36\left[-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right]=-18+18i\sqrt{3}$

8. $(\sqrt{18})^3\left[\cos(5\pi/2)+i\sin(5\pi/2)\right]=$
$18\sqrt{18}\left[0+i\right]=54i\sqrt{2}$

9. $4.3^5\left[\cos 61.5^\circ+i\sin 61.5^\circ\right]\approx$
$1470.1\left[0.4772+0.8788\right]\approx 701.5+1291.9i$

10. $4.9^6\left[\cos 224.4^\circ+i\sin 224.4^\circ\right]\approx$
$13841.3\left[-0.71447-0.69966\right]\approx$
$-9889.2-9684.2i$

11. $\left(2\sqrt{2}\left[\cos 45^\circ+i\sin 45^\circ\right]\right)^3=$
$16\sqrt{2}\left[\cos 135^\circ+i\sin 135^\circ\right]=$
$16\sqrt{2}\left[-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right]=-16+16i$

12. $\left(\sqrt{2}\left[\cos 315^\circ+i\sin 315^\circ\right]\right)^3=$
$2\sqrt{2}\left[\cos 945^\circ+i\sin 945^\circ\right]=$
$2\sqrt{2}\left[\cos 225^\circ+i\sin 225^\circ\right]=$
$2\sqrt{2}\left[-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right]=-2-2i$

13. $\left(2\left[\cos(-30^\circ)+i\sin(-30^\circ)\right]\right)^4=$
$16\left[\cos(-120^\circ)+i\sin(-120^\circ)\right]=$
$16\left[-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right]=-8-8i\sqrt{3}$

14. $\left(4\left[\cos 120^\circ+i\sin 120^\circ\right]\right)^4=$
$256\left[\cos 480^\circ+i\sin 480^\circ\right]=$

$256\,[\cos 120° + i \sin 120°] =$
$256\left[-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] = -128 + 128i\sqrt{3}$

15. $(6\,[\cos 240° + i \sin 240°])^5 =$
$7776\,[\cos 1200° + i \sin 1200°] =$
$7776\,[\cos 120° + i \sin 120°] =$
$7776\left[-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] = -3888 + 3888i\sqrt{3}$

16. $(4\,[\cos(-30°) + i \sin(-30°)])^5 =$
$1024\,[\cos(-150°) + i \sin(-150°)] =$
$1024\left[-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right] = -512\sqrt{3} - 512i$

17. Note $|2+3i| = \sqrt{13}$. If the terminal side of α goes through $(2,3)$ then $\cos\alpha = 2/\sqrt{13}$ and $\sin\alpha = 3/\sqrt{13}$. By using the double-angle identities one can successively obtain
$\cos 2\alpha = -5/13$, $\sin 2\alpha = 12/13$,
$\cos 4\alpha = -119/169$, $\sin 4\alpha = -120/169$.

So $(2+3i)^4 = \left(\sqrt{13}\,[\cos\alpha + i\sin\alpha]\right)^4 =$
$169(\cos 4\alpha + i \sin 4\alpha) =$
$169\,(-119/169 - 120i/169) = -119 - 120i$

18. Note $|4-i| = \sqrt{17}$. If the terminal side of α goes through $(4,-1)$ then $\cos\alpha = 4/\sqrt{17}$ and $\sin\alpha = -1/\sqrt{17}$. By using the double-angle identities one can successively obtain
$\cos 2\alpha = 15/17$, $\sin 2\alpha = -8/17$,
$\cos 4\alpha = 161/289$, $\sin 4\alpha = -240/289$.
By the sum identities one gets
$\cos 5\alpha = \cos(4\alpha + \alpha) = \frac{404}{289\sqrt{17}}$ and
$\sin 5\alpha = \sin(4\alpha + \alpha) = -\frac{1121}{289\sqrt{17}}$.

So $(4-i)^5 = \left(\sqrt{17}\,[\cos\alpha + i\sin\alpha]\right)^5 =$

$289\sqrt{17}(\cos 5\alpha + i\sin 5\alpha) =$
$289\sqrt{17}\left(\frac{404}{289\sqrt{17}} - i\cdot\frac{1121}{289\sqrt{17}}\right) =$
$404 - 1121i$.

19. Note $|2-i| = \sqrt{5}$. If the terminal side of α goes through $(2,-1)$ then $\cos\alpha = 2/\sqrt{5}$ and $\sin\alpha = -1/\sqrt{5}$. By using the double-angle identities one can successively obtain
$\cos 2\alpha = 3/5$, $\sin 2\alpha = -4/5$,
$\cos 4\alpha = -7/25$, $\sin 4\alpha = -24/25$.

So $(2-4i)^4 = \left(\sqrt{5}\,[\cos\alpha + i\sin\alpha]\right)^4 =$
$25(\cos 4\alpha + i\sin 4\alpha) =$
$25\,(-7/25 - 24i/25) = -7 - 24i$.

20. Note $|-1-2i| = \sqrt{5}$. If the terminal side of α goes through $(-1,-2)$ then $\cos\alpha = -1/\sqrt{5}$ and $\sin\alpha = -2/\sqrt{5}$. By using the double-angle identities one can successively obtain
$\cos 2\alpha = -3/5$, $\sin 2\alpha = 4/5$,
$\cos 4\alpha = -7/25$, $\sin 4\alpha = -24/25$.
By the sum identities one gets
$\cos 6\alpha = \cos(4\alpha + 2\alpha) = 117/125$ and
$\sin 6\alpha = \sin(4\alpha + 2\alpha) = 44/125$.

So $(-1-2i)^6 = \left(\sqrt{5}\,[\cos\alpha + i\sin\alpha]\right)^6 =$
$125(\cos 6\alpha + i\sin 6\alpha) =$
$125\,(117/125 + 44i/125) = 117 + 44i$.

21. Let $\omega = |1.2 + 3.6i|$. If the terminal side of α goes through $(1.2, 3.6)$ then $\cos\alpha = 1.2/\omega$ and $\sin\alpha = 3.6/\omega$. By using the double-angle identities one can obtain
$\cos 2\alpha = -11.52/\omega^2$ and $\sin 2\alpha = 8.64/\omega^2$.
By the sum identities one gets
$\cos 3\alpha = \cos(2\alpha + \alpha) = -44.928/\omega^3$ and
$\sin 3\alpha = \sin(2\alpha + \alpha) = -31.104/\omega^3$.

So $(1.2 + 3.6i)^3 = (\omega\,[\cos\alpha + i\sin\alpha])^3 =$
$\omega^3(\cos 3\alpha + i\sin 3\alpha) =$
$\omega^3\,(-44.928/\omega^3 - 31.104i/\omega^3) =$
$-44.928 - 31.104i$.

22. Let $\omega = |-2.3 - i|$. If the terminal side of α goes through $(-2.3, -1)$ then $\cos\alpha = -2.3/\omega$ and $\sin\alpha = -1/\omega$. By using the double-angle identities one can obtain
$\cos 2\alpha = 4.29/\omega^2$ and $\sin 2\alpha = 4.6/\omega^2$.
By the sum identities one gets
$\cos 3\alpha = \cos(2\alpha + \alpha) = -5.267/\omega^3$ and
$\sin 3\alpha = \sin(2\alpha + \alpha) = -14.87/\omega^3$.

So $(-2.3 - i)^3 = (\omega\,[\cos\alpha + i\sin\alpha])^3 =$
$\omega^3(\cos 3\alpha + i\sin 3\alpha) =$
$\omega^3\,(-5.267/\omega^3 - 14.87i/\omega^3) =$
$-5.267 - 14.87i$.

23. The square roots are given by

$2\left[\cos\left(\dfrac{90^\circ + k360^\circ}{2}\right) + i\sin\left(\dfrac{90^\circ + k360^\circ}{2}\right)\right] =$
$2\left[\cos(45^\circ + k\cdot 180^\circ) + i\sin(45^\circ + k\cdot 180^\circ)\right] =$
When $k = 0, 1$ one gets
$2\left(\cos 45^\circ + i\sin 45^\circ\right)$ and
$2\left(\cos 225^\circ + i\sin 225^\circ\right)$.

24. Cube roots are given by
$2\left[\cos\left(\dfrac{30^\circ + k360^\circ}{3}\right) + i\sin\left(\dfrac{30^\circ + k360^\circ}{3}\right)\right] =$
$2\left[\cos(10^\circ + k\cdot 120^\circ) + i\sin(10^\circ + k\cdot 120^\circ)\right] =$
When $k = 0, 1, 2$ one gets
$2\left(\cos\alpha + i\sin\alpha\right)$ where $\alpha = 10^\circ, 130^\circ, 250^\circ$.

25. Fourth roots are given by
$\cos\left(\dfrac{120^\circ + k360^\circ}{4}\right) + i\sin\left(\dfrac{120^\circ + k360^\circ}{4}\right) =$
$\cos(30^\circ + k\cdot 90^\circ) + i\sin(30^\circ + k\cdot 90^\circ) =$
When $k = 0, 1, 2, 3$ one gets $\cos\alpha + i\sin\alpha$
where $\alpha = 30^\circ, 120^\circ, 210^\circ, 300^\circ$.

26. Fifth roots are given by
$2\left[\cos\left(\dfrac{300^\circ + k360^\circ}{5}\right) + i\sin\left(\dfrac{300^\circ + k360^\circ}{5}\right)\right]$
$= 2\left[\cos(60^\circ + k\cdot 72^\circ) + i\sin(60^\circ + k\cdot 72^\circ)\right]$.
When $k = 0, 1, 2, 3, 4$ one gets

$$2\left(\cos\alpha + i\sin\alpha\right)$$

where $\alpha = 60^\circ, 132^\circ, 204^\circ, 276^\circ, 348^\circ$.

27. Sixth roots are given by
$2\left[\cos\left(\dfrac{\pi + 2k\pi}{6}\right) + i\sin\left(\dfrac{\pi + 2k\pi}{6}\right)\right]$.
When $k = 0, 1, 2, 3, 4, 5$ one gets

$$2\left(\cos\alpha + i\sin\alpha\right)$$

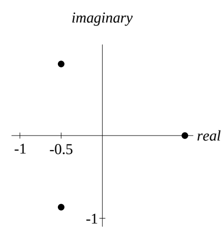
where $\alpha = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$.

28. Fourth roots are given by
$2\left[\cos\left(\dfrac{3\pi/2 + 2k\pi}{4}\right) + i\sin\left(\dfrac{3\pi/2 + 2k\pi}{4}\right)\right]$
$= 2\left[\cos\left(\dfrac{3\pi + 4k\pi}{8}\right) + i\sin\left(\dfrac{3\pi + 4k\pi}{8}\right)\right]$.
When $k = 0, 1, 2, 3$ one gets $2\left(\cos\alpha + i\sin\alpha\right)$
where $\alpha = \dfrac{3\pi}{8}, \dfrac{7\pi}{8}, \dfrac{11\pi}{8}, \dfrac{15\pi}{8}$.

29. Cube roots of 1 are given by

$$\cos\left(\frac{k360^\circ}{3}\right) + i\sin\left(\frac{k360^\circ}{3}\right).$$

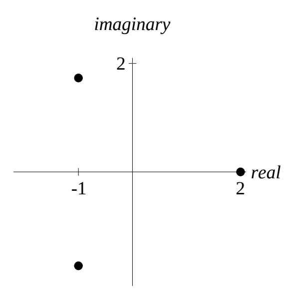
If $k = 0, 1, 2$ one obtains
$\cos 0^\circ + i\sin 0^\circ = 1$,
$\cos 120^\circ + i\sin 120^\circ = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$, and
$\cos 240^\circ + i\sin 240^\circ = -\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$.



30. Cube roots of 8 are given by

$$2\left[\cos\left(\frac{k360^\circ}{3}\right) + i\sin\left(\frac{k360^\circ}{3}\right)\right].$$

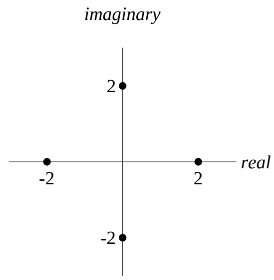
If $k = 0, 1, 2$ one obtains
$2\left[\cos 0^\circ + i\sin 0^\circ\right] = 2$,
$2\left[\cos 120^\circ + i\sin 120^\circ\right] = -1 + i\sqrt{3}$, and
$2\left[\cos 240^\circ + i\sin 240^\circ\right] = -1 - i\sqrt{3}$.



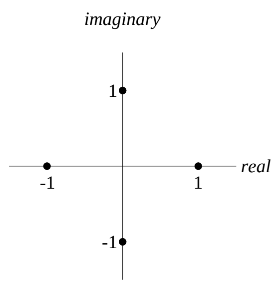
31. Fourth roots of 16 are given by

$$2\left[\cos\left(\frac{k360^\circ}{4}\right) + i\sin\left(\frac{k360^\circ}{4}\right)\right].$$

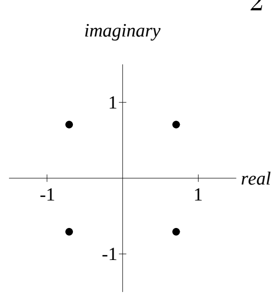
If $k = 0, 1, 2, 3$ one obtains
$2\left[\cos 0^\circ + i\sin 0^\circ\right] = 2$,
$2\left[\cos 90^\circ + i\sin 90^\circ\right] = 2i$,
$2\left[\cos 180^\circ + i\sin 180^\circ\right] = -2$, and
$2\left[\cos 270^\circ + i\sin 270^\circ\right] = -2i$.



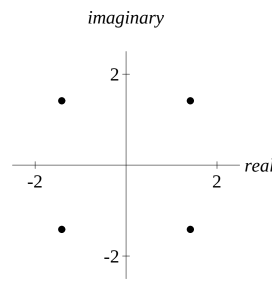
32. Fourth roots of 1 are given by
$\cos\left(\dfrac{k360^\circ}{4}\right) + i\sin\left(\dfrac{k360^\circ}{4}\right)$.
If $k = 0, 1, 2, 3$ one obtains
$\cos 0^\circ + i\sin 0^\circ = 1$,
$\cos 90^\circ + i\sin 90^\circ = i$,
$\cos 180^\circ + i\sin 180^\circ = -1$, and
$\cos 270^\circ + i\sin 270^\circ = -i$.



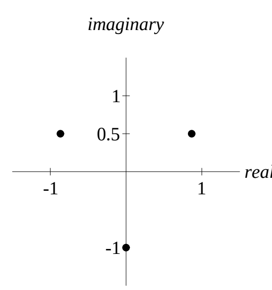
33. Fourth roots of -1 are given by
$\cos\left(\dfrac{180^\circ + k360^\circ}{4}\right) + i\sin\left(\dfrac{180^\circ + k360^\circ}{4}\right)$
$= \cos(45^\circ + k90^\circ) + i\sin(45^\circ + k90^\circ)$.
If $k = 0, 1, 2, 3$ one obtains
$\cos 45^\circ + i\sin 45^\circ = \dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}$,
$\cos 135^\circ + i\sin 135^\circ = -\dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}$,
$\cos 225^\circ + i\sin 225^\circ = -\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}$, and
$\cos 315^\circ + i\sin 315^\circ = \dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}$.



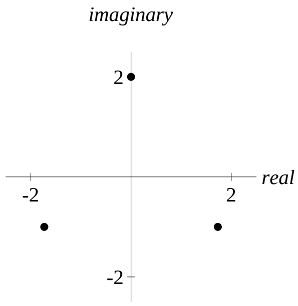
34. Fourth roots of -16 are given by
$2\left[\cos\left(\dfrac{180^\circ + k360^\circ}{4}\right) + i\sin\left(\dfrac{180^\circ + k360^\circ}{4}\right)\right]$
$= 2[\cos(45^\circ + k90^\circ) + i\sin(45^\circ + k90^\circ)]$.
If $k = 0, 1, 2, 3$ one obtains
$2[\cos 45^\circ + i\sin 45^\circ] = \sqrt{2} + i\sqrt{2}$,
$2[\cos 135^\circ + i\sin 135^\circ] = -\sqrt{2} + i\sqrt{2}$,
$2[\cos 225^\circ + i\sin 225^\circ] = -\sqrt{2} - i\sqrt{2}$ and
$2[\cos 315^\circ + i\sin 315^\circ] = \sqrt{2} - i\sqrt{2}$.



35. The cube roots of i are given by
$\cos\left(\dfrac{90^\circ + k360^\circ}{3}\right) + i\sin\left(\dfrac{90^\circ + k360^\circ}{3}\right)$
$= \cos(30^\circ + k120^\circ) + i\sin(30^\circ + k120^\circ)$.
If $k = 0, 1, 2$ one obtains
$\cos 30^\circ + i\sin 30^\circ = \dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}$,
$\cos 150^\circ + i\sin 150^\circ = -\dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}$, and
$\cos 270^\circ + i\sin 270^\circ = -i$.



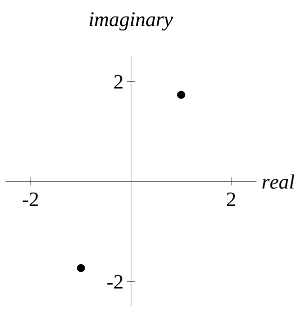
36. Cube roots of $-8i$ are given by
$2\left[\cos\left(\dfrac{-90^\circ + k360^\circ}{3}\right) + i\sin\left(\dfrac{-90^\circ + k360^\circ}{3}\right)\right]$
$= 2[\cos(-30^\circ + k120^\circ) + i\sin(-30^\circ + k120^\circ)]$
If $k = 0, 1, 2$ one obtains
$2[\cos(-30^\circ) + i\sin(-30^\circ)] = \sqrt{3} - i$,
$2[\cos 90^\circ + i\sin 90^\circ] = 2i$, and
$2[\cos 210^\circ + i\sin 210^\circ] = -\sqrt{3} - i$.



37. Since $|-2+2i\sqrt{3}| = 4$, the square roots are given by

$$2\left[\cos\left(\frac{120^\circ + k360^\circ}{2}\right) + i\sin\left(\frac{120^\circ + k360^\circ}{2}\right)\right]$$
$$= 2\left[\cos\left(60^\circ + k180^\circ\right) + i\sin\left(60^\circ + k180^\circ\right)\right].$$

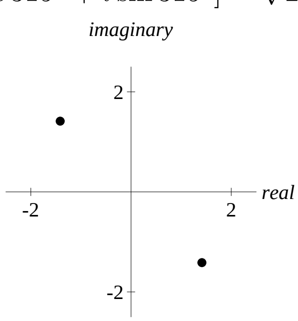
If $k = 0, 1$ one obtains
$2\left[\cos 60^\circ + i\sin 60^\circ\right] = 1 + i\sqrt{3}$, and
$2\left[\cos 240^\circ + i\sin 240^\circ\right] = -1 - i\sqrt{3}$.



38. Square roots are given by

$$2\left[\cos\left(\frac{270^\circ + k360^\circ}{2}\right) + i\sin\left(\frac{270^\circ + k360^\circ}{2}\right)\right]$$
$$= 2\left[\cos\left(135^\circ + k180^\circ\right) + i\sin\left(135^\circ + k180^\circ\right)\right].$$

If $k = 0, 1$ one obtains
$2\left[\cos 135^\circ + i\sin 135^\circ\right] = -\sqrt{2} + i\sqrt{2}$, and
$2\left[\cos 315^\circ + i\sin 315^\circ\right] = \sqrt{2} - i\sqrt{2}$.

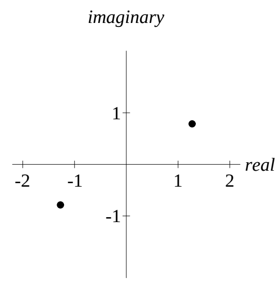


39. Note $|1+2i| = \sqrt{5}$. Since $\tan^{-1} 2 \approx 63.4^\circ$ and

$$\frac{63.4^\circ + k360^\circ}{2} = 31.7^\circ + k180^\circ$$

the square roots are given by
$\sqrt[4]{5}\left[\cos\left(31.7^\circ + k180^\circ\right) + i\sin\left(31.7^\circ + k180^\circ\right)\right]$.

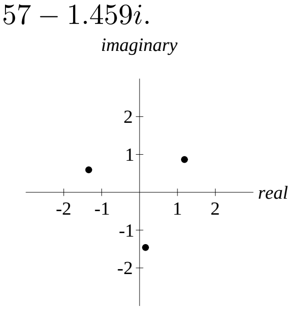
If $k = 0, 1$ one obtains
$\sqrt[4]{5}\left[\cos 31.7^\circ + i\sin 31.7^\circ\right] \approx 1.272 + 0.786i$, and
$\sqrt[4]{5}\left[\cos 211.7^\circ + i\sin 211.7^\circ\right] \approx -1.272 - 0.786i$.



40. Since $\tan^{-1}(-3) \approx -71.57^\circ$, an argument of $-1+3i$ is $180^\circ - 71.57^\circ = 108.43^\circ$.
Note $|-1+3i| = \sqrt{10}$. Choose $k = 0, 1, 2$ in

$$\frac{108.43^\circ + k360^\circ}{3} \approx 36.14^\circ + k120^\circ$$

to obtain the cube roots $\sqrt[6]{10}(\cos\alpha + i\sin\alpha)$ where $\alpha = 36.14^\circ, 156.14^\circ, 276.14^\circ$.
These roots, respectively, are approximately $1.185 + 0.866i$, $-1.342 + 0.594i$, and $0.157 - 1.459i$.



41. Solutions are the cube roots of -1. Namely,

$$\cos\left(\frac{180^\circ + k360^\circ}{3}\right) + i\sin\left(\frac{180^\circ + k360^\circ}{3}\right)$$
$$= \cos\left(60^\circ + k120^\circ\right) + i\sin\left(60^\circ + k120^\circ\right).$$

If $k = 0, 1, 2$ one obtains
$\cos 60^\circ + i\sin 60^\circ = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$,
$\cos 180^\circ + i\sin 180^\circ = -1$, and
$\cos 300^\circ + i\sin 300^\circ = \dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$.

42. Solutions are the cube roots of -125. Namely,
$$5\left[\cos\left(\frac{180^\circ + k360^\circ}{3}\right) + i\sin\left(\frac{180^\circ + k360^\circ}{3}\right)\right]$$
$$= 5\left[\cos\left(60^\circ + k120^\circ\right) + i\sin\left(60^\circ + k120^\circ\right)\right].$$
If $k = 0, 1, 2$ one obtains
$5\left[\cos 60^\circ + i\sin 60^\circ\right] = \frac{5}{2} + i\frac{5\sqrt{3}}{2}$,
$5\left[\cos 180^\circ + i\sin 180^\circ\right] = -5$, and
$5\left[\cos 300^\circ + i\sin 300^\circ\right] = \frac{5}{2} - i\frac{5\sqrt{3}}{2}$.

43. Solutions are the fourth roots of 81. Namely,
$$3\left[\cos\left(\frac{k360^\circ}{4}\right) + i\sin\left(\frac{k360^\circ}{4}\right)\right]$$
$$= 3\left[\cos\left(k90^\circ\right) + i\sin\left(k90^\circ\right)\right].$$
If $k = 0, 1, 2, 3$ one obtains
$3\left[\cos 0^\circ + i\sin 0^\circ\right] = 3$,
$3\left[\cos 90^\circ + i\sin 90^\circ\right] = 3i$,
$3\left[\cos 180^\circ + i\sin 180^\circ\right] = -3$ and
$3\left[\cos 270^\circ + i\sin 270^\circ\right] = -3i$.

44. Solutions are the fourth roots of -81.
$$3\left[\cos\left(\frac{180^\circ + k360^\circ}{4}\right) + i\sin\left(\frac{180^\circ + k360^\circ}{4}\right)\right]$$
$$= 3\left[\cos\left(45^\circ + k90^\circ\right) + i\sin\left(45^\circ + k90^\circ\right)\right].$$
If $k = 0, 1, 2, 3$ one obtains
$3\left[\cos 45^\circ + i\sin 45^\circ\right] = \frac{3\sqrt{2}}{2} + i\frac{3\sqrt{2}}{2}$,
$3\left[\cos 135^\circ + i\sin 135^\circ\right] = -\frac{3\sqrt{2}}{2} + i\frac{3\sqrt{2}}{2}$,
$3\left[\cos 225^\circ + i\sin 225^\circ\right] = -\frac{3\sqrt{2}}{2} - i\frac{3\sqrt{2}}{2}$, and
$3\left[\cos 315^\circ + i\sin 315^\circ\right] = \frac{3\sqrt{2}}{2} - i\frac{3\sqrt{2}}{2}$.

45. Solutions are the square roots of $-2i$.
If $k = 0, 1$ in $\alpha = \dfrac{-90^\circ + k360^\circ}{2}$ then
$\alpha = -45^\circ, 135^\circ$. These roots are
$\sqrt{2}\left[\cos(-45^\circ) + i\sin(-45^\circ)\right] = 1 - i$ and
$\sqrt{2}\left[\cos 135^\circ + i\sin 135^\circ\right] = -1 + i$.

46. Since $-3/i = 3i$, the solutions are the square roots of $3i$. Namely,
$$\sqrt{3}\left[\cos\left(\frac{90^\circ + k360^\circ}{2}\right) + i\sin\left(\frac{90^\circ + k360^\circ}{2}\right)\right]$$
$$= \sqrt{3}\left[\cos\left(45^\circ + k180^\circ\right) + i\sin\left(45^\circ + k180^\circ\right)\right]$$
If $k = 0, 1$ one obtains
$\sqrt{3}\left[\cos 45^\circ + i\sin 45^\circ\right] = \frac{\sqrt{6}}{2} + i\frac{\sqrt{6}}{2}$, and
$\sqrt{3}\left[\cos 225^\circ + i\sin 225^\circ\right] = -\frac{\sqrt{6}}{2} - i\frac{\sqrt{6}}{2}$.

47. Solutions of $x(x^6 - 64) = 0$ are $x = 0$ and the sixth roots of 64. The sixth roots are given by
$$2\left[\cos\left(\frac{k360^\circ}{6}\right) + i\sin\left(\frac{k360^\circ}{6}\right)\right]$$
$$= 2\left[\cos\left(k60^\circ\right) + i\sin\left(k60^\circ\right)\right].$$
If $k = 0, 1, 2, 3, 4, 5$ one obtains
$2\left[\cos 0^\circ + i\sin 0^\circ\right] = 2$,
$2\left[\cos 60^\circ + i\sin 60^\circ\right] = 1 + i\sqrt{3}$,
$2\left[\cos 120^\circ + i\sin 120^\circ\right] = -1 + i\sqrt{3}$,
$2\left[\cos 180^\circ + i\sin 180^\circ\right] = -2$,
$2\left[\cos 240^\circ + i\sin 240^\circ\right] = -1 - i\sqrt{3}$, and
$2\left[\cos 300^\circ + i\sin 300^\circ\right] = 1 - i\sqrt{3}$.

48. Since $x(x^8 - 1) = 0$, the solutions are $x = 0$ and the 8th roots of 1. The 8th roots are given by
$$\cos\left(\frac{k360^\circ}{8}\right) + i\sin\left(\frac{k360^\circ}{8}\right)$$
$$= \cos\left(k45^\circ\right) + i\sin\left(k45^\circ\right).$$
If $k = 0, 1, ..., 8$ one obtains
$\cos 0^\circ + i\sin 0^\circ = 1$,
$\cos 45^\circ + i\sin 45^\circ = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$,
$\cos 90^\circ + i\sin 90^\circ = i$,
$\cos 135^\circ + i\sin 135^\circ = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$,
$\cos 180^\circ + i\sin 180^\circ = -1$,
$\cos 225^\circ + i\sin 225^\circ = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$,
$\cos 270^\circ + i\sin 270^\circ = -i$, and
$\cos 315^\circ + i\sin 315^\circ = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$.

49. Factoring, we obtain

$$x^3(x^2+5)+8(x^2+5)=(x^2+5)(x^3+8)=0.$$

Then $x^2 = -5$ and $x^3 = -8$. Note, the square roots of -5 are $x = \pm i\sqrt{5}$. The cube roots of -8 are -2, $2(\cos 300°+i\sin 300°)$, and $2(\cos 60° + i\sin 60°)$. Thus, the solutions are $\pm i\sqrt{5}$, -2, and $1 \pm i\sqrt{3}$.

50. Factoring, we obtain

$$x^3(x^2+1)-27(x^2+1)=(x^2+1)(x^3-27)=0.$$

Then $x^2 = -1$ and $x^3 = 27$. Note, the square roots of -1 are $x = \pm i$. The cube roots of 27 are 3, $3(\cos 120°+i\sin 120°)$, and $3(\cos 240° + i\sin 240°)$. Thus, the solutions are $\pm i$, 3, and $\dfrac{-3\pm 3i\sqrt{3}}{2}$.

51. Solutions are the fifth roots of 2. Namely,
$\sqrt[5]{2}\left[\cos\left(\dfrac{k360°}{5}\right)+i\sin\left(\dfrac{k360°}{5}\right)\right]$
$= \sqrt[5]{2}\left[\cos(k72°)+i\sin(k72°)\right]$.
Solutions are $x = \sqrt[5]{2}\,[\cos\alpha + i\sin\alpha]$
where $\alpha = 0°, 72°, 144°, 216°, 288°$.

52. Solutions are the fifth roots of -3. Namely,
$\sqrt[5]{3}\left[\cos\left(\dfrac{180°+k360°}{5}\right)+i\sin\left(\dfrac{180°+k360°}{5}\right)\right]$
$= \sqrt[5]{3}\left[\cos(36°+k72°)+i\sin(36°+k72°)\right]$.
Solutions are $x = \sqrt[5]{3}\,[\cos\alpha + i\sin\alpha]$
where $\alpha = 36°, 108°, 180°, 252°, 324°$.

53. Solutions are the fourth roots of $-3+i$.
Since $|-3+i| = \sqrt{10}$, an argument of
$-3+i$ is $\cos^{-1}(-3/\sqrt{10}) \approx 161.6°$.
Arguments of the fourth roots are given by

$$\frac{161.6° + k360°}{4} = 40.4° + k90°.$$

By choosing $k = 0, 1, 2, 3$, the
solutions are $x = \sqrt[8]{10}\,[\cos\alpha + i\sin\alpha]$
where $\alpha = 40.4°, 130.4°, 220.4°, 310.4°$.

54. Note $x^3 = \dfrac{-2+i}{i}\cdot\dfrac{-i}{-i} = 2i+1$.
Solutions are the cube roots of $1+2i$.
Since $|1+2i| = \sqrt{5}$, an argument of
$1+2i$ is $\cos^{-1}(1/\sqrt{5}) \approx 63.4°$.
Arguments of the cube roots are

$$\frac{63.4° + k360°}{3} \approx 21.1° + k120°.$$

By choosing $k = 0, 1, 2$,
the solutions $x = \sqrt[6]{5}\,[\cos\alpha + i\sin\alpha]$
where $\alpha = 21.1°, 141.1°, 261.1°$.

55. $[\cos\pi/3 + i\sin\pi/6]^3 =$
$[1/2 + i(1/2)]^3 = \left[\dfrac{1}{2}(1+i)\right]^3 =$
$\dfrac{1}{8}\left[\sqrt{2}\,(\cos 45° + i\sin 45°)\right]^3 =$
$\dfrac{1}{8}\left[2\sqrt{2}\,(\cos 135° + i\sin 135°)\right] =$
$\dfrac{\sqrt{2}}{4}\left(-\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i\right) = -\dfrac{1}{4}+\dfrac{1}{4}i$

56. Factor as a difference of two squares.
Then factor as a sum and/or difference
of two cubes.

$$\begin{aligned}(x^3-1)(x^3+1) &= 0\\ (x-1)(x^2+x+1)(x+1)(x^2-x+1) &= 0\end{aligned}$$

Two solutions are $x = \pm 1$. Using the quadratic formula on the 2nd degree factors, we obtain

$$x = \frac{-1\pm\sqrt{1^2-4(1)(1)}}{2} \quad \text{or}$$

$$x = \frac{1\pm\sqrt{(-1)^2-4(1)(1)}}{2}.$$

Simplifying, we get

$$x = \frac{-1\pm\sqrt{-3}}{2} \quad \text{or} \quad x = \frac{1\pm\sqrt{-3}}{2}$$

$$x = \frac{-1\pm i\sqrt{3}}{2} \quad \text{or} \quad x = \frac{1\pm i\sqrt{3}}{2}.$$

Solutions are $x = \pm 1, -\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i, \dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$.

57. By the quadratic formula, we get

$$\begin{aligned} x &= \frac{-(-1+i) \pm \sqrt{(-1+i)^2 - 4(1)(-i)}}{2} \\ x &= \frac{1-i \pm \sqrt{(1-2i-1)+4i}}{2} \\ x &= \frac{1-i \pm \sqrt{2i}}{2}. \end{aligned}$$

Note the square roots of $2i$ are given by $\sqrt{2}(\cos 45^\circ + i \sin 45^\circ) = 1 + i$ and $\sqrt{2}(\cos 225^\circ + i \sin 225^\circ) = -1 - i$.
These two roots differ by a minus sign. So

$$x = \frac{1-i \pm (1+i)}{2}$$

$$x = \frac{1-i+1+i}{2} \quad \text{or} \quad x = \frac{1-i-1-i}{2}$$

$$x = 1 \quad \text{or} \quad x = -i.$$

Solutions are $x = 1, -i$.

58. By the quadratic formula, the solutions are

$$= \frac{-(-1-3i) \pm \sqrt{(-1-3i)^2 - 4(1)(-2+2i)}}{2}$$

$$\begin{aligned} &= \frac{1+3i \pm \sqrt{(1+6i-9)+8-8i}}{2} \\ &= \frac{1+3i \pm \sqrt{-2i}}{2}. \end{aligned}$$

Note, the square roots of $-2i$ are given by

$$\sqrt{2}(\cos 135^\circ + i \sin 135^\circ) = -1 + i$$

and

$$\sqrt{2}(\cos 315^\circ + i \sin 315^\circ) = 1 - i.$$

These two roots differ by a minus sign. Thus,

$$x = \frac{1+3i-1+i}{2} \quad \text{or} \quad x = \frac{1+3i+1-i}{2}$$

$$x = \frac{4i}{2} \quad \text{or} \quad x = \frac{2+2i}{2}.$$

The solutions are $x = 2i, 1+i$.

61. $\sqrt{3^2+5^2} = \sqrt{34}$

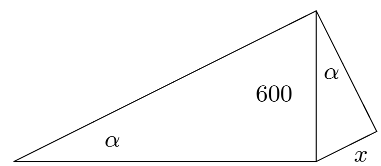
62. The magnitude is $\sqrt{4^2+(-4)^2} = 4\sqrt{2}$. Since $\arctan(-4/4) = -45^\circ$, an argument is 315°. The trigonometric form is

$$4\sqrt{2}\left(\cos 315^\circ + i \sin 315^\circ\right).$$

63. Since $\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2} = 90^\circ$, the product is

$$6\left(\cos 90^\circ + i \sin 90^\circ\right) = 6i.$$

64. In the diagram, $\alpha = 20^\circ$ and x is the required force.



Using right triangle trigonometry, we find

$$x = 600 \sin 20^\circ \approx 205.2 \text{ lb}.$$

65. The magnitude is

$$\sqrt{(-3)^2+(-9)^2} = \sqrt{90} = 3\sqrt{10}.$$

Note, $\alpha = \arctan\left(\frac{-9}{-3}\right) \approx 71.6^\circ$. Since the vector lies in the 3rd quadrant, the direction angle is

$$\alpha + 180^\circ \approx 251.6^\circ.$$

66. The largest angle is γ and is opposite the longest side. By the cosine law,

$$\begin{aligned} 12^2 &= 9^2 + 5^2 - 2(9)(5)\cos\gamma \\ 144 &= 106 - 90\cos\gamma \\ \gamma &= \cos^{-1}\left(-\frac{38}{90}\right) \\ \gamma &\approx 115.0^\circ \end{aligned}$$

The remaining angles are less than 90°.

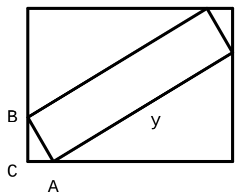
By the sine law,

$$\begin{aligned} \sin\beta &= \frac{9\sin\gamma}{12} \\ \beta &= \sin^{-1}\left(\frac{9\sin\gamma}{12}\right) \\ \beta &\approx 42.8^\circ \end{aligned}$$

Finally, $\alpha = 180^\circ - \gamma - \beta = 22.2^\circ$.

Thinking Outside the Box LXV

In the figure below, we have $AB = 1$, $AC = x$, and $BC = \sqrt{1 - x^2}$. Let y be the length of the statue.



By the Pythagorean Theorem, we get

$$(4 - x)^2 + (3 - \sqrt{1 - x^2})^2 = y^2.$$

Using the area of the rectangle, we find

$$12 = x\sqrt{1 - x^2} + y + (4 - x)(3 - \sqrt{1 - x^2}).$$

The solutions of the system of two equations are

$$x \approx 0.5262 \text{ ft}, y \approx 4.0851 \text{ ft}.$$

The length of the statue is $y \approx 4.0851$ ft.

7.5 Pop Quiz

1. Since $1 + i = \sqrt{2}\,(\cos 45^\circ + i \sin 45^\circ)$, we obtain
$\left[\sqrt{2}\,(\cos 45^\circ + i \sin 45^\circ)\right]^8 =$
$2^4\,(\cos 360^\circ + i \sin 360^\circ) = 16$

2. The fourth roots of -16 are
$2\left[\cos\left(\dfrac{180^\circ + k360^\circ}{4}\right) + i \sin\left(\dfrac{180^\circ + k360^\circ}{4}\right)\right]$
$= 2\left[\cos(45^\circ + k90^\circ) + i \sin(45^\circ + k90^\circ)\right].$
If $k = 0, 1, 2, 3$ one obtains
$2[\cos 45^\circ + i \sin 45^\circ] = \sqrt{2} + \sqrt{2}i = \sqrt{2}(1 + i)$,
$2[\cos 135^\circ + i \sin 135^\circ] = -\sqrt{2} + \sqrt{2}i = \sqrt{2}(-1 + i)$,
$2[\cos 225^\circ + i \sin 225^\circ] = -\sqrt{2} - \sqrt{2}i = \sqrt{2}(-1 - i)$, and
$2[\cos 315^\circ + i \sin 315^\circ] = \sqrt{2} - \sqrt{2}i = \sqrt{2}(1 - i)$.

For Thought

1. True **2.** False, the distance is $|r|$.

3. False

4. False, $x = r\cos\theta$, $y = r\sin\theta$, and $x^2 + y^2 = r^2$.

5. True, since $x = -4\cos 225^\circ = 2\sqrt{2}$ and $y = -4\sin 225^\circ = 2\sqrt{2}$.

6. True, $\theta = \pi/4$ is a straight line through the origin which makes an angle of $\pi/4$ with the positive x-axis.

7. True, since each circle is centered at the origin with radius 5.

8. False, since upon substitution one gets $\cos 2\pi/3 = -1/2$ while $r^2 = 1/2$.

9. False, $r = 1/\sin\theta$ is the vertical line $y = 1$.

10. False, since $(r, \theta) = (1, 1)$ lies on the graph $r = \theta$, but not on the graph of $r = -\theta$.

7.6 Exercises

1. polar

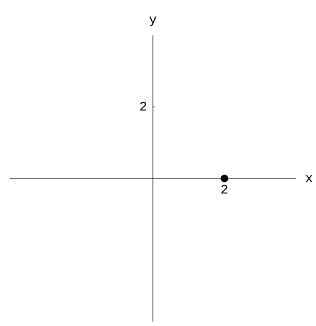
2. pole

3. $(3, \pi/2)$ **4.** $(2, 0)$

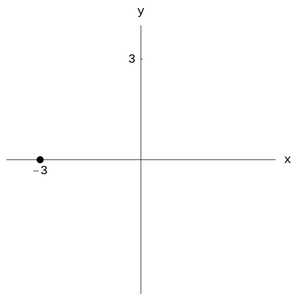
5. Since $r = \sqrt{3^2 + 3^2} = 3\sqrt{2}$ and $\theta = \tan^{-1}(3/3) = \pi/4$, $(r, \theta) = (3\sqrt{2}, \pi/4)$.

6. Since $r = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$ and
$$\theta = \cos^{-1}\left(-\frac{2}{2\sqrt{2}}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4},$$
we have $(r, \theta) = (2\sqrt{2}, 3\pi/4)$.

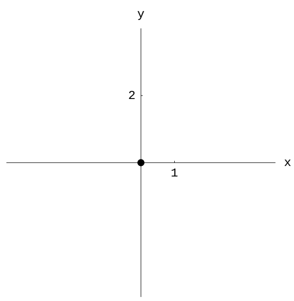
7. $(2, 0^\circ)$



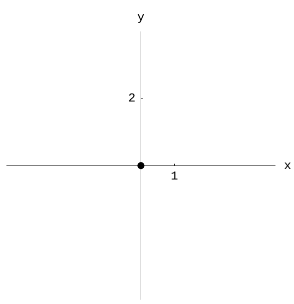
8. $(-3, 0°)$



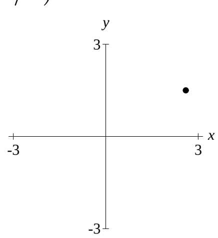
9. $(0, 35°)$



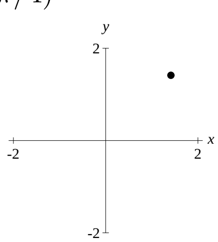
10. $(0, 90°)$



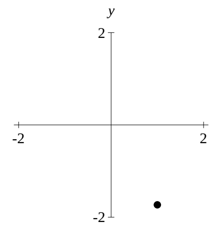
11. $(3, \pi/6)$



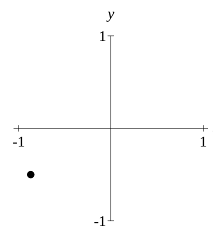
12. $(2, \pi/4)$



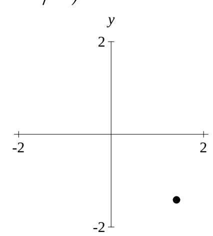
13. $(-2, 2\pi/3)$



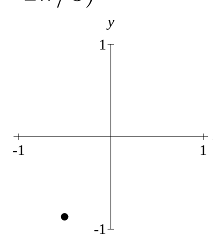
14. $(-1, \pi/6)$



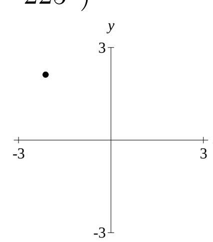
15. $(2, -\pi/4)$



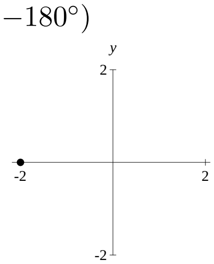
16. $(1, -2\pi/3)$



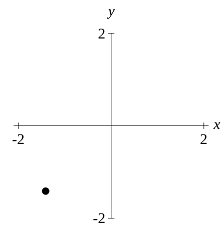
17. $(3, -225°)$



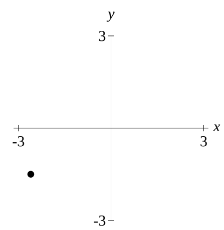
18. $(2, -180°)$



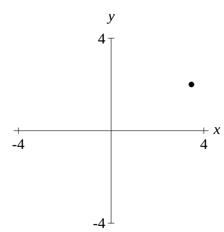
19. $(-2, 45°)$



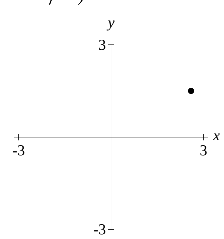
20. $(-3, 30°)$



21. $(4, 390°)$



22. $(3, 13\pi/6)$



23. $(x, y) = (4 \cdot \cos(0°), 4 \cdot \sin(0°)) = (4, 0)$

24. $(x, y) = (-5 \cdot \cos(0°), -5 \cdot \sin(0°)) = (-5, 0)$

25. $(x, y) = (0 \cdot \cos(\pi/4), 0 \cdot \sin(\pi/4)) = (0, 0)$

26. $(x, y) = (0 \cdot \cos(-\pi), 0 \cdot \sin(-\pi)) = (0, 0)$

27. $(x, y) = (1 \cdot \cos(\pi/6), 1 \cdot \sin(\pi/6)) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

28. $(x, y) = (2\cos(\pi/4), 2\sin(\pi/4)) = \left(\sqrt{2}, \sqrt{2}\right)$

29. $(x, y) = (-3\cos(3\pi/2), -3\sin(3\pi/2)) = (0, 3)$

30. $(x, y) = (-2\cos(2\pi), -2\sin(2\pi)) = (-2, 0)$

31. $(x, y) = \left(\sqrt{2}\cos 135°, \sqrt{2}\sin 135°\right) = (-1, 1)$

32. $(x, y) = \left(\sqrt{3}\cos 150°, \sqrt{3}\sin 150°\right) = \left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$

33. $(x, y) = \left(-\sqrt{6}\cos(-60°), -\sqrt{6}\sin(-60°)\right) = \left(-\frac{\sqrt{6}}{2}, \frac{3\sqrt{2}}{2}\right)$

34. $(x, y) = \left(-\frac{\sqrt{2}}{2}\cos(-45°), -\frac{\sqrt{2}}{2}\sin(-45°)\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)$

35. Since $r = \sqrt{(\sqrt{3})^2 + 3^2} = 2\sqrt{3}$ and $\cos\theta = \frac{x}{r} = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2}$, one can choose $\theta = 60°$. So $(r, \theta) = (2\sqrt{3}, 60°)$.

36. Since $r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ and $\tan\theta = \frac{x}{y} = \frac{4}{4} = 1$, one can choose $\theta = 45°$. So $(r, \theta) = (4\sqrt{2}, 45°)$.

37. Since $r = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$ and $\cos\theta = \frac{x}{r} = \frac{-2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}$, one can choose $\theta = 135°$. So $(r, \theta) = (2\sqrt{2}, 135°)$.

38. Since $r = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$ and $\cos\theta = \frac{x}{r} = \frac{-2}{4} = -\frac{1}{2}$, one can choose $\theta = 120°$. So $(r, \theta) = (4, 120°)$.

39. $(r, \theta) = (2, 90°)$ **40.** $(r, \theta) = (2, 180°)$

41. Note $r = \sqrt{(-3)^2 + (-3)^2} = 3\sqrt{2}$. Since $\tan\theta = \frac{y}{x} = \frac{-3}{-3} = 1$ and $(-3, -3)$ is in quadrant III, one can choose $\theta = 225°$. Thus, $(r, \theta) = (3\sqrt{2}, 225°)$.

42. Note $r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$.
Since $\tan\theta = \dfrac{y}{x} = \dfrac{-2}{2} = -1$ and $(2,-2)$ is in quadrant IV, one can choose $\theta = -45°$.
Then $(r,\theta) = (2\sqrt{2}, -45°)$.

43. Since $r = \sqrt{1^2 + 4^2} = \sqrt{17}$ and
$\theta = \cos^{-1}\left(\dfrac{x}{r}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{17}}\right) \approx 75.96°$
one gets $(r,\theta) = (\sqrt{17}, 75.96°)$.

44. Since $r = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$ and
$\theta = \cos^{-1}\left(\dfrac{x}{r}\right) = \cos^{-1}\left(\dfrac{-2}{\sqrt{13}}\right) \approx 123.7°$,
one gets $(r,\theta) = (\sqrt{13}, 123.7°)$.

45. Since $r = \sqrt{(\sqrt{2})^2 + (-2)^2} = \sqrt{6}$ and
$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{-2}{\sqrt{2}}\right) \approx -54.7°$,
one obtains $(r,\theta) = (\sqrt{6}, -54.7°)$.

46. Note $r = \sqrt{(-2)^2 + (-\sqrt{3})^2} = \sqrt{7}$.
Since $\tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{-\sqrt{3}}{2}\right) \approx 40.9°$
and $(-2,-\sqrt{3})$ is in quadrant III, one can choose $\theta = 180° + 40.9° = 220.9°$.
Thus, $(r,\theta) = (\sqrt{7}, 220.9°)$.

47. $(x,y) = (4\cos 26°, 4\sin 26°) \approx (3.60, 1.75)$

48. $(x,y) = (-5\cos 33°, -5\sin 33°) \approx (-4.19, -2.72)$

49. $(x,y) = (2\cos(\pi/7), 2\sin(\pi/7)) \approx (1.80, 0.87)$

50. $(x,y) = (3\cos(2\pi/9), 3\sin(2\pi/9)) \approx (2.30, 1.93)$

51.] $(x,y) = (-2\cos(1.1), -2\sin(1.1)) \approx (-0.91, -1.78)$

52. $(x,y) = (6\cos(2.3), 6\sin(2.3)) \approx (-4.00, 4.47)$

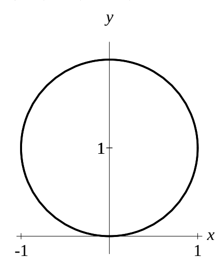
53. Since $r = \sqrt{4^2 + 5^2} \approx 6.4$ and
$\tan\theta = \dfrac{y}{x} = \dfrac{5}{4}$, we get $\theta = \tan^{-1}\left(\dfrac{5}{4}\right) \approx 51.34°$. Then $(r,\theta) \approx (6.4, 51.34°)$.

54. Note, $r = \sqrt{(-5)^2 + 3^2} \approx 5.8$ and
$\tan^{-1}\left(\dfrac{3}{-5}\right) \approx -31.0°$. Since $(-5,3)$ is a point in the 2nd quadrant, choose
$\theta = -31.0° + 180° = 149.0°$.
Then $(r,\theta) \approx (5.8, 149.0°)$.

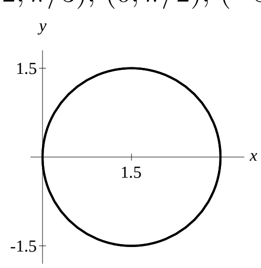
55. Note, $r = \sqrt{(-2)^2 + (-7)^2} \approx 7.3$ and
$\tan^{-1}\left(\dfrac{-7}{-2}\right) \approx 74.1°$. Since $(-2,-7)$ is a point in the 3rd quadrant, we choose
$\theta = 74.1° + 180° = 254.1°$.
Then $(r,\theta) \approx (7.3, 254.1°)$.

56. Note, $r = \sqrt{3^2 + *(-8)^2} \approx 8.5$ and
$\tan^{-1}\left(\dfrac{-8}{3}\right) \approx -69.4°$. Since $(3,-8)$ is a point in the 4th quadrant, we let
$\theta = -69.4° + 360° = 290.6°$.
Then $(r,\theta) \approx (8.5, 290.6°)$.

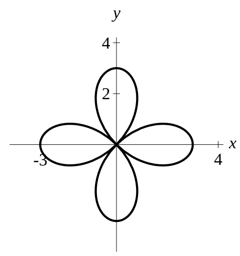
57. $r = 2\sin\theta$ is a circle centered at $(x,y) = (0,1)$. It goes through the following points in polar coordinates: $(0,0)$, $(1,\pi/6)$, $(2,\pi/2)$, $(1,5\pi/6)$, $(0,\pi)$.



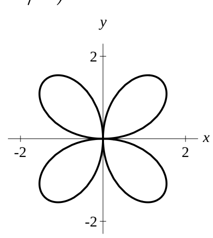
58. $r = 3\cos\theta$ is a circle centered at $(x,y) = (3/2,0)$. It goes through the following points in polar coordinates: $(3,0)$, $(3/2,\pi/3)$, $(0,\pi/2)$, $(-3/2,2\pi/3)$, $(-3,\pi)$.



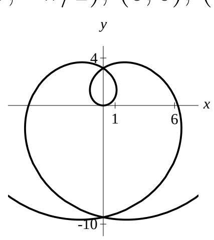
59. $r = 3\cos 2\theta$ is a four-leaf rose that goes through the following points in polar coordinates $(3, 0)$, $(3/2, \pi/6)$, $(0, \pi/4)$, $(-3, \pi/2)$, $(0, 3\pi/4)$, $(3, \pi)$, $(-3, 3\pi/2)$, $(0, 7\pi/4)$.



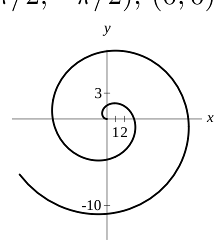
60. $r = -2\sin 2\theta$ is a four-leaf rose that goes through the following points in polar coordinates $(0, 0)$, $(-\sqrt{3}, \pi/6)$, $(-2, \pi/4)$, $(0, \pi/2)$, $(2, 3\pi/4)$,$(0, \pi)$, $(-2, 5\pi/4)$, $(2, 7\pi/4)$.



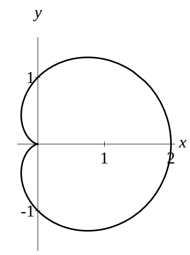
61. $r = 2\theta$ is spiral-shaped and goes through the following points in polar coordinates $(-\pi, -\pi/2)$, $(0, 0)$, $(\pi, \pi/2)$, $(2\pi, \pi)$



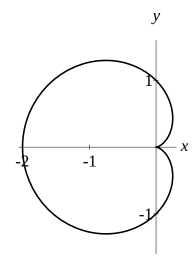
62. $r = \theta$ for $\theta \leq 0$ goes through the following points in polar coordinates $(-\pi, -\pi)$, $(-\pi/2, -\pi/2)$, $(0, 0)$



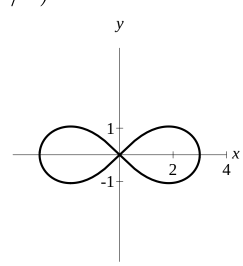
63. $r = 1 + \cos\theta$ goes through the following points in polar coordinates $(2, 0)$, $(1.5, \pi/3)$, $(1, \pi/2)$, $(0.5, 2\pi/3)$, $(0, \pi)$.



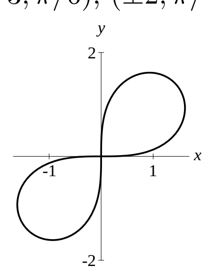
64. $r = 1 - \cos\theta$ goes through the following points in polar coordinates $(0, 0)$, $(0.5, \pi/3)$, $(1, \pi/2)$, $(1.5, 2\pi/3)$, $(2, \pi)$.



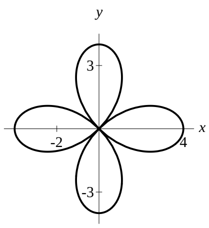
65. $r^2 = 9\cos 2\theta$ goes through the following points in polar coordinates $(0, -\pi/4)$, $(\pm\sqrt{3}/2, -\pi/6)$, $(\pm 3, 0)$, $(\pm\sqrt{3}/2, \pi/6)$, $(0, \pi/4)$.



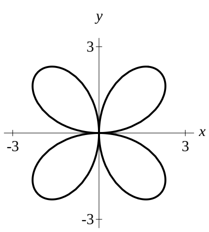
66. $r^2 = 4\sin 2\theta$ goes through the following points in polar coordinates $(0, 0)$, $(\pm 2\sqrt{3}, \pi/6)$, $(\pm 2, \pi/4)$, $(0, \pi/2)$.



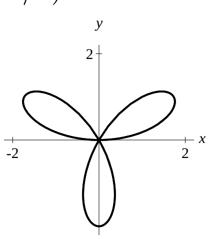
67. $r = 4\cos 2\theta$ is a four-leaf rose that goes through the following points in polar coordinates $(4, 0)$, $(2, \pi/6)$, $(0, \pi/4)$, $(-4, \pi/2)$, $(2, 5\pi/6)$, $(4, \pi)$, $(0, 5\pi/4)$, $(-4, 3\pi/2)$, $(0, 7\pi/4)$.



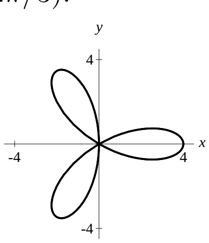
68. $r = 3\sin 2\theta$ is a four-leaf rose that goes through the following points in polar coordinates $(0, 0)$, $(3\sqrt{3}/2, \pi/6)$, $(3, \pi/4)$, $(0, \pi/2)$, $(-3, 3\pi/4)$, $(0, \pi)$, $(3, 5\pi/4)$, $(0, 3\pi/2)$, $(-3, 7\pi/4)$, $(0, 2\pi)$.



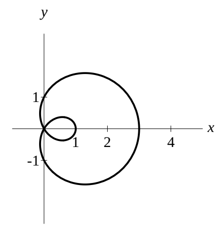
69. $r = 2\sin 3\theta$ is a three-leaf rose that goes through the following points in polar coordinates $(0, 0)$, $(2, \pi/6)$, $(-2, \pi/2)$, $(2, 5\pi/6)$.



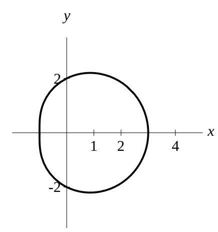
70. $r = 4\cos 3\theta$ is a three-leaf rose that goes through the following points in polar coordinates $(4, 0)$, $(0, \pi/6)$, $(-4, \pi/3)$, $(4, 2\pi/3)$.



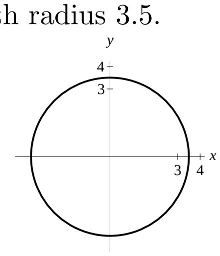
71. $r = 1 + 2\cos\theta$ goes through the following points in polar coordinates $(3, 0)$, $(2, \pi/3)$, $(1, \pi/2)$, $(0, 2\pi/3)$, $(1 - \sqrt{3}, 5\pi/6)$, $(-1, \pi)$.



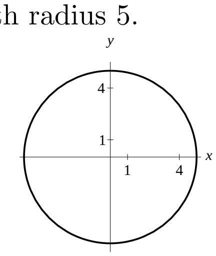
72. $r = 2 + \cos\theta$ goes through the following points in polar coordinates $(3, 0)$, $(2.5, \pi/3)$, $(2, \pi/2)$, $(1.5, 2\pi/3)$, $(1, \pi)$.



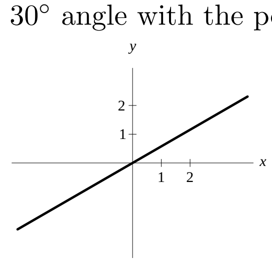
73. $r = 3.5$ is a circle centered at the origin with radius 3.5.



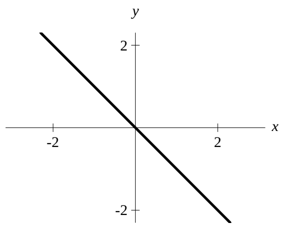
74. $r = -5$ is a circle centered at the origin with radius 5.



75. $\theta = 30°$ is a line through the origin that makes a $30°$ angle with the positive x-axis.



76. $\theta = 3\pi/4$ is a line through the origin that makes a $3\pi/4$ angle with the positive x-axis.



77. Multiply equation by r.

$$\begin{aligned} r^2 &= 4r\cos\theta \\ x^2+y^2 &= 4x \\ x^2-4x+y^2 &= 0 \end{aligned}$$

78. Multiply equation by r.

$$\begin{aligned} r^2 &= 2r\sin\theta \\ x^2+y^2 &= 2y \\ x^2+y^2-2y &= 0 \end{aligned}$$

79. Multiply equation by $\sin\theta$.

$$\begin{aligned} r\sin\theta &= 3 \\ y &= 3 \end{aligned}$$

80. Multiply equation by $\cos\theta$.

$$\begin{aligned} r\cos\theta &= -2 \\ x &= -2 \end{aligned}$$

81. Multiply equation by $\cos\theta$.

$$\begin{aligned} r\cos\theta &= 3 \\ x &= 3 \end{aligned}$$

82. Multiply equation by $\sin\theta$.

$$\begin{aligned} r\sin\theta &= 2 \\ y &= 2 \end{aligned}$$

83. Since $r = 5$, $\sqrt{x^2+y^2} = 5$ and by squaring one gets $x^2+y^2 = 25$.

84. Since $r = -3$, $-\sqrt{x^2+y^2} = -3$ and by squaring one gets $x^2+y^2 = 9$.

85. Note $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y/r}{x/r} = \dfrac{y}{x}$.
Since $\tan\pi/4 = 1$, $\dfrac{y}{x} = 1$ or $y = x$.

86. Note $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y/r}{x/r} = \dfrac{y}{x}$.
Since $\tan 0 = 0$, $\dfrac{y}{x} = 0$ or $y = 0$.

87. Multiply equation by $1-\sin\theta$.

$$\begin{aligned} r(1-\sin\theta) &= 2 \\ r - r\sin\theta &= 2 \\ r - y &= 2 \\ \pm\sqrt{x^2+y^2} &= y+2 \\ x^2+y^2 &= y^2+4y+4 \\ x^2-4y &= 4 \end{aligned}$$

88. Multiply equation by $1+\cos\theta$.

$$\begin{aligned} r(1+\cos\theta) &= 3 \\ r + r\cos\theta &= 3 \\ r + x &= 3 \\ \pm\sqrt{x^2+y^2} &= 3-x \\ x^2+y^2 &= 9-6x+x^2 \\ y^2+6x &= 9 \end{aligned}$$

89. $r\cos\theta = 4$ **90.** $r\sin\theta = -6$

91. Note $\tan\theta = y/x$.

$$\begin{aligned} y &= -x \\ \frac{y}{x} &= -1 \\ \tan\theta &= -1 \\ \theta &= -\pi/4 \end{aligned}$$

92. Note $\tan\theta = y/x$.

$$\begin{aligned} y &= x\sqrt{3} \\ \frac{y}{x} &= \sqrt{3} \\ \tan\theta &= \sqrt{3} \\ \theta &= \pi/3 \end{aligned}$$

93. Note $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} (r\cos\theta)^2 &= 4r\sin\theta \\ r^2\cos^2\theta &= 4r\sin\theta \\ r &= \frac{4\sin\theta}{\cos^2\theta} \\ r &= 4\tan\theta\sec\theta \end{aligned}$$

94. Note $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} (r\sin\theta)^2 &= 2r\cos\theta \\ r^2\sin^2\theta &= 2r\cos\theta \\ r &= \frac{2\cos\theta}{\sin^2\theta} \\ r &= 2\cot\theta\csc\theta \end{aligned}$$

95. $r = 2$

96. Note $x = r\cos\theta$ and $x^2 + y^2 = r^2$.

$$\begin{aligned} x^2 + x^2 + y^2 &= 1 \\ (r\cos\theta)^2 + r^2 &= 1 \\ r^2(\cos^2\theta + 1) &= 1 \\ r^2 &= \frac{1}{\cos^2\theta + 1}. \end{aligned}$$

97. Note $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} y &= 2x - 1 \\ r\sin\theta &= 2r\cos\theta - 1 \\ r(\sin\theta - 2\cos\theta) &= -1 \\ r(2\cos\theta - \sin\theta) &= 1 \\ r &= \frac{1}{2\cos\theta - \sin\theta} \end{aligned}$$

98. Note $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} y &= -3x + 5 \\ r\sin\theta &= -3r\cos\theta + 5 \\ r(\sin\theta + 3\cos\theta) &= 5 \\ r &= \frac{5}{\sin\theta + 3\cos\theta} \end{aligned}$$

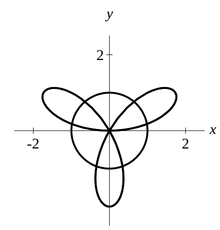
99. Note that $y = r\sin\theta$ and $x^2 + y^2 = r^2$.

$$\begin{aligned} x^2 + (y^2 - 2y + 1) &= 1 \\ x^2 + y^2 - 2y &= 0 \\ r^2 - 2r\sin\theta &= 0 \\ r^2 &= 2r\sin\theta \\ r &= 2\sin\theta \end{aligned}$$

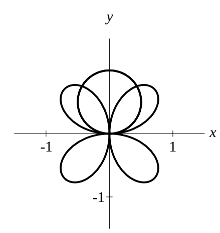
100. Note that $x = r\cos\theta$ and $x^2 + y^2 = r^2$.

$$\begin{aligned} (x^2 + 2x + 1) + y^2 &= 4 \\ r^2 + 2x &= 3 \\ r^2 + 2r\cos\theta &= 3 \end{aligned}$$

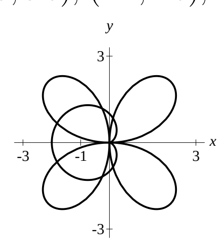
101. There are six points of intersection and in polar coordinates these are approximately $(1, 0.17)$, $(1, 2.27)$, $(1, 4.36)$, $(1, 0.87)$, $(1, 2.97)$, $(1, 5.06)$



102. There are three points of intersection and in polar coordinates these are approximately $(0, 0)$, $(0.9, 1.0)$, $(0.9, 2.09)$.



103. There are seven points of intersection and in polar coordinates these are approximately $(0, 0)$, $(0.9, 1.4)$, $(1.2, 1.8)$, $(1.9, 2.8)$, $(1.9, 3.5)$, $(1.2, 4.5)$, $(0.8, 4.9)$



104. By solving $3\sin(4\theta) = \pm 2$ or $\sin(4\theta) = \pm\dfrac{2}{3}$, one finds

$$4\theta = \pm\sin^{-1}\left(\frac{2}{3}\right) + 2\pi k \quad \text{or}$$

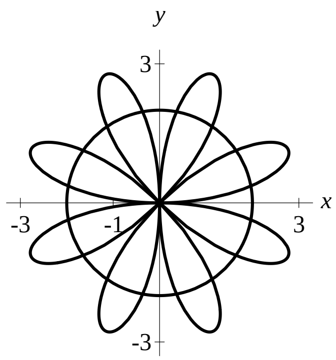
$$4\theta = \pi \pm \sin^{-1}\left(\frac{2}{3}\right) + 2\pi k$$

where k is an integer. Then

$$\theta = \frac{\pm \sin^{-1}\left(\frac{2}{3}\right)}{4} + \frac{\pi k}{2} \quad \text{or}$$

$$\theta = \frac{\pi \pm \sin^{-1}\left(\frac{2}{3}\right)}{4} + \frac{\pi k}{2}.$$

There are 16 points of intersection, namely, $(2, \theta)$ where $\theta \approx 0.2$, $\theta \approx 0.6$,$\theta \approx 1.0$, $\theta \approx 1.4$, $\theta \approx 1.8$, $\theta \approx 2.2$, $\theta \approx 2.5$, $\theta \approx 3.0$, $\theta \approx 3.3$, $\theta \approx 3.7$, $\theta \approx 4.1$, $\theta \approx 4.5$, $\theta \approx 4.9$, $\theta \approx 5.3$, $\theta \approx 5.7$, $\theta \approx 6.1$.



107. Since $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, we find

$$\begin{aligned}(1+i)^{12} &= (\sqrt{2})^{12}\left(\cos\frac{12\pi}{4} + i\sin\frac{12\pi}{4}\right)\\ &= 64\left(\cos 3\pi + i\sin 3\pi\right)\\ &= -64\end{aligned}$$

108. Note, $-8 - 8i\sqrt{3} = 16\left(\cos 240^\circ + i\sin 240^\circ\right)$. Then the fourth roots are

$$2\left(\cos(60^\circ + k90^\circ) + i\sin(60^\circ + k90^\circ)\right)$$

where $k = 0, 1, 2, 3$.

If $k = 0$, the fourth root is

$$2\left(\cos 60^\circ + i\sin 60^\circ\right) = 1 + i\sqrt{3}$$

If $k = 1$, the fourth root is

$$2\left(\cos 150^\circ + i\sin 150^\circ\right) = -\sqrt{3} + i$$

If $k = 2$, the fourth root is

$$2\left(\cos 240^\circ + i\sin 240^\circ\right) = -1 - i\sqrt{3}$$

If $k = 3$, the fourth root is

$$2\left(\cos 330^\circ + i\sin 330^\circ\right) = \sqrt{3} - i$$

109. Note, $\arcsin(0.88) \approx 1.08$ and $\pi - \arcsin(0.88) \approx 2.07$. Then

$$x \approx 0.88 + 2k\pi \quad \text{or} \quad x \approx 2.07 + 2k\pi$$

where k is an integer.

110. Separate the fraction as follows:

$$\frac{1 - 2\sin^2 x}{\sin^2 x} - \csc^2 x =$$

$$\frac{1}{\sin^2 x} - \frac{2\sin^2 x}{\sin^2 x} - \csc^2 x =$$

$$\csc^2 x - 2 - \csc^2 x = -2$$

111. Use a cofunction identity and the fact that cosine is an even function.

$$\begin{aligned}\sin\left(2x - \frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{2} - \left(2x - \frac{\pi}{4}\right)\right)\\ &= \cos\left(-2x + \frac{3\pi}{4}\right)\\ &= \cos\left(2x - \frac{3\pi}{4}\right)\\ \sin\left(2x - \frac{\pi}{4}\right) &= \cos\left(2\left(x - \frac{3\pi}{8}\right)\right)\end{aligned}$$

112. Since the y-values of the key points are $-1, -2, -3, -2, -1$ in that order, choose $A = 1$ and $D = -2$.

Since the first and last x-values are $\pi/3$ and $5\pi/3$, choose $C = \pi/3$.

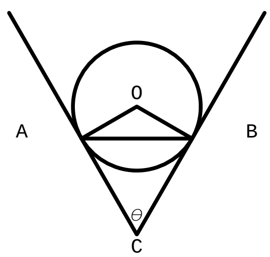
Since the difference of the first and last x-values is $5\pi/3 - \pi/3 = 4\pi/3$ which is the period, choose $B = 3/2$ for $\frac{2\pi}{3/2} = \frac{4\pi}{3}$.

Then

$$\begin{aligned}y &= A\cos\left(B(x - C)\right) + D\\ y &= \cos\left(1.5(x - \pi/3)\right) - 2\end{aligned}$$

Thinking Outside the Box LXVI

In the figure below,



the angle at the center O is

$$\langle AOB = \pi - \theta.$$

Then the area of triangle ΔAOB is

$$A_t = \frac{\sin\theta}{2}$$

and the area of sector $\widetilde{AB}$ is

$$A_s = \frac{\pi - \theta}{2}.$$

a) The area inside the circle and below line AB is

$$\begin{aligned} A_s - A_t &= \frac{\pi}{2} - \frac{\theta}{2} - \frac{\sin\theta}{2} \\ &= \frac{\pi}{2} - \frac{\theta}{2} - \sin(\theta/2)\cos(\theta/2) \end{aligned}$$

b) Applying the Pythagorean Theorem to ΔAOC, we find

$$AC = 1/\tan(\theta/2).$$

Then the area of ΔAOC is

$$\frac{1}{2\tan(\theta/2)}$$

and the area of the quadrilateral $ABCO$ is

$$A_q = \frac{1}{\tan(\theta/2)}.$$

Hence, the area below the circle and inside the trench is

$$A_q - A_s = \frac{1}{\tan(\theta/2)} - \frac{\pi}{2} + \frac{\theta}{2}$$

c) Using a calculator, we find that as θ approaches π, the ratio

$$\frac{A_s - A_t}{A_q - A_s}$$

approaches 2.

7.6 Pop Quiz

1. Since $-15\pi/4$ lies in Quadrant 4 and $r = -1$ is negative, the point lies in Quadrant 2.

2. $(x, y) = (4\cos 150°, 4\sin 150°) = \left(-2\sqrt{3}, 2\right)$

3. Since the angle is $3\pi/4$ and $r = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$, the polar coordinates are $(2\sqrt{2}, 3\pi/4)$.

4. Multiply both sides of $r = 4\cos\theta$ by r.

$$\begin{aligned} r^2 &= 4r\cos\theta \\ x^2 + y^2 &= 4x \\ x^2 - 4x + y^2 &= 0 \end{aligned}$$

5. Multiply both sides of $r = -16\sin\theta$ by r.

$$\begin{aligned} r^2 &= -16r\sin\theta \\ x^2 + y^2 &= -16y \\ x^2 + y^2 + 16y &= 0 \\ x^2 + (y+8)^2 &= 64 \end{aligned}$$

The center is $(0, -8)$ and the radius is 8.

6. Substitute $y = r\sin\theta$ and $x = r\cos\theta$, and solve for r

$$\begin{aligned} y &= x + 1 \\ r\sin\theta &= r\cos\theta + 1 \\ r(\sin\theta - \cos\theta) &= 1 \\ r &= \frac{1}{\sin\theta - \cos\theta} \end{aligned}$$

For Thought

1. False, t is the parameter.

2. True, graphs of parametric equations are sketched in a rectangular coordinate system in this book.

3. True, since $2x = t$ and $y = 2t + 1 = 2(2x) + 1 = 4x + 1$.

4. False, it is a circle of radius 1.

5. True, since if $t = \dfrac{1}{3}$ then $x = 3\left(\dfrac{1}{3}\right) + 1 = 2$ and $y = 6\left(\dfrac{1}{3}\right) - 1 = 1$.

6. False, for if $w^2 - 3 = 1$ then $w = \pm 2$ and this does not satisfy $-2 < w < 2$.

7. True, since x and y take only positive values.

8. True

9. False, since e^t is non-negative while $\ln(t)$ can be negative.

10. True

7.7 Exercises

1. parametric

2. parameter

3. If $t = 0$, then $x = 4(0) + 1 = 1$ and $y = 0-2 = -2$. If $t = 1$, then $x = 4(1)+1 = 5$ and $y = 1 - 2 = -1$.

If $x = 7$, then $7 = 4t + 1$. Solving for t, we get $t = 1.5$. Substitute $t = 1.5$ into $y = t - 2$. Then $y = 1.5 - 2 = -0.5$.

If $y = 1$, then $1 = t - 2$. Solving for t, we get $t = 3$. Consequently $y = 4(3) + 1 = 13$.

We tabulate the results as follows.

t	x	y
0	1	-2
1	5	-1
1.5	7	-0.5
3	13	1

4. If $t = 2$, then $x = 3-2 = 1$ and $y = 2(2)+5 = 9$. Similarly, if $t = 3$, then $x = 0$ and $y = 11$.

If $x = -2$, then $-2 = 3 - t$. Solving for t, we get $t = 5$. Substitute $t = 5$ into $y = 2t + 5$. Then $y = 15$.

Similarly, if $y = 19$, then $19 = 2t + 5$ and so $t = 7$. Consequently $x = 3 - 7 = -4$.

We tabulate the results as follows.

t	x	y
2	1	9
3	0	11
5	-2	15
7	-4	19

5. If $t = 1$, then $x = 1^2 = 1$ and $y = 3(1) - 1 = 2$. If $t = 2.5$, then $x = (2.5)^2 = 6.25$ and $y = 3(2.5) - 1 = 6.5$.

If $x = 5$, then $5 = t^2$ and $t = \sqrt{5}$. Consequently, $y = 3\sqrt{5} - 1$.

If $y = 11$, then $11 = 3t - 1$. Solving for t, we get $t = 4$. Consequently $x = 4^2 = 16$.

If $x = 25$, then $25 = t^2$ and $t = 5$. Consequently, $y = 3(5) - 1 = 14$.

We tabulate the results as follows.

t	x	y
1	1	2
2.5	6.25	6.5
$\sqrt{5}$	5	$3\sqrt{5} - 1$
4	16	11
5	25	14

6. If $t = 0$, then $x = \sqrt{0} = 0$ and $y = 0 + 4 = 4$. If $t = 2$, then $x = \sqrt{2}$ and $y = 2 + 4 = 6$. If $t = 4$, then $x = \sqrt{4} = 2$ and $y = 4 + 4 = 8$.

If $y = 12$, then $12 = t + 4$ and $t = 8$. Consequently, $x = \sqrt{8} = 2\sqrt{2}$.

If $x = 3$, then $3 = \sqrt{t}$ and $t = 9$. Consequently $y = 9 + 4 = 13$.

We tabulate the results as follows.

t	x	y
0	0	4
2	$\sqrt{2}$	6
4	2	8
8	$2\sqrt{2}$	12
9	3	13

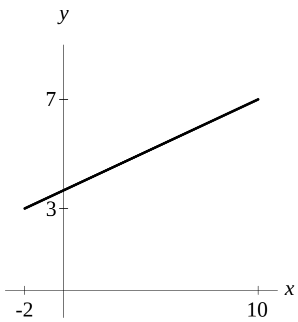
7.

Some points are given by

t	x	y
0	-2	3
4	10	7

.

The domain is $[-2, 10]$ and the range is $[3, 7]$.



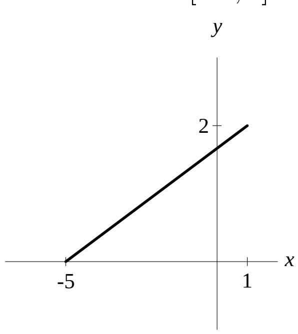
8.

Some points are

t	x	y
1	1	2
3	-5	0

.

The domain is $[-5, 1]$ and the range is $[0, 2]$.



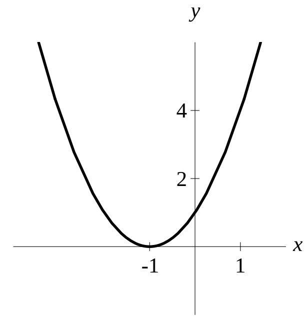
9.

Some points are given by

t	x	y
0	-1	0
2	1	4

.

The domain is $(-\infty, \infty)$ and the range is $[0, \infty)$.

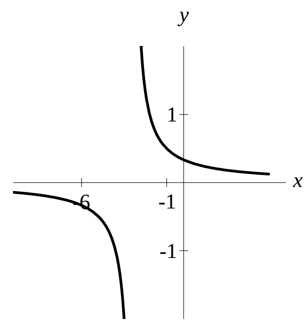


10. Some points are

t	x	y
-3	-6	$-1/3$
2	-1	$1/2$

.

The domain is $(-\infty, -3) \cup (-3, \infty)$ and the range is $(-\infty, 0) \cup (0, \infty)$.

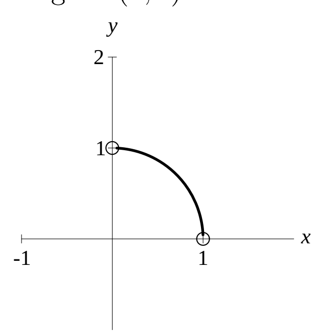


11. A few points are approximated by

w	x	y
0.2	0.4	0.9
0.8	0.9	0.4

.

The domain is $(0, 1)$ and the range is $(0, 1)$.

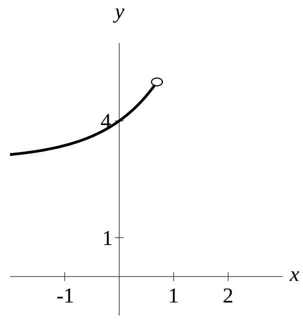


12. Since $x = \ln(t)$ is undefined for $-2 < t \leq 0$, the parametric equations are valid for $0 < t < 2$. Some points are approximated by

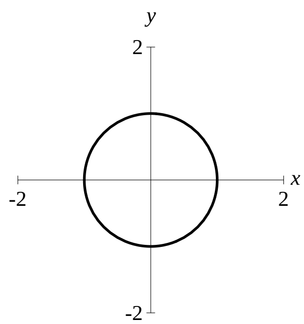
t	x	y
1	0	4
2	0.7	5

.

The domain is $(-\infty, \ln 2)$ and the range is $(3, 5)$.



13. A circle of radius 1 and centered at the origin. The domain is $[-1, 1]$ and the the range is $[-1, 1]$.

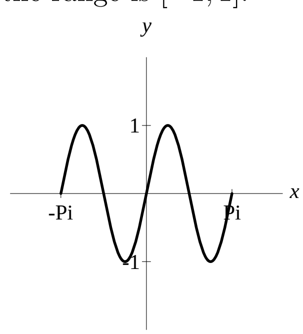


14. Some points are given by

t	x	y
π	$\pi/2$	0
$-\pi/2$	$-\pi/4$	-1

.

The domain is $(-\infty, \infty)$ and the the range is $[-1, 1]$.



15. Since $t = \dfrac{x+5}{4}$, we obtain

$y = 3 - 4\left(\dfrac{x+5}{4}\right) = -x - 2$ or $x + y = -2$.

The graph is a straight line with domain $(-\infty, \infty)$ and range $(-\infty, \infty)$.

16. Since $t = \dfrac{x+1}{5}$, we obtain

$y = 4\left(\dfrac{x+1}{5}\right) + 6 = \dfrac{4x}{5} + \dfrac{34}{5}$ or $5y - 4x = 34$.

The graph is a straight line with domain $(-\infty, \infty)$ and range $(-\infty, \infty)$.

17. Since $x^2 + y^2 = 16\sin^2(3t) + 16\cos^2(3t) = 16$, the graph is a circle with radius 4 and with center at the origin.

Domain $[-4, 4]$ and range $[-4, 4]$

18. Since $x = \sin 2t$, $y = 3x$ and the graph is a straight line segment with domain $[-1, 1]$ and range $[-3, 3]$.

19. Since $t = 4x$, we find $y = e^{4x}$ and the graph is an exponential graph.

Domain $(-\infty, \infty)$ and range $(0, \infty)$

20. $y = (t-5)^2 = x^2$ and its graph is a parabola

Domain $(-\infty, \infty)$ and range $[0, \infty)$

21. $y = 2x + 3$ represents the graph of a straight line

Domain $(-\infty, \infty)$ and range $(-\infty, \infty)$

22. $y = -x^2 + 3$ and its graph is a parabola

Domain $(-\infty, \infty)$ and range $(-\infty, 3]$

23. An equation (in terms of x and y) of the line through $(2, 3)$ and $(5, 9)$ is $y = 2x - 1$.

An equation (in terms of t and x) of the line through $(0, 2)$ and $(2, 5)$ is $x = \dfrac{3}{2}t + 2$.

Parametric equations are $x = \dfrac{3}{2}t + 2$ and

$y = 2\left(\dfrac{3}{2}t + 2\right) - 1 = 3t + 3$ where $0 \le t \le 2$.

24. An equation (in terms of x and y) of the line through $(-2, 4)$ and $(5, -9)$ is $y = \dfrac{-13}{7}x + \dfrac{2}{7}$.

An equation (in terms of t and x) of the line through $(3, -2)$ and $(7, 5)$ is

$$x = \frac{7t}{4} - \frac{29}{4}.$$

Parametric equations are $x = \frac{7t}{4} - \frac{29}{4}$ and
$y = \frac{-13}{7}\left(\frac{7t}{4} - \frac{29}{4}\right) + \frac{2}{7} = -\frac{13}{4}t + \frac{55}{4}$
where $3 \le t \le 7$.

25. $x = 2\cos t$, $y = 2\sin t$, $\pi < t < \frac{3\pi}{2}$

26. $x = 3\cos t$, $y = 3\sin t$, $\pi < t < 2\pi$

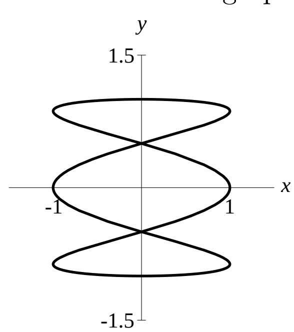
27. $x = 3$, $y = t$, $-\infty < t < \infty$

28. $x = t$, $y = 2$, $-\infty < t < \infty$

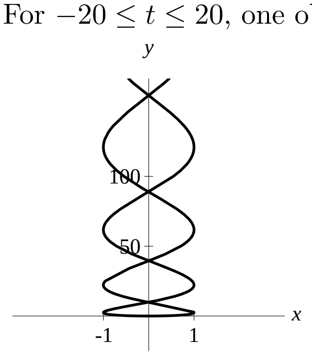
29. Since $x = r\cos t$, $y = r\sin t$, and $r = 2\sin t$,
we get $x = 2\sin t\cos t = \sin 2t$ and
$y = 2\sin t\sin t = 2\sin^2 t$ where $0 \le t \le 2\pi$.
Then $x = \sin 2t$ and $y = 2\sin^2 t$.

30. $x = 5\sin 2t\cos t$, $y = 5\sin 2t\sin t$,
$0 \le t \le 2\pi$

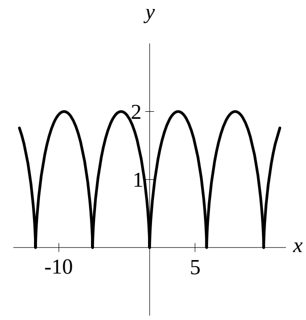
31. For $-\pi \le t \le \pi$, one obtains the given graph (for a larger range of values for t, more points are filled and the graph would be different)



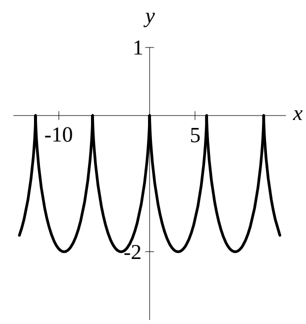
32. For $-20 \le t \le 20$, one obtains



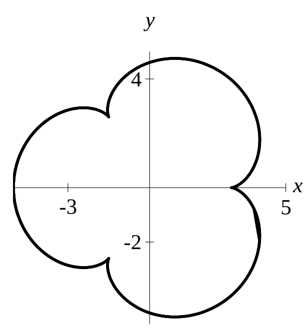
33. For $-15 \le t \le 15$, one finds



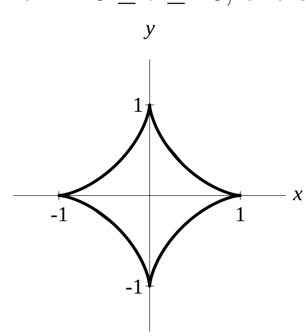
34. For $-15 \le t \le 15$, one finds



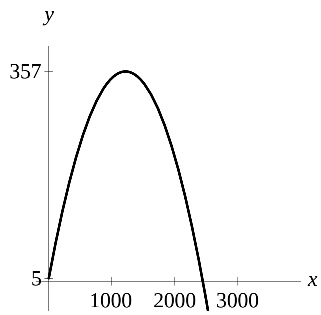
35. For $-10 \le t \le 10$, one obtains



36. For $-10 \le t \le 10$, one obtains



37. A graph of the parametric equations $x = 150\sqrt{3}t$ and $y = -16t^2 + 150t + 5$ (for $0 \le t \le 10$) is given



38. If $y = 5$, then $-16t^2 + 150t = t(150 - 16t) = 0$. (Note, $t = \dfrac{150}{16}$). The distance of the target from the archer is given by

$$x = 150\sqrt{3}\left(\frac{150}{16}\right) \approx 2435.7 \text{ feet.}$$

39. Solving $y = -16t^2 + 150t + 5 = 0$, one finds

$$\frac{-150 \pm \sqrt{150^2 - 4(-16)(5)}}{-32} \approx 9.41, -0.03$$

The arrow is in the air for 9.4 seconds.

40. In finding the vertex of $y = -16t^2 + 150t + 5$, one obtains $t = -\dfrac{b}{2a} = -\dfrac{150}{-32} = \dfrac{75}{16}$ and the maximum height is

$$-16\left(\frac{75}{16}\right)^2 + 150\left(\frac{75}{16}\right) + 5 \approx 356.6 \text{ feet.}$$

41. Multiply both sides by r:

$$\begin{aligned} r^2 &= 8r\cos\theta \\ x^2 + y^2 &= 8x \\ x^2 - 8x + y^2 &= 0 \end{aligned}$$

42. Note, $-1 = \cos 180° + i \sin 180°$. Then the fourth roots of -1 are

$$\cos(45° + k90°) + i\sin(45° + k90°)$$

where $k = 0, 1, 2, 3$.

If $k = 0$, the fourth root is

$$\cos 45° + i \sin 45° = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

If $k = 1$, the fourth root is

$$\cos 135° + i \sin 135° = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

If $k = 2$, the fourth root is

$$\cos 225° + i \sin 225° = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

If $k = 3$, the fourth root is

$$\cos 315° + i \sin 315° = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

43. The magnitude is

$$\sqrt{3^2 + (-3\sqrt{3})^2} = \sqrt{9 + 27} = \sqrt{36} = 6.$$

Note, $\arctan \frac{-3\sqrt{3}}{3} = \arctan(-\sqrt{3}) = -\pi/3$. Since $(3, -3\sqrt{3})$ lies in quadrant 4, we may take the argument to be $5\pi/3$. Then

$$3 - 3\sqrt{3} = 6\left(\cos(5\pi/3) + i\sin(5\pi/3)\right).$$

44. Since $\cos 4b = 3w$, we obtain $4b = \arccos(3w)$. Then $b = \frac{1}{4}\arccos(3w)$.

45. Solve by factoring as follows:

$$\begin{aligned} (2\cos x - 1)(\cos x + 1) &= 0 \\ \cos x &= -1, \frac{1}{2} \end{aligned}$$

Then $x = \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$.

46. Note, α lies in quadrant 3. Then

$$\cos\alpha = -\sqrt{1 - \sin^2\alpha} = -\sqrt{1 - 49/64} = -\frac{\sqrt{15}}{8}.$$

Thus,

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-7/8}{-\sqrt{15}/8} = \frac{7}{\sqrt{15}} = \frac{7\sqrt{15}}{15}$$

Thinking Outside the Box LXVII

The sides of the three squares are $\sqrt{8}$, $\sqrt{13}$, and $\sqrt{17}$. These are also the sides of the triangle. We use Heron's formula to find the area of the triangle. Let

$$s = \frac{\sqrt{8}+\sqrt{13}+\sqrt{17}}{2}.$$

The area of the triangle is

$$\begin{aligned} \text{Area} &= \sqrt{s(s-\sqrt{8})(s-\sqrt{13})(s-\sqrt{17})} \\ &= 5 \text{ acres} \\ &= 5(43,560) \text{ ft}^2 \\ \text{Area} &= 217,800 \text{ ft}^2. \end{aligned}$$

7.7 Pop Quiz

1. Substitute $t = 3, 5$ in $x = 2t + 5$ and $y = 3t - 7$.

t	x	y
3	11	2
5	15	8

The endpoints are $(11, 2)$ and $(15, 8)$.

2. Since $x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9$, we get

$$x^2 + y^2 = 9.$$

The graph is a circle with radius 3.

3. The line passing through $(0, 1)$ and $(3, 5)$ is

$$y = \frac{4}{3}x + 1.$$

The linear function $y = f(t)$ that satisfies

i) $y = 1$ if $t = 0$, and

ii) $y = 5$ if $t = 4$

is

$$y = t + 1.$$

Then we obtain

$$\frac{4}{3}x + 1 = y = t + 1 \quad \text{or} \quad \frac{4}{3}x = t.$$

Thus, the parametric equations are

$$x = \frac{3}{4}t, \quad y = t + 1.$$

7.7 Linking Concepts

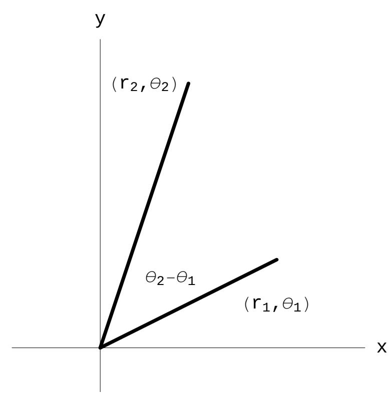
a) The rectangular coordinates of (r_1, θ_1) and (r_2, θ_2) are $(x_1, y_1) = (r_1\cos\theta_1, r_1\sin\theta_1)$ and $(x_2, y_2) = (r_2\cos\theta_2, r_2\sin\theta_2)$, respectively. Note, for all r and θ we have

$$r^2\cos^2\theta + r^2\sin^2\theta = r^2.$$

Consequently, we obtain the distance d as shown below.

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(r_2\cos\theta_2 - r_1 cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2} = \sqrt{r_1^2 + r_2^2 - 2r_1r_2[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2]} = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2-\theta_1)}$$

b) Suppose the polar coordinates satisfy $r_1, r_2 > 0$ and $0 \le \theta_1 < \theta_2$, as shown below.



By using the law of cosines, we have

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2-\theta_1)}.$$

The distance formula also holds true and can be proved similarly for the remaining cases such as when θ_1 or θ_2 are greater than 2π or negative, and when r_1 or r_2 are negative.

c) In three-dimensional space, the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in rectangular coordinates is

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}.$$

Note, the conversion of the polar coordinates (r, θ, z) into rectangular coordinates is

$$(r\cos\theta, r\sin\theta, z).$$

By using the results in Part (a), we obtain the distance d between (r_1, θ_1, z_1) and (r_2, θ_2, z_2).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1) + (z_2 - z_1)^2}$$

d) Note, a point on the earth with longtitude α and latitude β has rectangular coordinates

$$(x, y, z) = (r \cos\alpha \cos\beta, r \sin\alpha \cos\beta, r \sin\beta)$$

assuming r is the radius of the earth. Recall, the angle θ formed by two vectors v and w satisfies

$$\cos\theta = \frac{\langle v, w\rangle}{|v| \cdot |w|}.$$

Let v be the vector whose initial point is the center of the earth and whose terminal point is (α_1, β_1). Similarly, let w be the vector that emanates from the center of the earth and whose terminal point is (α_2, β_2). Then $|v| = |w| = r$ and

$$\begin{aligned}
\cos\theta &= \Big(r^2 \cos\beta_1 \cos\beta_2 \cos\alpha_1 \cos\alpha_2 \\
&\quad + r^2 \cos\beta_1 \cos\beta_2 \sin\alpha_1 \sin\alpha_2 \\
&\quad + r^2 \sin\beta_1 \sin\beta_2\Big) \div r^2 \\
&= \cos\beta_1 \cos\beta_2 \cos\alpha_1 \cos\alpha_2 \\
&\quad + \cos\beta_1 \cos\beta_2 \sin\alpha_1 \sin\alpha_2 \\
&\quad + \sin\beta_1 \sin\beta_2 \\
\cos\theta &= \cos\beta_1 \cos\beta_2 \cos(\alpha_1 - \alpha_2) + \sin\beta_1 \sin\beta_2.
\end{aligned}$$

Note, the length of the arc, S, subtended by a central angle θ (radians) in a circle with radius r is given by

$$S = r\theta.$$

Therefore, the distance S between two points (α_1, β_1) and (α_2, β_2) on the surface of the earth is given by

$$S = r \cos^{-1}\left(\sin\beta_1 \sin\beta_2 + \cos\beta_1 \cos\beta_2 \cos(\alpha_1 - \alpha_2)\right).$$

e) The latitude and longtitude of Chicago, Illinois are approximately 89° west and 43° north, respectively. Using the (α, β)-coordinates described in Part (d), for Chicago we have $(\alpha_1, \beta_1) = (89°, 43°)$.

The latitude and longtitude of Paris, France are approximately 2° east and 49° north, respectively. The (α, β)-coordinates for Paris are $(\alpha_2, \beta_2) = (358°, 49°)$.

Suppose the radius of the earth is $r = 3963$ miles. Then by using the formulas for θ (radians) and S in Part (d), we obtain the distance S between Chicago and Paris.

$$\begin{aligned}
\theta &= \cos^{-1}\left(\sin\beta_1 \sin\beta_2 + \right. \\
&\quad \left. \cos\beta_1 \cos\beta_2 \cos(\alpha_1 - \alpha_2)\right) \\
&= \cos^{-1}\left(\sin 43° \sin 49° + \right. \\
&\quad \left. \cos 43° \cos 49° \cos(89° - 358°)\right) \\
\theta &\approx 1.03986
\end{aligned}$$

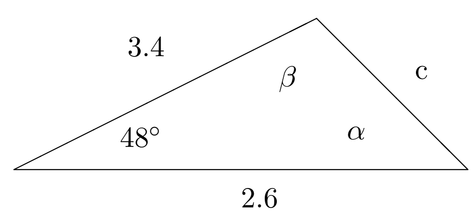
Hence, the distance between Chicago and Paris is

$$S \approx 3963(1.03986) \approx 4121 \text{ miles}.$$

The discrepancy of the distance obtained is due to the estimates for the latitude, longtitude, and the points in Chicago and Paris where the distance of 4140 miles was determined.

Review Exercises

1. Draw a triangle with $\gamma = 48°$, $a = 3.4$, $b = 2.6$.

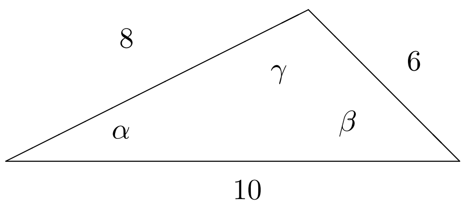


By the cosine law, we obtain $c = \sqrt{2.6^2 + 3.4^2 - 2(2.6)(3.4)\cos 48°} \approx 2.5475 \approx 2.5$. By the sine law, we find

$$\begin{aligned}
\frac{2.5475}{\sin 48°} &= \frac{2.6}{\sin\beta} \\
\sin\beta &= \frac{2.6 \sin 48°}{2.5} \\
\sin\beta &\approx 0.75846 \\
\beta &\approx \sin^{-1}(0.75846) \\
\beta &\approx 49.3°.
\end{aligned}$$

Also, $\alpha = 180° - (49.3° + 48°) = 82.7°$.

2. Draw a triangle with sides $a = 6$, $b = 8$, $c = 10$.

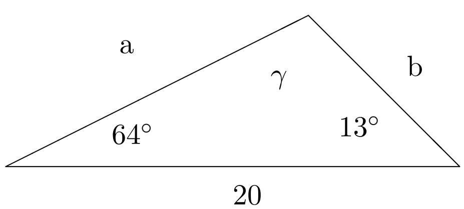


By the cosine law,

$$\cos\gamma = \frac{6^2 + 8^2 - 10^2}{2(6)(8)} = 0.$$

So $\gamma = 90°$. This is a right triangle.
Since $\alpha = \sin^{-1}(6/10) \approx 36.9°$,
we have $\beta = 90° - 36.9° = 53.1°$.

3. Draw a triangle with $\alpha = 13°$, $\beta = 64°$, $c = 20$.



Note $\gamma = 180° - (64° + 13°) = 103°$.

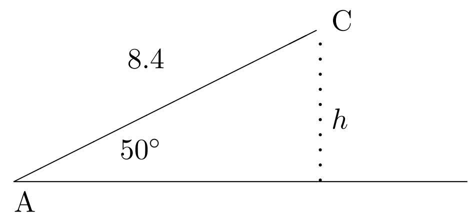
By the sine law, we get $\dfrac{20}{\sin 103°} = \dfrac{a}{\sin 13°}$

and $\dfrac{20}{\sin 103°} = \dfrac{b}{\sin 64°}$.

So $a = \dfrac{20}{\sin 103°}\sin 13° \approx 4.6$

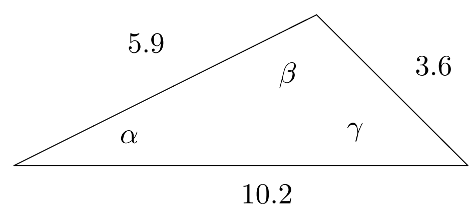
and $b = \dfrac{20}{\sin 103°}\sin 64° \approx 18.4$.

4. Draw angle $\alpha = 50°$.



Since $h = 8.4\sin 50° \approx 6.4$ and $a = 3.2 < 6.4$, no triangle exists.

5. Draw a triangle with $a = 3.6$, $b = 10.2$, $c = 5.9$.

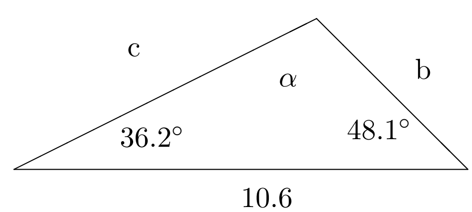


By the cosine law one gets

$$\cos\beta = \frac{5.9^2 + 3.6^2 - 10.2^2}{2(5.9)(3.6)} \approx -1.3.$$

This is a contradiction since the range of cosine is $[-1, 1]$. No triangle exists.

6. Draw a triangle with $\beta = 36.2°$, $\gamma = 48.1°$, and $a = 10.6$.



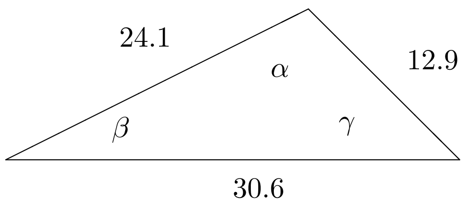
Note $\alpha = 180° - (36.2° + 48.1°) = 95.7°$.
By the sine law,

$$\frac{b}{\sin 36.2°} = \frac{10.6}{\sin 95.7°} \text{ and } \frac{c}{\sin 48.1°} = \frac{10.6}{\sin 95.7°}$$

So $b = \dfrac{10.6}{\sin 95.7°}\sin 36.2° \approx 6.3$ and

$c = \dfrac{10.6}{\sin 95.7^\circ} \sin 48.1^\circ \approx 7.9.$

7. Draw a triangle with sides $a = 30.6$, $b = 12.9$, and $c = 24.1$.



By the cosine law, we get

$\cos\alpha = \dfrac{24.1^2 + 12.9^2 - 30.6^2}{2(24.1)(12.9)} \approx -0.3042.$

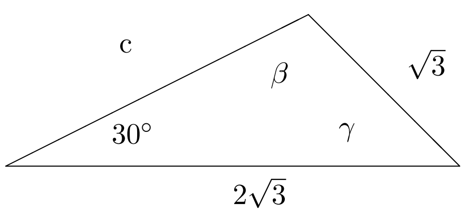
So $\alpha = \cos^{-1}(-0.3042) \approx 107.7^\circ$.

Similarly, we find

$\cos\beta = \dfrac{24.1^2 + 30.6^2 - 12.9^2}{2(24.1)(30.6)} \approx 0.9158.$

So $\beta = \cos^{-1}(0.9158) \approx 23.7^\circ$.
Also, $\gamma = 180^\circ - (107.7^\circ + 23.7^\circ) = 48.6^\circ$.

8. Draw a triangle with $\alpha = 30^\circ$, $a = \sqrt{3}$, $b = 2\sqrt{3}$.



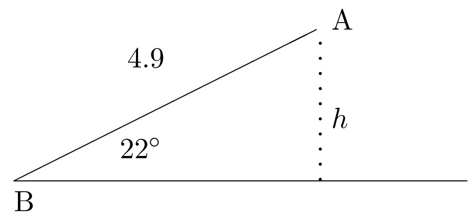
By the sine law,

$$\begin{aligned}
\frac{\sqrt{3}}{\sin 30^\circ} &= \frac{2\sqrt{3}}{\sin\beta} \\
\sin\beta &= 2\sin 30^\circ \\
\sin\beta &= 1 \\
\beta &= 90^\circ.
\end{aligned}$$

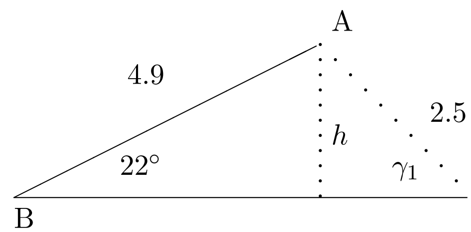
This is a right triangle and $\gamma = 60^\circ$. Then

$$\begin{aligned}
(\sqrt{3})^2 + c^2 &= (2\sqrt{3})^2 \\
3 + c^2 &= 12 \\
c^2 &= 9 \\
c &= 3.
\end{aligned}$$

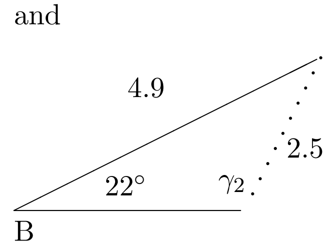
9. Draw angle $\beta = 22^\circ$ and let h be the height.



Since $h = 4.9 \sin 22^\circ \approx 1.8$ and $1.8 < b < 4.9$, we have two triangles and they are given by



and



Apply the sine law to case 1.

$$\begin{aligned}
\frac{4.9}{\sin\gamma_1} &= \frac{2.5}{\sin 22^\circ} \\
\sin\gamma_1 &= \frac{4.9\sin 22^\circ}{2.5} \\
\sin\gamma_1 &\approx 0.7342 \\
\gamma_1 = \sin^{-1}(0.7342) &\approx 47.2^\circ
\end{aligned}$$

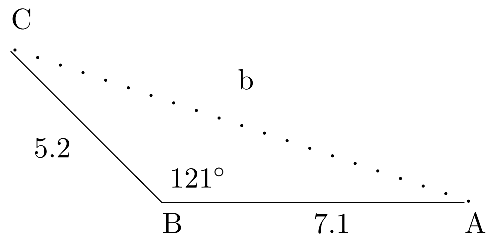
So $\alpha_1 = 180^\circ - (22^\circ + 47.2^\circ) = 110.8^\circ$.

By the sine law, $a_1 = \dfrac{2.5}{\sin 22^\circ} \sin 110.8^\circ \approx 6.2$.

In case 2, $\gamma_2 = 180^\circ - \gamma_1 = 132.8^\circ$
and $\alpha_2 = 180^\circ - (22^\circ + 132.8^\circ) = 25.2^\circ$.

By the sine law, $a_2 = \dfrac{2.5}{\sin 22^\circ} \sin 25.2^\circ \approx 2.8$.

10. Draw angle $\beta = 121°$.



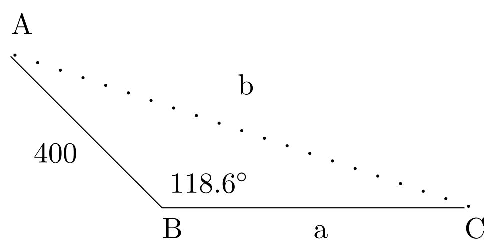
By the cosine law, we find
$b = \sqrt{5.2^2 + 7.1^2 - 2(5.2)(7.1)\cos 121°} \approx$
$10.746 \approx 10.7$. Apply the sine law.

$$\begin{aligned} \frac{10.746}{\sin 121°} &= \frac{7.1}{\sin\gamma} \\ \sin\gamma &= \frac{7.1\sin 121°}{10.746} \\ \sin\gamma &\approx 0.5663 \\ \gamma = \sin^{-1}(0.5663) &\approx 34.5° \end{aligned}$$

So $\alpha = 180° - (34.5° + 121°) = 24.5°$.

11. Area is $A = \dfrac{1}{2}(12.2)(24.6)\sin 38° \approx 92.4$ ft^2.

12. Draw angle $\beta = 118.6°$.



Note $\gamma = 180° - (118.6° + 12.4°) = 49°$.
By the sine law,

$$\begin{aligned} \frac{a}{\sin 12.4°} &= \frac{400}{\sin 49°} \\ a &= \frac{400}{\sin 49°}\sin 12.4° \\ a &\approx 113.811. \end{aligned}$$

The area is $A = \dfrac{1}{2}(113.811)(400)\sin 118.6°$
$\approx 19,984.8$ m^2.

13. Since $S = \dfrac{5.4 + 12.3 + 9.2}{2} = 13.45$, the area is
$\sqrt{13.45(13.45 - 5.4)(13.45 - 12.3)(13.45 - 9.2)}$
≈ 23.0 km^2.

14. Since $S = \dfrac{20 + 22 + 3}{2} = 22.5$, the area is
$\sqrt{22.5(22.5 - 20)(22.5 - 22)(22.5 - 3)}$
≈ 23.4 ft^2.

15. $|\,\boldsymbol{v_x}\,| = |6\cos 23.3°| \approx 5.5$,
$|\,\boldsymbol{v_y}\,| = |6\sin 23.3°| \approx 2.4$

16. $|\,\boldsymbol{v_x}\,| = |4.5\cos 156°| \approx 4.1$,
$|\,\boldsymbol{v_y}\,| = |4.5\sin 156°| \approx 1.8$

17. $|\,\boldsymbol{v_x}\,| = |3.2\cos 231.4°| \approx 2.0$,
$|\,\boldsymbol{v_y}\,| = |3.2\sin 231.4°| \approx 2.5$

18. $|\,\boldsymbol{v_x}\,| = |7.3\cos 344°| \approx 7.0$,
$|\,\boldsymbol{v_y}\,| = |7.3\sin 344°| \approx 2.0$

19. Magnitude $\sqrt{2^2 + 3^2} = \sqrt{13}$, direction angle $\tan^{-1}(3/2) \approx 56.3°$

20. Magnitude $\sqrt{(-4)^2 + 3^2} = 5$, direction angle $\cos^{-1}(-4/5) \approx 143.1°$

21. The magnitude is $\sqrt{(-3.2)^2 + (-5.1)^2}$ ≈ 6.0. Since $\tan^{-1}(5.1/3.2) \approx 57.9°$, the direction angle is $180° + 57.9° = 237.9°$.

22. The magnitude is $\sqrt{(2.1)^2 + (-3.8)^2} \approx 4.3$. Since $\tan^{-1}(-3.8/2.1) \approx -61.1°$, the direction angle is $360° - 61.1° = 298.9°$.

23. $\langle\sqrt{2}\cos 45°, \sqrt{2}\sin 45°\rangle = \langle 1, 1\rangle$

24. $\langle 6\cos 60°, 6\sin 60°\rangle = \langle 3, 3\sqrt{3}\rangle$

25. $\langle 9.1\cos 109.3°, 9.1\sin 109.3°\rangle \approx \langle -3.0, 8.6\rangle$

26. $\langle 5.5\cos 344.6°, 5.5\sin 344.6°\rangle \approx \langle 5.3, -1.5\rangle$

27. $\langle -6, 8\rangle$ **28.** $\langle -12, 3\rangle$

29. $\langle 2 - 2, -5 - 12\rangle = \langle 0, -17\rangle$

30. $\langle 3 - 4, 6 - 8\rangle = \langle -1, -2\rangle$

31. $\langle -1, 5\rangle \cdot \langle 4, 2\rangle = -4 + 10 = 6$

32. $\langle -4, 7\rangle \cdot \langle 7, 4\rangle = -28 + 28 = 0$

33. $-4\,\boldsymbol{i} + 8\,\boldsymbol{j}$ **34.** $3.2\,\boldsymbol{i} - 4.1\,\boldsymbol{j}$

35. $(7.2\cos 30°)\,\boldsymbol{i} + (7.2\sin 30°)\,\boldsymbol{j} \approx 3.6\sqrt{3}\,\boldsymbol{i} + 3.6\,\boldsymbol{j}$

36. The magnitude of $\langle 2,5\rangle$ is $\sqrt{2^2+5^2} = \sqrt{29}$.

So $\boldsymbol{v} = \dfrac{6}{\sqrt{29}}\langle 2,5\rangle = \dfrac{12\sqrt{29}}{29}\,\boldsymbol{i} + \dfrac{30\sqrt{29}}{29}\,\boldsymbol{j}$

37. $|3-5i| = \sqrt{3^2+(-5)^2} = \sqrt{34}$

38. $|3.6+4.8i| = \sqrt{(3.6)^2+(4.8)^2} = 6$

39. $|\sqrt{5}+i\sqrt{3}| = \sqrt{(\sqrt{5})^2+(\sqrt{3})^2} = \sqrt{8} = 2\sqrt{2}$

40. $|-2\sqrt{2}+3i\sqrt{5}| = \sqrt{(-2\sqrt{2})^2+(3\sqrt{5})^2} = \sqrt{8+45} = \sqrt{53}$

41. Note $\sqrt{(-4.2)^2+(4.2)^2} \approx 5.94$. If the terminal side of α goes through $(-4.2, 4.2)$, then $\tan\alpha = -1$. Choose $\alpha = 135°$. So $-4.2+4.2i = 5.94\,[\cos 135° + i\sin 135°]$.

42. Note $\sqrt{3^2+(-\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$. If the terminal side of α goes through $(3, -\sqrt{3})$, then $\tan\alpha = -\sqrt{3}/3$. Choose $\alpha = 330°$. So $3 - i\sqrt{3} = 2\sqrt{3}\,[\cos 330° + i\sin 330°]$.

43. Note $\sqrt{(-2.3)^2+(-7.2)^2} \approx 7.6$. If the terminal side of α goes through $(-2.3, -7.2)$ and since $\tan^{-1}(7.2/2.3) \approx 72.3°$, then one can choose $\alpha = 180° + 72.3° = 252.3°$. So $-2.3 - 7.2i \approx 7.6\,[\cos 252.3° + i\sin 252.3°]$.

44. Note $\sqrt{4^2+(9.2)^2} \approx 10.0$. If the terminal side of α goes through $(4, 9.2)$, then $\alpha = \tan^{-1}(9.2/4) \approx 66.5°$. So $4 + 9.2i \approx 10.0\,[\cos 66.5° + i\sin 66.5°]$.

45. $\sqrt{3}\left(-\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) = -\dfrac{3}{2} + \dfrac{\sqrt{3}}{2}i$

46. $\sqrt{2}\left(-\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}\right) = -1 - i$

47. $6.5[0.8377 + (0.5461)i] \approx 5.4 + 3.5i$

48. $14.9[0.3322 - (0.9432)i] \approx 4.9 - 14.1i$

49. Since $z_1 = 2.5\sqrt{2}\,[\cos 45° + i\sin 45°]$ and $z_2 = 3\sqrt{2}\,[\cos 225° + i\sin 225°]$, we have $z_1z_2 = 15\,[\cos 270° + i\sin 270°] = -15i$ and

$$\frac{z_1}{z_2} = \frac{(5/2)\sqrt{2}}{3\sqrt{2}}\,[\cos(-180°) + i\sin(-180°)] = -\frac{5}{6}.$$

50. Since $z_1 = 2\,[\cos 150° + i\sin 150°]$ and $z_2 = 4\,[\cos 240° + i\sin 240°]$, we have $z_1z_2 = 8\,[\cos 390° + i\sin 390°] =$

$$z_1z_2 = 8\left[\frac{\sqrt{3}}{2} + i\frac{1}{2}\right] = 4\sqrt{3} + 4i \text{ and}$$

$$\frac{z_1}{z_2} = \frac{1}{2}\,[\cos(-90°) + i\sin(-90°)] = -\frac{1}{2}i.$$

51. Let α and β be angles whose terminal sides go through $(2,1)$ and $(3,-2)$, respectively. Since $|2+i| = \sqrt{5}$ and $|3-2i| = \sqrt{13}$, we get $\cos\alpha = 2/\sqrt{5}$, $\sin\alpha = 1/\sqrt{5}$, $\cos\beta = 3/\sqrt{13}$, and $\sin\beta = -2/\sqrt{13}$. From the sum and difference identities, we obtain

$$\cos(\alpha+\beta) = \frac{8}{\sqrt{65}},\ \sin(\alpha+\beta) = -\frac{1}{\sqrt{65}},$$

$$\cos(\alpha-\beta) = \frac{4}{\sqrt{65}},\ \sin(\alpha-\beta) = \frac{7}{\sqrt{65}}.$$

Note $z_1 = \sqrt{5}(\cos\alpha + i\sin\alpha)$ and $z_2 = \sqrt{13}(\cos\beta + i\sin\beta)$. Then

$$\begin{aligned} z_1z_2 &= \sqrt{65}\,(\cos(\alpha+\beta) + i\sin(\alpha+\beta)) \\ &= \sqrt{65}\left(\frac{8}{\sqrt{65}} - \frac{1}{\sqrt{65}}i\right) \\ z_1z_2 &= 8 - i \end{aligned}$$

and

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{\sqrt{5}}{\sqrt{13}}\,(\cos(\alpha-\beta) + i\sin(\alpha-\beta)) \\ &= \frac{\sqrt{65}}{13}\left(\frac{4}{\sqrt{65}} + \frac{7}{\sqrt{65}}i\right) \\ \frac{z_1}{z_2} &= \frac{4}{13} + \frac{7}{13}i. \end{aligned}$$

52. Let α and β be angles whose terminal sides go through $(-3,1)$ and $(2,-1)$, respectively. Since $|-3+i| = \sqrt{10}$ and $|2-i| = \sqrt{5}$,

we have $\cos\alpha = -3/\sqrt{10}$, $\sin\alpha = 1/\sqrt{10}$, $\cos\beta = 2/\sqrt{5}$, and $\sin\beta = -1/\sqrt{5}$. From the sum and difference identities, we find
$\cos(\alpha+\beta) = -\dfrac{5}{\sqrt{50}}$, $\sin(\alpha+\beta) = \dfrac{5}{\sqrt{50}}$,
$\cos(\alpha-\beta) = -\dfrac{7}{\sqrt{50}}$, $\sin(\alpha-\beta) = -\dfrac{1}{\sqrt{50}}$.
Note $z_1 = \sqrt{10}(\cos\alpha + i\sin\alpha)$ and $z_2 = \sqrt{5}(\cos\beta + i\sin\beta)$. So

$$\begin{aligned} z_1z_2 &= \sqrt{50}\left(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right) \\ &= \sqrt{50}\left(-\frac{5}{\sqrt{50}} + \frac{5}{\sqrt{50}}i\right) \\ z_1z_2 &= -5+5i \end{aligned}$$

and

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{\sqrt{10}}{\sqrt{5}}\left(\cos(\alpha-\beta) + i\sin(\alpha-\beta)\right) \\ &= \sqrt{2}\left(-\frac{7}{\sqrt{50}} - \frac{1}{\sqrt{50}}i\right) \\ \frac{z_1}{z_2} &= -\frac{7}{5} - \frac{1}{5}i. \end{aligned}$$

53. $2^3\left[\cos 135^\circ + i\sin 135^\circ\right] =$
$8\left[-\dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}\right] = -4\sqrt{2} + 4\sqrt{2}i$

54. $(\sqrt{3})^4\left[\cos 840^\circ + i\sin 840^\circ\right] =$
$9\left[\cos 120^\circ + i\sin 120^\circ\right] =$
$9\left[-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right] = -4.5 + 4.5i\sqrt{3}$

55. $(4+4i)^3 = (4\sqrt{2})^3\left[\cos 45^\circ + i\sin 45^\circ\right]^3 =$
$128\sqrt{2}\left[\cos 135^\circ + i\sin 135^\circ\right] =$
$128\sqrt{2}\left[-\dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}\right] = -128 + 128i$

56. $(1-i\sqrt{3})^4 = 2^4\left[\cos 300^\circ + i\sin 300^\circ\right]^4 =$
$16\left[\cos 1200^\circ + i\sin 1200^\circ\right]^4 =$
$16\left[\cos 120^\circ + i\sin 120^\circ\right]^4 =$
$16\left[-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right] = -8 + 8i\sqrt{3}$

57. Square roots of i are given by
$\cos\left(\dfrac{90^\circ + k360^\circ}{2}\right) + i\sin\left(\dfrac{90^\circ + k360^\circ}{2}\right) =$
$\cos(45^\circ + k\cdot 180^\circ) + i\sin(45^\circ + k\cdot 180^\circ)$.
When $k = 0, 1$ one gets
$\cos 45^\circ + i\sin 45^\circ = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i$ and
$\cos 225^\circ + i\sin 225^\circ = -\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i$.

58. Cube roots of $-i$ are given by
$\cos\left(\dfrac{270^\circ + k360^\circ}{3}\right) + i\sin\left(\dfrac{270^\circ + k360^\circ}{3}\right) =$
$\cos(90^\circ + k\cdot 120^\circ) + i\sin(90^\circ + k\cdot 120^\circ)$.
When $k = 0, 1, 2$ one gets
$\cos 90^\circ + i\sin 90^\circ = i$,
$\cos 210^\circ + i\sin 210^\circ = -\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$, and
$\cos 330^\circ + i\sin 330^\circ = \dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$.

59. Since $|\sqrt{3} + i| = 2$, the cube roots are
$\sqrt[3]{2}\left[\cos\left(\dfrac{30^\circ + k360^\circ}{3}\right) + i\sin\left(\dfrac{30^\circ + k360^\circ}{3}\right)\right]$
When $k = 0, 1, 2$ one gets $\sqrt[3]{2}\left[\cos\alpha + i\sin\alpha\right]$ where $\alpha = 10^\circ, 130^\circ, 250^\circ$.

60. Since $|3+3i| = \sqrt{18}$, the square roots of $3+3i$ are
$\sqrt[4]{18}\left[\cos\left(\dfrac{45^\circ + k360^\circ}{2}\right) + i\sin\left(\dfrac{45^\circ + k360^\circ}{2}\right)\right]$
When $k = 0, 1$ one gets $\sqrt[4]{18}\left[\cos\alpha + i\sin\alpha\right]$ where $\alpha = 22.5^\circ, 202.5^\circ$.

61. Since $|2+i| = \sqrt{5}$ and $\tan^{-1}(1/2) \approx 26.6^\circ$, the arguments of the cube roots are

$$\frac{26.6^\circ + k360^\circ}{3} \approx 8.9^\circ + k120^\circ$$

When $k = 0, 1, 2$ one gets $\sqrt[6]{5}\left[\cos\alpha + i\sin\alpha\right]$ where $\alpha = 8.9^\circ, 128.9^\circ, 248.9^\circ$.

62. Note $|3-i| = \sqrt{10}$. Since $\tan^{-1}(-1/3) \approx -18.4^\circ$, an argument for $3-i$ is $360^\circ - 18.4^\circ = 341.6^\circ$. Arguments of the cube roots are
$\dfrac{341.6^\circ + k360^\circ}{3} \approx 113.9^\circ + k120^\circ$.
When $k = 0, 1, 2$ the cube roots are are $\sqrt[6]{10}\left[\cos\alpha + i\sin\alpha\right]$ where $\alpha = 113.9^\circ, 233.9^\circ, 353.9^\circ$.

63. Since $\sqrt[4]{625} = 5$, the fourth roots of $625i$ are
$5\left[\cos\left(\frac{90^\circ + k360^\circ}{4}\right) + i\sin\left(\frac{90^\circ + k360^\circ}{4}\right)\right]$
When $k = 0, 1, 2, 3$ the fourth roots are
$5\left[\cos\alpha + i\sin\alpha\right]$ where
$\alpha = 22.5^\circ, 112.5^\circ, 202.5^\circ, 292.5^\circ$.

64. Since $\sqrt[4]{625} = 5$, the fourth roots of $-625i$ are
$5\left[\cos\left(\frac{270^\circ + k360^\circ}{4}\right) + i\sin\left(\frac{270^\circ + k360^\circ}{4}\right)\right]$
When $k = 0, 1, 2, 3$ the fourth roots are
$5\left[\cos\alpha + i\sin\alpha\right]$ where
$\alpha = 67.5^\circ, 157.5^\circ, 247.5^\circ, 337.5^\circ$.

65. $(5\cos 60^\circ, 5\sin 60^\circ) = \left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$

66. $(-4\cos 30^\circ, -4\sin 30^\circ) = \left(-2\sqrt{3}, -2\right)$

67. $(\sqrt{3}\cos 100^\circ, \sqrt{3}\sin 100^\circ) \approx (-0.3, 1.7)$

68. $(\sqrt{5}\cos 230^\circ, \sqrt{5}\sin 230^\circ) \approx (-1.4, -1.7)$

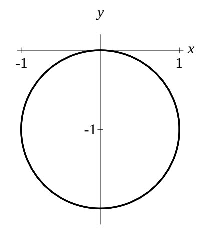
69. Note $r = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{16} = 4$.
Since $\tan\theta = \sqrt{3}$ and the terminal side of θ goes through $(-2, -2\sqrt{3})$, we have
$\theta = 4\pi/3$. Then $(r, \theta) = \left(4, \frac{4\pi}{3}\right)$.

70. Note $r = \sqrt{(-3\sqrt{2})^2 + (3\sqrt{2})^2} = \sqrt{36} = 6$.
Since $\tan\theta = -1$ and the terminal side of θ goes through $(-3\sqrt{2}, 3\sqrt{2})$, we have
$\theta = 3\pi/4$. So $(r, \theta) = \left(6, \frac{3\pi}{4}\right)$.

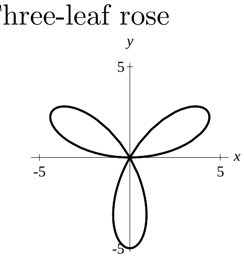
71. Note $r = \sqrt{2^2 + (-3)^2} = \sqrt{13}$.
Since $\theta = \tan^{-1}(-3/2) \approx -0.98$,
we have $(r, \theta) \approx \left(\sqrt{13}, -0.98\right)$.

72. Note $r = \sqrt{(-4)^2 + (-5)^2} = \sqrt{41}$.
Since $\tan^{-1}(5/4) \approx 0.896$ and the terminal side of θ goes through $(-4, -5)$, one can choose $\theta = \pi + 0.896 \approx 4.04$.
Then $(r, \theta) \approx \left(\sqrt{41}, 4.04\right)$.

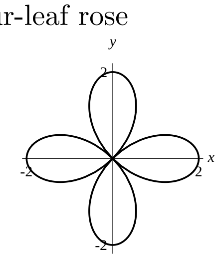
73. Circle centered at $(r, \theta) = (1, -\pi/2)$



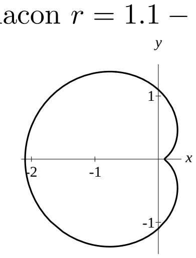
74. Three-leaf rose



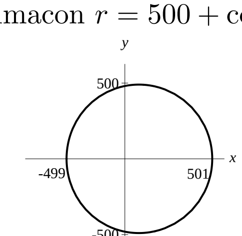
75. four-leaf rose



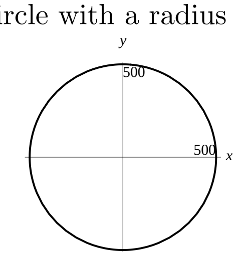
76. Limacon $r = 1.1 - \cos\theta$



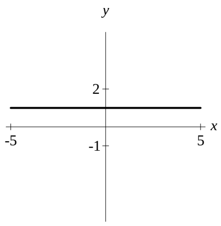
77. Limacon $r = 500 + \cos\theta$



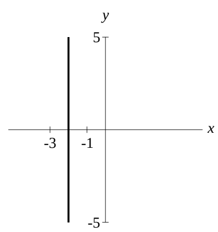
78. Circle with a radius of 500.



79. Horizontal line $y = 1$



80. Vertical line $x = -2$



81. Since $r = \dfrac{1}{\sin\theta + \cos\theta}$, we obtain

$$\begin{aligned} r\sin\theta + r\cos\theta &= 1 \\ y + x &= 1. \end{aligned}$$

82. Since $r = -6\cos\theta$, we find

$$\begin{aligned} r^2 &= -6r\cos\theta \\ x^2 + y^2 &= -6x \\ x^2 + 6x + y^2 &= 0 \end{aligned}$$

83. $x^2 + y^2 = 25$

84. Since $r = \dfrac{1}{1 + \sin\theta}$, we get

$$\begin{aligned} r + r\sin\theta &= 1 \\ r &= 1 - y \\ r^2 &= (1-y)^2 \\ x^2 + y^2 &= 1 - 2y + y^2 \\ x^2 &= 1 - 2y. \end{aligned}$$

85. Since $y = 3$, we find $r\sin\theta = 3$ and $r = \dfrac{3}{\sin\theta}$.

86. Since $x^2 + (y+1)^2 = 1$,

$$\begin{aligned} x^2 + y^2 + 2y + 1 &= 1 \\ x^2 + y^2 + 2y &= 0 \\ r^2 + 2r\sin\theta &= 0 \\ r^2 &= -2r\sin\theta \\ r &= -2\sin\theta. \end{aligned}$$

87. $r = 7$

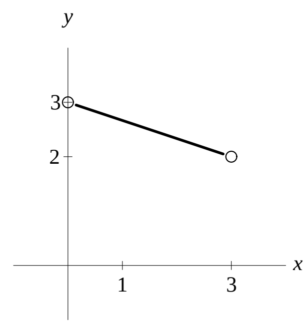
88. Since $2x + 3y = 6$, we obtain

$$\begin{aligned} 2r\cos\theta + 3r\sin\theta &= 6 \\ r(2\cos\theta + 3\sin\theta) &= 6 \\ r &= \frac{6}{2\cos\theta + 3\sin\theta}. \end{aligned}$$

89. The boundary points are given by

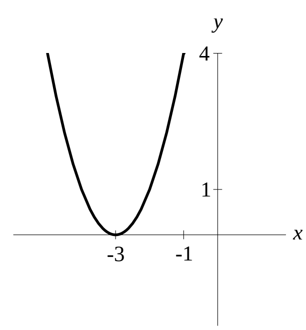
t	x	y
0	0	3
1	3	2

Note, the boundary points do not lie on the graph.

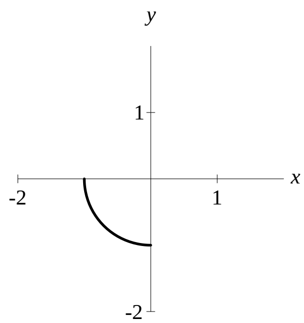


90. Some points on the graph are given by

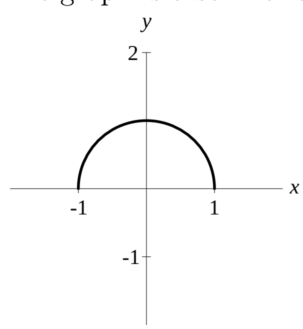
t	x	y
0	-3	0
2	-1	4



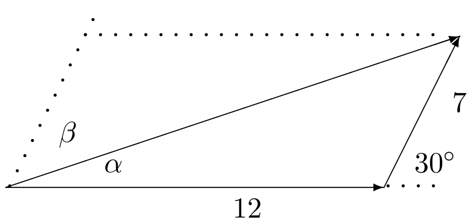
91. The graph is a quarter of a circle.



92. The graph is a semi-circle.



93. Draw two vectors with magnitudes 7 and 12 that act at an angle of 30° with each other.



By the cosine law, the magnitude of the resultant force is

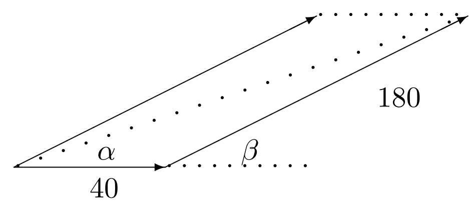
$$\sqrt{12^2 + 7^2 - 2(12)(7)\cos 150^\circ} \approx 18.4 \text{ lb.}$$

By the sine law, we find

$$\begin{aligned} \frac{7}{\sin\alpha} &= \frac{18.4}{\sin 150^\circ} \\ \sin\alpha &= \frac{7\sin 150^\circ}{18.4} \\ \sin\alpha &\approx 0.19 \\ \alpha \approx \sin^{-1}(0.19) &\approx 11.0^\circ. \end{aligned}$$

The angles between the resultant and each force are 11.0° and $\beta = 180^\circ - 150^\circ - 11^\circ = 19.0^\circ$.

94. Draw two vectors with magnitudes 40 and 180.



Since the bearing is 35°, we get $\beta = 90^\circ - 35^\circ = 55^\circ$. By the cosine law, the ground speed of the airplane which is equal to the magnitude of the resultant force is $\sqrt{40^2 + 180^2 - 2(40)(180)\cos 125^\circ} \approx 205.6$ mph. By the sine law, we obtain

$$\begin{aligned} \frac{180}{\sin\alpha} &= \frac{205.6}{\sin 125^\circ} \\ \sin\alpha &= \frac{180\sin 125^\circ}{205.6} \\ \sin\alpha &\approx 0.717 \\ \alpha \approx \sin^{-1}(0.717) &\approx 45.8^\circ. \end{aligned}$$

The bearing of the course is $90^\circ - 45.8^\circ = 44.2^\circ$.

95.

Since $\dfrac{482 + 364 + 241}{2} = 543.5$, then by using Heron's formula the area of Susan's lot is

$\sqrt{543.5(543.5 - 482)(543.5 - 364)(543.5 - 241)}$
$\approx 42,602 \text{ ft}^2$.

Similarly, since $\dfrac{482 + 369 + 238}{2} = 544.5$, the area of Seth's lot is

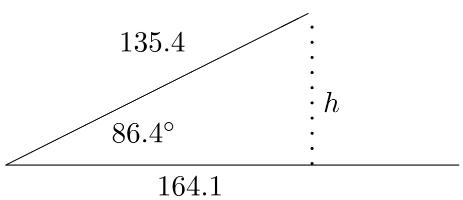
$\sqrt{544.5(544.5 - 482)(544.5 - 369)(544.5 - 238)}$
$\approx 42,785 \text{ ft}^2$.

Then Seth got the larger piece.

96. Since an included angle is given, the area is $\frac{1}{2}(135.4)(164.1)\sin 86.4^\circ \approx 11,087.6 \text{ ft}^2$.
To use Heron's formula, first find the length of the third side. It is

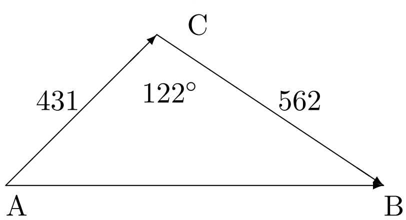
$\sqrt{135.4^2 + 164.1^2 - 2(135.4)(164.1)\cos 86.4^\circ}$
$\approx 206.086553 \approx 206.1$ ft.
Since $s = \dfrac{206.086553 + 135.4 + 164.1}{2} \approx$
252.7932765, the area is
$\sqrt{s(s - 206.086553)(s - 135.4)(s - 164.1)} \approx$
$11,087.6$ ft^2.

For the third formula, draw a triangle where h is the height.



Since $h = 135.4 \sin 86.4$, the area is $A = \dfrac{1}{2}bh$
$= \dfrac{1}{2}(164.1)(135.4 \sin 86.4) \approx 11,087.6$ ft^2.

97. Consider triangle below.



The distance between A and B is

$\sqrt{431^2 + 562^2 - 2(431)(562)\cos 122^\circ} \approx 870.82$ ft.

The extra amount spent is

$$(431 + 562 - 870.82)(\$21.60) \approx \$2639.$$

98. The distance of the top of the boom from the ground is $60 \sin 53^\circ \approx 47.9$ feet. Since the ball is extended 40 feet from the top of the boom, the ball is at least 7.9 feet above the ground at all times. A 6-ft tall pedestrian cannot be struck by the wrecking ball provided the radius of the ball is under 1.9 feet.

99.

a) By the cosine law and by the method of completing the square, one derives

$$\begin{aligned} c^2 &= a^2 + r^2 - 2ar\cos\theta \\ c^2 - r^2 &= a^2 - 2ar\cos\theta \\ c^2 - r^2 + r^2\cos^2\theta &= (a - r\cos\theta)^2 \\ c^2 - r^2(1 - \cos^2\theta) &= (a - r\cos\theta)^2 \\ c^2 - r^2\sin^2\theta &= (a - r\cos\theta)^2 \\ \sqrt{c^2 - r^2\sin^2\theta} &= a - r\cos\theta. \end{aligned}$$

Then $a = \sqrt{c^2 - r^2\sin^2\theta} + r\cos\theta$.

b) When $t = 0.1$ minute, the number of revolutions is 42.6.
Then $\theta = (0.6)(360^\circ) = 216^\circ$ and
$a = \sqrt{12^2 - 2^2\sin^2 216^\circ} + 2\cos 216^\circ$
≈ 10.3 in.

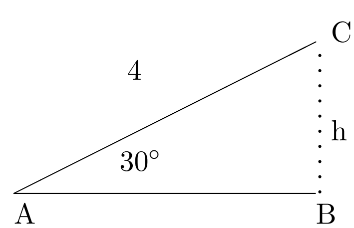
Thinking Outside the Box LXVIII

a) First, use four tiles to make a 4-by-4 square. Then construct three more 4-by-4 square squares. Now, you have four 4-by-4 squares. Then put these four squares together to make a 8-by-8 square. Yes, it is possible.

b) By elimination, you will not be able to make a 6-by-6 square. There are only a few possibilities and none of them will make a 6-by-6 square.

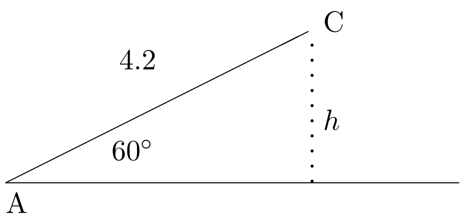
Chapter 7 Test

1. Draw a triangle with $\alpha = 30^\circ$, $b = 4$, $a = 2$.

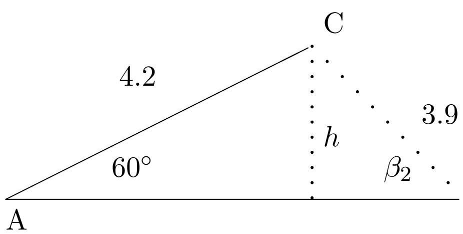


Since $h = 4\sin 30° = 2$ and $a = 2$, there is only one triangle and $\beta = 90°$. So $\gamma = 90° - 30° = 60°$. Since $c^2 + 2^2 = 4^2$, we get $c = 2\sqrt{3}$.

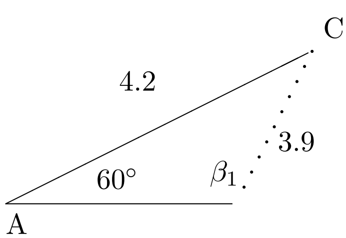
2. Draw angle $\alpha = 60°$ and let h be the height.



Since $h = 4.2 \sin 60° \approx 3.6$ and $3.6 < a < 4.2$, there are two triangles and they are given by



and



Apply the sine law to the acute triangle.

$$\begin{aligned}
\frac{3.9}{\sin 60°} &= \frac{4.2}{\sin \beta_2} \\
\sin \beta_2 &= \frac{4.2 \sin 60°}{3.9} \\
\sin \beta_2 &\approx 0.93264 \\
\beta_2 = \sin^{-1}(0.93264) &\approx 68.9°
\end{aligned}$$

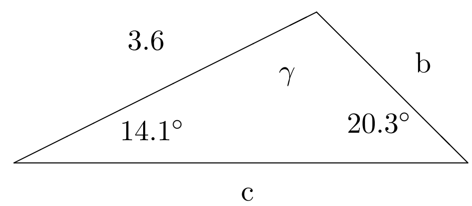
So $\gamma_2 = 180° - (68.9° + 60°) = 51.1°$.

By the sine law, $c_2 = \dfrac{3.9}{\sin 60°} \sin 51.1° \approx 3.5$.

In the obtuse triangle, $\beta_1 = 180° - \beta_2 = 111.1°$ and $\gamma_1 = 180° - (111.1° + 60°) = 8.9°$.

By the sine law, $c_1 = \dfrac{3.9}{\sin 60°} \sin 8.9° \approx 0.7$.

3. Draw the only triangle with $a = 3.6$, $\alpha = 20.3°$, and $\beta = 14.1°$.



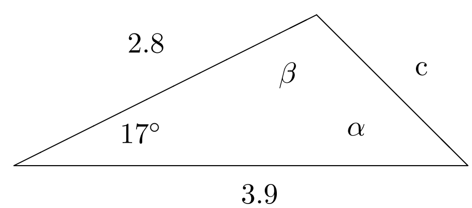
Note $\gamma = 180° - 14.1° - 20.3° = 145.6°$. By using the sine law, we find

$$\frac{b}{\sin 14.1°} = \frac{3.6}{\sin 20.3°} \text{ and } \frac{c}{\sin 145.6°} = \frac{3.6}{\sin 20.3°}.$$

Then $b = \dfrac{3.6}{\sin 20.3°} \sin 14.1° \approx 2.5$ and

$c = \dfrac{3.6}{\sin 20.3°} \sin 145.6° \approx 5.9$.

4. Draw the only triangle with $a = 2.8$, $b = 3.9$, and $\gamma = 17°$.

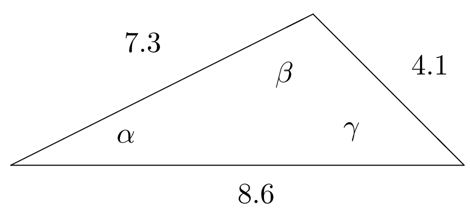


By the cosine law, we get $c = \sqrt{3.9^2 + 2.8^2 - 2(3.9)(2.8)\cos 17°} \approx 1.47 \approx 1.5$. By the sine law,

$$\begin{aligned}
\frac{1.47}{\sin 17°} &= \frac{2.8}{\sin \alpha} \\
\sin \alpha &= \frac{2.8 \sin 17°}{1.47} \\
\sin \alpha &\approx 0.5569 \\
\alpha \approx \sin^{-1}(0.5569) &\approx 33.8°.
\end{aligned}$$

Also, $\beta = 180° - (33.8° + 17°) = 129.2°$.

5. Draw the only triangle with the given sides $a = 4.1$, $b = 8.6$, and $c = 7.3$.



First, find the largest angle β by the cosine law.

$$\begin{aligned} \cos\beta &= \frac{7.3^2 + 4.1^2 - 8.6^2}{2(7.3)(4.1)} \\ \cos\beta &\approx -0.06448 \\ \beta &\approx \cos^{-1}(-0.06448) \\ \beta &\approx 93.7°. \end{aligned}$$

By the sine law,

$$\begin{aligned} \frac{8.6}{\sin 93.7°} &= \frac{7.3}{\sin\gamma} \\ \sin\gamma &= \frac{7.3 \sin 93.7°}{8.6} \\ \sin\gamma &\approx 0.8471 \\ \gamma \approx \sin^{-1}(0.8471) &\approx 57.9°. \end{aligned}$$

Also $\alpha = 180° - (57.9° + 93.7°) = 28.4°$.

6. The magnitude of $\boldsymbol{A} + \boldsymbol{B} = \langle -2, 6\rangle$ is $\sqrt{(-2)^2 + 6^2} = \sqrt{40} = 2\sqrt{10}$.
Direction angle is $\cos^{-1}(-2/\sqrt{40}) \approx 108.4°$.

7. The magnitude of $\boldsymbol{A} - \boldsymbol{B} = \langle -4, -2\rangle$ is $\sqrt{(-4)^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}$.
Since $\tan^{-1}(2/4) \approx 26.6°$, the direction angle is $180° + 26.6° = 206.6°$.

8. Magnitude of $\boldsymbol{3B} = \langle 3, 12\rangle$ is $\sqrt{3^2 + 12^2} = \sqrt{153} = 3\sqrt{17}$.
Direction angle is $\tan^{-1}(12/3) \approx 76.0°$.

9. Since $|3 + 3i| = 3\sqrt{2}$ and $\tan^{-1}(3/3) = 45°$, we have $3 + 3i = 3\sqrt{2}\,[\cos 45° + i \sin 45°]$.

10. Since $|-1+i\sqrt{3}| = 2$ and $\cos^{-1}(-1/2) = 120°$, we have $|-1 + i\sqrt{3}| = 2\,[\cos 120° + i \sin 120°]$.

11. Note $|-4 - 2i| = \sqrt{20} = 2\sqrt{5}$.
Since $\tan^{-1}(2/4) = 26.6°$, the direction angle of $-4 - 2i$ is $180° + 26.6° = 206.6°$.
So $-4 - 2i = 2\sqrt{5}\,[\cos 206.6° + i \sin 206.6°]$.

12. $6\,[\cos 45° + i \sin 45°] =$

$$6\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right] = 3\sqrt{2} + 3i\sqrt{2}$$

13. $2^9\,[\cos 90° + i \sin 90°] =$
$512\,[0 + i] = 512i$

14. $\frac{3}{2}\,[\cos 45° + i \sin 45°] =$

$$\frac{3}{2}\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right] = \frac{3\sqrt{2}}{4} + \frac{3\sqrt{2}}{4}i$$

15. $(5\cos 30°, 5\sin 30°) = \left(5\frac{\sqrt{3}}{2}, 5\frac{1}{2}\right) =$

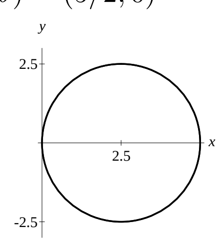
$$\left(\frac{5\sqrt{3}}{2}, \frac{5}{2}\right)$$

16. $(-3\cos(-\pi/4), -3\sin(-\pi/4)) =$

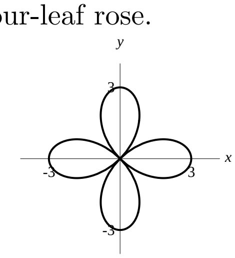
$$\left(-3\frac{\sqrt{2}}{2}, 3\frac{\sqrt{2}}{2}\right) = \left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$$

17. $(33\cos 217°, 33\sin 217°) \approx (-26.4, -19.9)$

18. Circle of radius $5/2$ with center at $(r, \theta) = (5/2, 0)$



19. Four-leaf rose.



20. We apply Heron's formula.
Since $\frac{4.1+6.8+9.5}{2} = 10.2$, the area is

$\sqrt{10.2(10.2-4.1)(10.2-6.8)(10.2-9.5)}$
$\approx 12.2 \text{ m}^2$.

21. Since $a_1 = 4.6\cos 37.2° \approx 3.66$ and $a_2 = 4.6\sin 37.2° \approx 2.78$, we have $\boldsymbol{v} \approx 3.66\,\boldsymbol{i} + 2.78\,\boldsymbol{j}$.

22. Fourth roots of -81 are given by

$$3\left[\cos\left(\frac{180° + k360°}{4}\right) + i\sin\left(\frac{180° + k360°}{4}\right)\right]$$
$$= 3\left[\cos(45° + k90°) + i\sin(45° + k90°)\right].$$

When $k = 0, 1, 2, 3$ one gets

$$3[\cos 45° + i\sin 45°] = \frac{3\sqrt{2}}{2} + i\frac{3\sqrt{2}}{2},$$
$$3[\cos 135° + i\sin 135°] = -\frac{3\sqrt{2}}{2} + i\frac{3\sqrt{2}}{2},$$
$$3[\cos 225° + i\sin 225°] = -\frac{3\sqrt{2}}{2} - i\frac{3\sqrt{2}}{2},$$
$$3[\cos 315° + i\sin 315°] = \frac{3\sqrt{2}}{2} - i\frac{3\sqrt{2}}{2}.$$

23. Since $x^2 + y^2 + 5y = 0$, we obtain

$$\begin{aligned} r^2 + 5r\sin\theta &= 0 \\ r + 5\sin\theta &= 0 \\ r &= -5\sin\theta \end{aligned}$$

24. Since $r = 5(2\sin\theta\cos\theta)$, we find

$$\begin{aligned} r^3 &= 10(r\sin\theta)(r\cos\theta) \\ r^3 &= 10yx \\ (x^2 + y^2)^{3/2} &= 10xy. \end{aligned}$$

25. The line that passes through through $(-2, -3)$ and $(4, 5)$ is given by

$$y = \frac{4}{3}x - \frac{1}{3}.$$

The linear function $x = f(t)$ that satisfies

i) $x = -2$ when $t = 0$, and
ii) $x = 4$ when $t = 1$

is defined by

$$x = 6t - 2.$$

Then

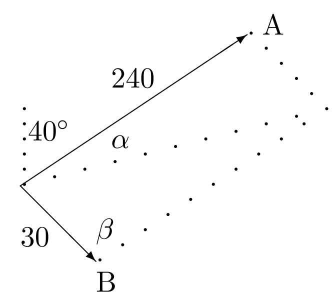
$$y = \frac{4}{3}(6t - 2) - \frac{1}{3} = 8t - 3.$$

Thus, the parametric equations are

$$x = 6t - 2, \; y = 8t - 3$$

for $0 \le t \le 1$.

26. Draw two vectors with magnitudes 240 and 30.



Ground speed of the airplane is the magnitude of the resultant force. From the difference of the two bearings, one sees that 95° is the angle between the two vectors. Since the sum of the angles in a parallelogram is 360°, we get $\beta = \dfrac{360° - 2\cdot 95°}{2} = 85°$. By the cosine law, the ground speed is

$$\sqrt{30^2 + 240^2 - 2(30)(240)\cos 85°} \approx 239.3 \text{ mph}.$$

By the sine law, we find

$$\begin{aligned} \frac{30}{\sin\alpha} &= \frac{239.3}{\sin 85°} \\ \sin\alpha &= \frac{30\sin 85°}{239.3} \\ \sin\alpha &\approx 0.12489 \\ \alpha &\approx \sin^{-1}(0.12489) \\ \alpha &\approx 7.2°. \end{aligned}$$

The bearing of the course is $\alpha + 40° = 47.2°$.

Tying It All Together

1. Use the quadratic formula to find the zeros of the second degree factor in

$$x(x^3-1)=x(x-1)(x^2+x+1).$$

Then

$$x = \frac{-1\pm\sqrt{1^2-4(1)(1)}}{2}$$
$$x = \frac{-1\pm\sqrt{-3}}{2}.$$

Roots are $x=0,1,-\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$.

2. The zeros are $-2,1,3$ since

1	1	-2	-5	6
		1	-1	-6
	1	-1	-6	0

and $x^2-x-6=(x-3)(x+2)$

3. Factoring by grouping, we obtain $x^5(x+2)-(x+2)=(x^5-1)(x+2)=0$. Then $x^5=1$ and $x=-2$. The fifth roots of 1 are given by $\cos\left(\dfrac{k360^\circ}{5}\right)+i\sin\left(\dfrac{k360^\circ}{5}\right)=$ $\cos(k72^\circ)+i\sin(k72^\circ)$.
When $k=0,1,2,3,4$ one obtains $\cos 0^\circ+i\sin 0^\circ=1$, $\cos 72^\circ+i\sin 72^\circ$, $\cos 144^\circ+i\sin 144^\circ$, $\cos 216^\circ+i\sin 216^\circ$, and $\cos 288^\circ+i\sin 288^\circ$.

4. Factoring by grouping, we get $x^4(x^3-1)+2(x^3-1)=(x^4+2)(x^3-1)=0$. So $x^4=-2$ and $x^3=1$. The fourth roots of -2 are given by

$$\sqrt[4]{2}\left[\cos\left(\frac{180^\circ+k360^\circ}{4}\right)+i\sin\left(\frac{180^\circ+k360^\circ}{4}\right)\right]=$$

$$\sqrt[4]{2}\left[\cos\left(45^\circ+k90^\circ\right)+i\sin\left(45^\circ+k90^\circ\right)\right].$$

When $k=0,1,2,3$ one obtains

$$\sqrt[4]{2}\left[\cos 45^\circ+i\sin 45^\circ\right]=\frac{1}{\sqrt[4]{2}}+i\frac{1}{\sqrt[4]{2}},$$

$$\sqrt[4]{2}\left[\cos 135^\circ+i\sin 135^\circ\right]=-\frac{1}{\sqrt[4]{2}}+i\frac{1}{\sqrt[4]{2}},$$

$$\sqrt[4]{2}\left[\cos 225^\circ+i\sin 225^\circ\right]=-\frac{1}{\sqrt[4]{2}}-i\frac{1}{\sqrt[4]{2}}, \text{ and}$$

$$\sqrt[4]{2}\left[\cos 315^\circ+i\sin 315^\circ\right]=\frac{1}{\sqrt[4]{2}}-i\frac{1}{\sqrt[4]{2}}.$$

The cube roots of 1 are given by

$$\cos\left(\frac{k360^\circ}{3}\right)+i\sin\left(\frac{k360^\circ}{3}\right)$$
$$=\cos(k120^\circ)+i\sin(k120^\circ)$$

When $k=0,1,2$ one obtains

$$\cos 0^\circ+i\sin 0^\circ=1,$$

$$\cos 120^\circ+i\sin 120^\circ=-\frac{1}{2}+\frac{\sqrt{3}}{2}i, \text{ and}$$

$$\cos 240^\circ+i\sin 240^\circ=-\frac{1}{2}-\frac{\sqrt{3}}{2}i.$$

The solutions are the fourth roots of -2 and the cube roots of 1.

5. By a double-angle identity, we find

$$2(2\sin x\cos x)-2\cos x+2\sin x-1=0$$

$$\begin{aligned} 2\cos x(2\sin x-1)+(2\sin x-1) &= 0\\ (2\cos x+1)(2\sin x-1) &= 0\\ \cos x=-\frac{1}{2} \text{ or } \sin x=\frac{1}{2}&. \end{aligned}$$

The solutions are $x=\dfrac{2\pi}{3}+2k\pi$, $\dfrac{4\pi}{3}+2k\pi$, $\dfrac{\pi}{6}+2k\pi$, and $\dfrac{5\pi}{6}+2k\pi$.

6.

$$\begin{aligned} 4x\sin x-2x+2\sin x-1 &= 0\\ 2x(2\sin x-1)+(2\sin x-1) &= 0\\ (2x+1)(2\sin x-1) &= 0\\ x=-\frac{1}{2} \text{ or } \sin x=\frac{1}{2}& \end{aligned}$$

The solutions are $x=-\dfrac{1}{2}$, $\dfrac{\pi}{6}+2k\pi$, $\dfrac{5\pi}{6}+2k\pi$.

7. Since $e^{\sin x}=1$, we get $\sin x=0$.
The solution are $x=k\pi$ where k is an integer.

8. Since $\sin(e^x) = 1/2$, the solutions are given by

$$e^x = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad e^x = \frac{5\pi}{6} + 2k\pi$$
$$x = \ln\left(\frac{\pi}{6} + 2k\pi\right) \quad \text{or} \quad x = \ln\left(\frac{5\pi}{6} + 2k\pi\right)$$

where k is a nonnegative integer.

9. Using a common base, we obtain

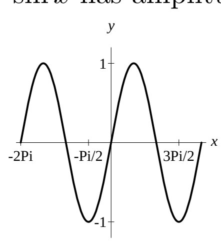
$$\begin{aligned} 2^{2x-3} &= 2^5 \\ 2x - 3 &= 5 \\ 2x &= 8 \\ x &= 4. \end{aligned}$$

10. Express the left-hand side as a single logarithm.

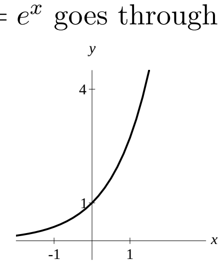
$$\begin{aligned} \log\left(\frac{x-1}{x+2}\right) &= 2 \\ \frac{x-1}{x+2} &= 10^2 \\ x - 1 &= 100x + 200 \\ -201 &= 99x \end{aligned}$$

Checking $x = -201/99$ one gets $\log(-201/99 - 1)$ which is undefined. The solution set is $\emptyset$.

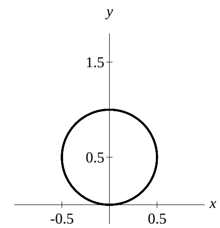
11. $y = \sin x$ has amplitude 1 and period 2π.



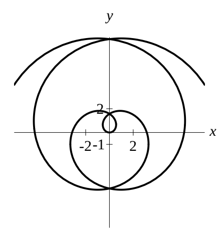
12. $y = e^x$ goes through $(-1, 1/e)$, $(0, 1)$, $(1, e)$.



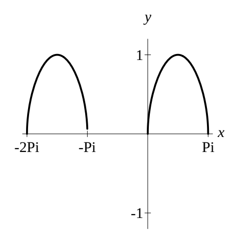
13. $r = \sin\theta$ in polar coordinates is a circle through the origin with radius $1/2$.



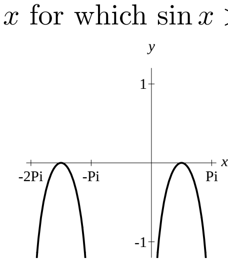
14. $r = \theta$ in polar coordinates goes through $(-\pi, -\pi)$, $(0, 0)$, (π, π).



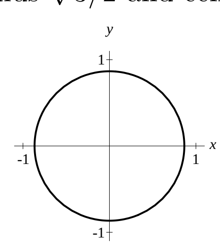
15. $y = \sqrt{\sin x}$ is only defined for values of x for which $\sin x$ is nonnegative.



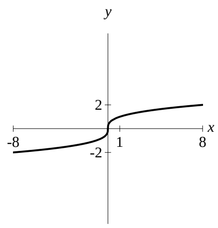
16. $y = \ln(\sin x)$ has vertical asymptotes at $x = k$ where $\sin k = 0$ and is only defined for values of x for which $\sin x > 0$.



17. $r = \sin(\pi/3) = \sqrt{3}/2$ is a circle with radius $\sqrt{3}/2$ and center at the origin.



18. $y = x^{1/3}$ goes through $(-8, -2)$, $(0, 0)$, $(8, 2)$.



19. $\log(1) = 0$ **20.** $\sin(0) = 0$

21. $\cos(\pi) = -1$ **22.** $\ln(1) = 0$

23. $\sin^{-1}(1/2) = \pi/6$ **24.** $\cos^{-1}(1/2) = \pi/3$

25. $\tan^{-1}(-1) = -\pi/4$ **26.** $\tan^{-1}(1) = \pi/4$

27. $\dfrac{(x-2)+(x+2)}{(x+2)(x-2)} = \dfrac{2x}{x^2-4}$

28. $\dfrac{(1+\sin x)+(1-\sin x)}{1-\sin^2 x} = \dfrac{2}{\cos^2 x} = 2\sec^2 x$

29. $\dfrac{(2-\sqrt{3})+(2+\sqrt{3})}{2^2-3} = \dfrac{4}{1} = 4$

30.
$$\log\left(\frac{1}{(x+2)(x-2)}\right) = \log\left(\frac{1}{x^2-4}\right) =$$
$$-\log(x^2-4)$$

31.
$$\frac{(x-3)(x+3)\cdot 4(x+3)}{2(x-3)\cdot(x+3)^2} = 2$$

32.
$$\frac{\cos^2 x(\cos x - \sin x)(\cos x + \sin x)}{(2\cos^2 x - 2\sin x\cos x)(2\cos^2 x + 2\sin x\cos x)}$$
$$= \frac{\cos^2 x(\cos x - \sin x)(\cos x + \sin x)}{4\cos^2 x(\cos x - \sin x)(\cos x + \sin x)} = \frac{1}{4}$$

33.
$$\frac{\sqrt{16}}{(2-\sqrt{3})(4+2\sqrt{3})} =$$
$$\frac{4}{(2-\sqrt{3})\cdot 2(2+\sqrt{3})} =$$
$$\frac{2}{(2-\sqrt{3})(2+\sqrt{3})} = \frac{2}{4-3} = 2$$

34.
$$\log\left(\frac{(x+1)(x+2)}{(x+3)(x+2)}\right) - \log\left(\frac{x+1}{x+3}\right) =$$
$$\log\left(\frac{x+1}{x+3}\right) - \log\left(\frac{x+1}{x+3}\right) = 0$$

35. Since $x^2 - 4 = (x-2)(x+2)$, we find

$$\frac{\dfrac{1}{x-2} + \dfrac{1}{x+2}}{\dfrac{1}{x^2-4} - \dfrac{1}{x+2}} \cdot \frac{x^2-4}{x^2-4} = \frac{(x+2)+(x-2)}{1-(x-2)}$$
$$\frac{2x}{3-x} = \frac{-2x}{x-3}.$$

36. Express in terms of $\sin x$ and $\cos x$.

$$\begin{aligned} \frac{\dfrac{1}{2\sin x\cos x} - \dfrac{\sin x}{\cos x}}{\dfrac{\cos x}{2\sin x} - \dfrac{\sin x}{2\cos x}} \cdot \frac{2\sin x\cos x}{2\sin x\cos x} &= \\ \frac{1-2\sin^2 x}{\cos^2 x - \sin^2 x} &= \\ \frac{\cos(2x)}{\cos(2x)} &= \\ 1 & \end{aligned}$$

37. frequency

38. vertical asymptote

39. even

40. odd

41. opposite, hypotenuse

42. adjacent, hypotenuse

43. one

44. period

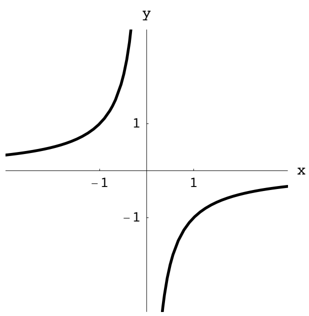
45. Pythagorean

46. even, odd

Concepts of Calculus

1a. Note, the graph of $f(x) = -1/x$ approaches the vertical asymptote $x = 0$ going downward as x approaches 0 through real numbers larger than 0, as seen below. Thus,

$$\lim_{x\to 0^+} \frac{-1}{x} = -\infty.$$

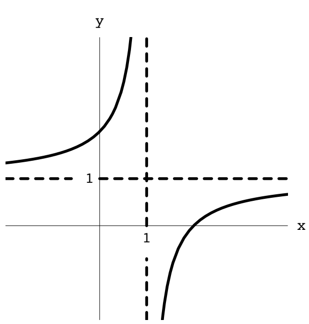


b. Based on the given graph in Part 1(a), the graph of $f(x) = -1/x$ approaches the vertical asymptote $x = 0$ going upward as x approaches 0 through real numbers smaller than 0. Thus,

$$\lim_{x\to 0^-} \frac{-1}{x} = \infty.$$

c. Note, the graph of $f(x) = (x-2)/(x-1)$ approaches the vertical asymptote $x = 1$ going downward as x approaches 1 through real numbers larger than 1, as seen below. Thus,

$$\lim_{x\to 1^+} \frac{x-2}{x-1} = -\infty.$$

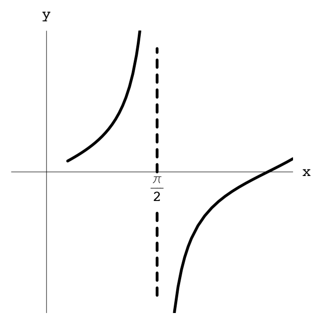


d. The graph of $f(x) = (x-2)/(x-1)$ approaches the vertical asymptote $x = 1$ going upward (see the graph in Part 1c) as x approaches 1 through real numbers less than 1. Hence,

$$\lim_{x\to 1^-} \frac{x-2}{x-1} = \infty.$$

e. The graph of $f(x) = \tan x$ approaches the vertical asymptote $x = \pi/2$ going downward as x approaches $\pi/2$ through real numbers larger than $\pi/2$, as shown below. Thus,

$$\lim_{x\to \pi/2^+} \tan x = -\infty.$$

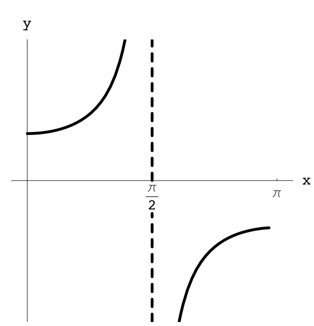


f. The graph of $f(x) = \tan x$ approaches the vertical asymptote $x = \pi/2$ going upward (see the graph in Part 1e) as x approaches $\pi/2$ through real numbers less than 1. Hence,

$$\lim_{x\to \pi/2^-} \tan x = \infty.$$

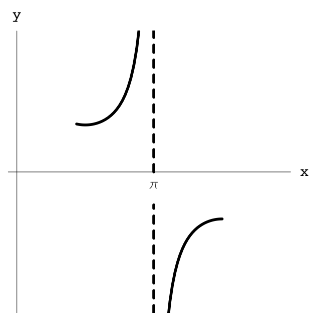
g. The graph of $f(x) = \sec x$ approaches the vertical asymptote $x = \pi/2$ going downward as x approaches $\pi/2$ through real numbers larger than $\pi/2$, as shown below. Hence,

$$\lim_{x\to \pi/2^+} \sec x = -\infty.$$



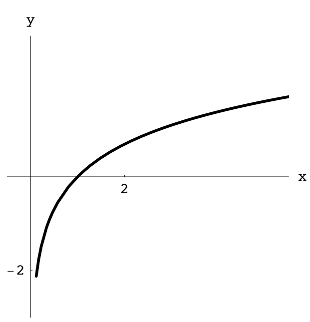
h. The graph of $f(x) = \csc x$ approaches the vertical asymptote $x = \pi$ going upward as x approaches π through real numbers less than π, as seen below. Hence,

$$\lim_{x\to\pi^-} \csc x = \infty.$$



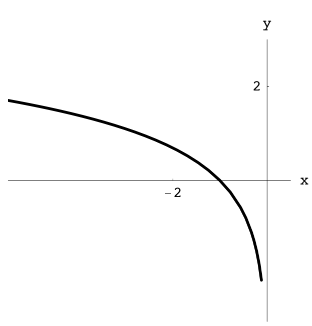
i. The graph of $f(x) = \ln x$ approaches the vertical asymptote $x = 0$ going downward as x approaches 0 through real numbers greater than 0, as seen below. Hence,

$$\lim_{x\to 0^+} \ln x = -\infty.$$



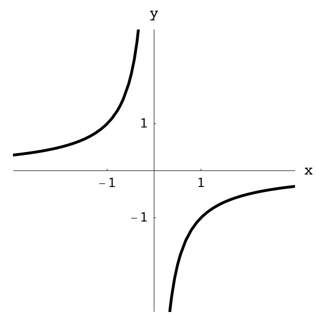
j. The graph of $f(x) = \ln(-x)$ approaches the vertical asymptote $x = 0$ going downward as x approaches 0 through real numbers less than 0, as seen below. Hence,

$$\lim_{x\to 0^-} \ln(-x) = -\infty.$$



2a. Note, the graph of $f(x) = -1/x$ approaches the horizontal asymptote $y = 0$ as x increases without bound, as seen below. Thus,

$$\lim_{x\to\infty} \frac{-1}{x} = 0.$$

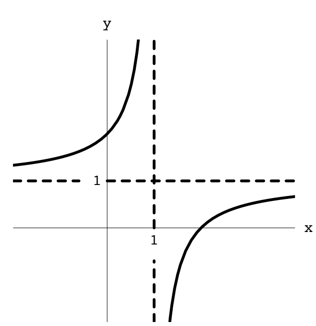


b. As seen in Part 2a, the graph of $f(x) = -1/x$ approaches the horizontal asymptote $y = 0$ as x decreases without bound. Hence,

$$\lim_{x\to-\infty} \frac{-1}{x} = 0.$$

c. The graph of $f(x) = (x-2)/(x-1)$ approaches the horizontal asymptote $y = 1$ as x increases without bound, as seen below. Thus,

$$\lim_{x\to\infty} \frac{x-2}{x-1} = 1.$$

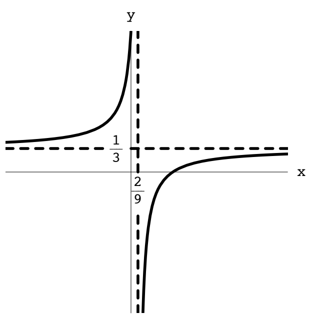


d. The graph of $f(x) = (x-2)/(x-1)$ approaches the horizontal asymptote $y = 1$ as x decreases without bound, as seen in Part 2c. Hence,

$$\lim_{x\to-\infty} \frac{x-2}{x-1} = 1.$$

e. The graph of $f(x) = (3x-4)/(9x-2)$ approaches the horizontal asymptote $y = 1/3$ as x increases without bound, as seen below. Thus, we obtain

$$\lim_{x\to\infty} \frac{3x-4}{9x-2} = \frac{1}{3}.$$

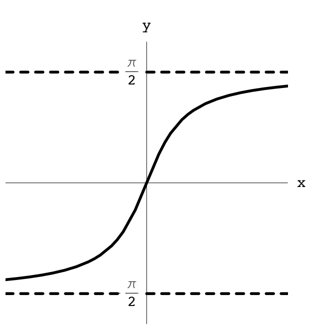


f. The graph of $f(x) = (3x-4)/(9x-2)$ approaches the horizontal asymptote $y = 1/3$ as x decreases without bound, see the graph in Part 2e. Thus,

$$\lim_{x\to-\infty} \frac{3x-4}{9x-2} = \frac{1}{3}.$$

g. The graph of $f(x) = \tan^{-1}(x)$ approaches the horizontal asymptote $y = \pi/2$ as x increases without bound, as seen below. Thus,

$$\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}.$$

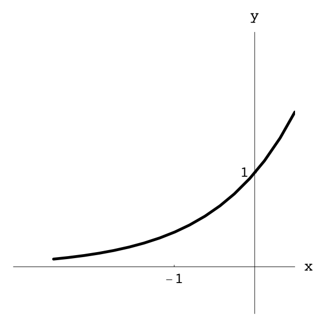


h. The graph of $f(x) = \tan^{-1}(x)$ approaches the horizontal asymptote $y = -\pi/2$ as x decreases without bound, see the graph in Part 2g. Hence, we have

$$\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}.$$

i. The graph of $f(x) = e^x$ approaches the horizontal asymptote $y = 0$ as x decreases without bound, as shown below. Hence,

$$\lim_{x\to-\infty} e^{-x} = 0.$$



j. The graph of $f(x) = e^{-x^2}$ approaches the horizontal asymptote $y = 0$ as x increases without bound, as shown below. Hence,

$$\lim_{x\to\infty} e^{-x^2} = 0.$$