For Thought

1. False, for example $\sin 0 = 0$ but $\cos 0 = 1$.

2. True, since $\dfrac{\sin(x)}{\cos(x)} \cdot \dfrac{\cos(x)}{\sin(x)} = 1$.

3. True, since $f(-x) = (\sin(-x))^2 = (-\sin(x))^2 = \sin^2(x) = f(x)$.

4. False, it is even since $f(-x) = (\cos(-x))^3 = (\cos(x))^3 = f(x)$.

5. True **6.** False, since $(\sin x + \cos x)^2 = 1 + 2\sin(x)\cos(x) \neq 1 = \sin^2 x + \cos^2 x$.

7. False, since $\tan(x) = \pm\sqrt{\sec^2(x) - 1}$.

8. True, since $[\sin(-3)\csc(-3)] \cdot [\cos(-3)\sec(3)] \cdot [\tan(-3)\cot(3)] = [1] \cdot [1] \cdot [-1] = -1$.

9. False, $\sin^2(-\pi/9) + \cos^2(-\pi/9) = 1$.

10. True, $1 - \sin^2(-\pi/7) = \cos^2(-\pi/7) = \cos^2(\pi/7)$.

6.1 Exercises

1. even

2. odd

3. Pythagorean

4. identity

5. odd

6. even

7. $\dfrac{\sin x}{\cos x} \cdot \cos x = \sin x$

8. $\sin x \cdot \dfrac{\cos x}{\sin x} = \cos x$

9. $\dfrac{1}{\cos x} \cdot \cos x = 1$

10. $\sin x \cdot \dfrac{1}{\sin x} = 1$

11. $\dfrac{1/\cos(x)}{\sin(x)/\cos(x)} = \dfrac{1}{\cos(x)} \cdot \dfrac{\cos(x)}{\sin(x)} = \dfrac{1}{\sin(x)} = \csc(x)$

12. $\dfrac{\cos(x)/\sin(x)}{1/\sin(x)} = \dfrac{\cos(x)}{\sin(x)} \cdot \dfrac{\sin(x)}{1} = \cos(x)$

13. $\dfrac{\sin x}{1/\sin x} + \cos^2 x = \sin^2 x + \cos^2 x = 1$

14. $\dfrac{\cos x}{1/\cos x} + \sin^2 x = \cos^2 x + \sin^2 x = 1$

15. $\dfrac{\sin(x)}{1/\sin(x)} + \dfrac{\cos(x)}{1/\cos(x)} = \sin^2(x) + \cos^2(x) = 1$

16. $\csc^2 x - \cot^2 x = 1$ **17.** $1 - \sin^2\alpha = \cos^2\alpha$

18. $\sec^2\alpha - 1 = \tan^2\alpha$

19. $(\sin\beta + 1)(\sin\beta - 1) = \sin^2\beta - 1 = -\cos^2\beta$

20. $(1 + \cos\beta)(1 - \cos\beta) = 1 - \cos^2\beta = \sin^2\beta$

21.
$$\frac{1 + \cos\alpha \cdot \dfrac{\sin\alpha}{\cos\alpha} \cdot \dfrac{1}{\sin\alpha}}{1/\sin\alpha} = \frac{1+1}{1/\sin\alpha} = 2\sin\alpha$$

22.
$$\frac{(\sin\alpha + 1)(\sin\alpha - 1)}{\cos^2\alpha} = \frac{\sin^2\alpha - 1}{\cos^2\alpha} = \frac{-\cos^2\alpha}{\cos^2\alpha} = -1$$

23. Since $\cot^2 x = \csc^2 x - 1$, we obtain $\cot x = \pm\sqrt{\csc^2 x - 1}$.

24. Since $\sec^2(x) = \tan^2(x) + 1$, we have $\sec(x) = \pm\sqrt{\tan^2(x) + 1}$.

25. $\sin(x) = \dfrac{1}{\csc(x)} = \dfrac{1}{\pm\sqrt{1 + \cot^2(x)}}$

26. $\cos(x) = \dfrac{1}{\sec(x)} = \dfrac{1}{\pm\sqrt{\tan^2(x) + 1}}$

27. Since $\cot^2(x) = \csc^2(x) - 1$, we obtain $\tan(x) = \dfrac{1}{\cot(x)} = \dfrac{1}{\pm\sqrt{\csc^2(x) - 1}}$

28. $\cot(x) = \dfrac{1}{\tan(x)} = \dfrac{1}{\pm\sqrt{\sec^2(x) - 1}}$

29. Since $\sec\alpha = \sqrt{1 + (1/2)^2} = \sqrt{5}/2$, $\cos\alpha = 2/\sqrt{5}$, $\sin\alpha = \sqrt{1 - (2/\sqrt{5})^2} = 1/\sqrt{5}$. So $\csc\alpha = \sqrt{5}$ and $\cot\alpha = 2$.

30. Since α is in Quadrant II,
$\cos\alpha = -\sqrt{1-(3/4)^2} = -\sqrt{7}/4$,
$\sec\alpha = -4/\sqrt{7}$, $\tan\alpha = \dfrac{3/4}{-\sqrt{7}/4} = -3/\sqrt{7}$,
$\cot\alpha = -\sqrt{7}/3$, and $\csc\alpha = 4/3$.

31. Since $\sin\alpha = -\sqrt{1-(-\sqrt{3}/5)^2} = -\sqrt{1-3/25} = -\sqrt{22}/5$, $\csc\alpha = -5/\sqrt{22}$,
$\sec\alpha = -5/\sqrt{3}$, $\tan\alpha = \dfrac{-\sqrt{22}/5}{-\sqrt{3}/5} = \sqrt{22}/\sqrt{3}$,
and $\cot\alpha = \sqrt{3}/\sqrt{22}$.

32. $\cos\alpha = \dfrac{5}{-4\sqrt{5}}\cdot\dfrac{\sqrt{5}}{\sqrt{5}} = -\dfrac{5\sqrt{5}}{-20} = -\sqrt{5}/4$,
$\sin\alpha = \sqrt{1-(-\sqrt{5}/4)^2} = \sqrt{11}/4$
Also, $\tan\alpha = \dfrac{\sqrt{11}/4}{-\sqrt{5}/4} = -\sqrt{11}/\sqrt{5} = -\sqrt{55}/5$, $\cot\alpha = -5/\sqrt{55}$, and
$\csc\alpha = 4/\sqrt{11}$.

33. Since α is in Quadrant IV, we get
$\csc\alpha = -\sqrt{1+(-1/3)^2} = -\sqrt{10}/3$,
$\sin\alpha = -3/\sqrt{10}$, $\cos\alpha = \sqrt{1-(-3/\sqrt{10})^2} = \sqrt{1-9/10} = 1/\sqrt{10}$, $\sec\alpha = \sqrt{10}$,
and $\tan\alpha = -3$.

34. Since $\sin\alpha = 1/\sqrt{3}$, we obtain
$\cos(\alpha) = \sqrt{1-(1/\sqrt{3})^2} = \sqrt{2}/\sqrt{3}$,
$\sec(\alpha) = \sqrt{3}/\sqrt{2}$, $\tan(\alpha) = \dfrac{1/\sqrt{3}}{\sqrt{2}/\sqrt{3}} = 1/\sqrt{2}$,
and $\cot(\alpha) = \sqrt{2}$.

35. Let $\theta = \arccos x$. Then $\cos\theta = x$ and θ lies in quadrant 1 or 2. Since $\sin^2\theta = 1-\cos^2\theta = 1-x^2$, we obtain $\sin(\arccos x) = \sin\theta = \pm\sqrt{1-x^2}$. Since sine is positive in both quadrants 1 and 2, we have $\sin(\arccos x) = \sqrt{1-x^2}$.

36. Let $\theta = \arcsin x$. Then $\sin\theta = x$ and θ lies in quadrant 1 or 4. Since $\cos^2\theta = 1-\sin^2\theta = 1-x^2$, we obtain $\cos(\arcsin x) = \cos\theta = \pm\sqrt{1-x^2}$. Since cosine is positive in both quadrants 1 and 4, we have $\cos(\arcsin x) = \sqrt{1-x^2}$.

37. Note, by Exercise 26, $\cos(x) = \dfrac{\pm 1}{\sqrt{\tan^2(x)+1}}$.

Since $\arctan x$ is an angle in quadrant 1 or 4, and cosine is positive in both quadrants 1 and 4, we get

$$\cos(\arctan x) = \frac{1}{\sqrt{\tan^2(\arctan x)+1}} = \frac{1}{\sqrt{x^2+1}}.$$

38. Note, $\tan x = \pm\sqrt{\sec^2 x - 1}$. Then

$$\begin{aligned}\tan(\arccos x) &= \pm\sqrt{\sec^2(\arccos x)-1}\\ &= \pm\sqrt{1/x^2-1}\\ &= \pm\frac{\sqrt{1-x^2}}{x}.\end{aligned}$$

Observe that $\tan(\arccos x)$ is positive exactly when $x > 0$, and $\tan(\arccos x)$ is negative exactly when $x < 0$.

Thus, $\tan(\arccos x) = \dfrac{\sqrt{1-x^2}}{x}$.

39. Note, $\tan x = \pm\sqrt{\sec^2 x - 1}$ and $\cos(\arcsin x) = \sqrt{1-x^2}$. Then

$$\begin{aligned}\tan(\arcsin x) &= \pm\sqrt{\sec^2(\arcsin x)-1}\\ &= \pm\sqrt{\left(\frac{1}{\sqrt{1-x^2}}\right)^2-1}\\ &= \pm\sqrt{\frac{1}{1-x^2}-1}\\ &= \pm\sqrt{\frac{x^2}{1-x^2}}\\ &= \pm\frac{\sqrt{x^2}}{\sqrt{1-x^2}}\\ &= \pm\frac{\pm x}{\sqrt{1-x^2}}\\ &= \pm\frac{x}{\sqrt{1-x^2}}.\end{aligned}$$

Note, $\tan(\arcsin x)$ is positive exactly when $x > 0$, and $\tan(\arcsin x)$ is negative exactly when $x < 0$. Thus, $\tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}}$.

40. $\sec(\arcsin x) = \dfrac{1}{\cos(\arcsin x)} = \dfrac{1}{\sqrt{1-x^2}}$

41. Note, $\arctan x$ is an angle in quadrant 1 or 4, and secant is positive in both quadrants 1 and 4. Since $\sec(\theta) = \pm\sqrt{\tan^2(\theta)+1}$, we have $\sec(\arctan x) = \sqrt{\tan^2(\arctan x)+1} = \sqrt{x^2+1}$.

42. $\csc(\arcsin x) = \dfrac{1}{\sin(\arcsin x)} = \dfrac{1}{x}$

43. $(-\sin x)\cdot(-\cot x) = \sin(x)\cdot\dfrac{\cos x}{\sin x} = \cos(x)$

44. $\sec x - \sec x = 0$ **45.** $\sin(y) + (-\sin(y)) = 0$

46. $\cos y + \cos y = 2\cos y$

47. $\dfrac{\sin(x)}{\cos(x)} + \dfrac{-\sin(x)}{\cos(x)} = 0$

48. $\dfrac{\cos(x)}{-\sin(x)} - \dfrac{\cos(x)}{\sin(x)} = -2\cot(x)$

49. $(1+\sin\alpha)(1-\sin\alpha) = 1-\sin^2\alpha = \cos^2\alpha$

50. $(1-\cos\alpha)(1+\cos\alpha) = 1-\cos^2\alpha = \sin^2\alpha$

51. $(-\sin\beta)(\cos\beta)(1/\sin\beta) = -\cos\beta$

52. $(-\tan\beta)\left(-\dfrac{1}{\sin\beta}\right)\cos\beta = \tan\beta\cot\beta = 1$

53. Odd, since $\sin(-y) = -\sin(y)$ for any y, even if $y = 2x$.

54. Even, since $\cos(-y) = \cos(y)$ for any y, even if $y = 2x$.

55. Neither, since $f(-\pi/6) \neq f(\pi/6)$ and $f(-\pi/6) \neq -f(\pi/6)$.

56. Odd, $f(-x) = 2\sin(-x)\cos(-x) = 2(-\sin(x))\cos(x) = -f(x)$.

57. Even, since $\sec^2(-t) - 1 = \sec^2(t) - 1$.

58. Neither, since $f(-\pi/6) \neq f(\pi/6)$ and $f(-\pi/6) \neq -f(\pi/6)$.

59. Even, $f(-\alpha) = 1+\sec(-\alpha) = 1+\sec(\alpha) = f(\alpha)$

60. Neither, since $f(-\pi/6) \neq f(\pi/6)$ and $f(-\pi/6) \neq -f(\pi/6)$.

61. Even, $f(-x) = \dfrac{\sin(-x)}{-x} = \dfrac{-\sin(x)}{-x} = f(x)$

62. Odd, $f(-x) = (-x)\cos(-x) = -x\cos(x) = -f(x)$

63. Odd, $f(-x) = -x+\sin(-x) = -x-\sin(x) = -f(x)$

64. Even, $f(-x) = \csc((-x)^2) = \csc(x^2) = f(x)$

65. h, since $\dfrac{1}{\csc(x)} = \dfrac{1}{1/\sin(x)} = 1\cdot\dfrac{\sin(x)}{1} = \sin(x)$

66. g, for $\dfrac{1}{\sec(x)} = \dfrac{1}{1/\cos(x)} = \cos(x)$

67. n **68.** o **69.** m **70.** i

71. k, since $\sin^2(x)+\cos^2(x) = 1$

72. j, since $\sin^2(x)+\cos^2(x) = 1$

73. l, since $\sec^2(x) = 1+\tan^2(x)$

74. a, since $\sin(x)$ is an odd function

75. g, since $\cos(-x) = \cos(x)$ and $\cos(x) = \dfrac{1}{\sec(x)}$

76. c, since $\tan(-x) = \dfrac{\sin(-x)}{\cos(-x)} = \dfrac{-\sin(x)}{\cos(x)} = -\tan(x)$

77. b, since $\cot(-x) = \dfrac{\cos(-x)}{\sin(-x)} = \dfrac{\cos(x)}{-\sin(x)} = -\cot(x)$

78. m, since $\sec(-x) = \dfrac{1}{\cos(-x)} = \dfrac{1}{\cos(x)}$

79. f, since $\csc(-x) = \dfrac{1}{\sin(-x)} = \dfrac{1}{-\sin(x)} = -\csc(x)$

80. p, since $\csc^2(x) = 1+\cot^2(x)$

81. d, since $\sec^2(x) = 1+\tan^2(x)$

82. e, since $\csc^2(x) = 1 + \cot^2(x)$

83. It is not an identity. If $\gamma = \pi/3$ then $(\sin(\pi/3) + \cos(\pi/3))^2 = (\sqrt{3}/2 + 1/2)^2 = \dfrac{(\sqrt{3}+1)^2}{4} = \dfrac{4+2\sqrt{3}}{4} \neq 1 = \sin^2(\pi/3) + \cos^2(\pi/3)$.

84. It is not an identity. If $x = \pi/4$ then $\tan^2(\pi/4) - 1 = 1 - 1 = 0$ and $\sec^2(\pi/4) = (\sqrt{2})^2 = 2$.

85. It is not an identity. If $\beta = \pi/6$ then $(1 + \sin(\pi/6))^2 = (1 + 1/2)^2 = (3/2)^2 = 9/4$ and $1 + \sin^2(\pi/6) = 1 + (1/2)^2 = 5/4$.

86. It is not an identity. If $\alpha = \pi/6$ then $\sin(2\pi/6) = \sin(\pi/3) = \sqrt{3}/2$ and $\sin(\pi/6)\cos(\pi/6) = (1/2)\cdot(\sqrt{3}/2) = \sqrt{3}/4$.

87. It is not an identity. If $\alpha = 7\pi/6$ then $\sin(7\pi/6) = -1/2$ while $\sqrt{1 - \cos^2(7\pi/6)}$ is a positive number.

88. It is not an identity. If $\alpha = 3\pi/4$ then $\tan(3\pi/4) = -1$ while $\sqrt{\sec^2(3\pi/4) - 1}$ is a positive number.

89. It is not an identity. If $y = \pi/6$ then $\sin(\pi/6) = 1/2$ and $\sin(-\pi/6) = -1/2$.

90. It is not an identity. If $y = \pi/3$ then $\cos(-\pi/3) = 1/2$ and $-\cos(\pi/3) = -1/2$.

91. It is not an identity. If $y = \pi/6$ then $\cos^2(\pi/6) - \sin^2(\pi/6) = (\sqrt{3}/2)^2 - (1/2)^2 = 3/4 - 1/4 = 1/2$ and $\sin(2\cdot\pi/6) = \sin\pi/3 = \sqrt{3}/2$.

92. It is not an identity. If $x = \pi/6$ then $\cos(2\cdot\pi/6) = \cos\pi/3 = 1/2$ and $2\cos(\pi/6)\sin(\pi/6) = 2(\sqrt{3}/2)(1/2) = \sqrt{3}/2$.

93. $1 - \dfrac{1}{\cos^2(x)} = 1 - \sec^2(x) = -\tan^2(x)$

94. $\dfrac{\sin^2 x(\sin^2 x - 1)}{1/\cos x} = \sin^2 x(-\cos^2 x)\cdot\cos x = -\sin^2(x)\cos^3(x)$

95. $\dfrac{-(\tan^2 t + 1)}{\sec^2 t} = \dfrac{-\sec^2 t}{\sec^2 t} = -1$

96. $\dfrac{\cos w(\sin^2 w + \cos^2 w)}{\sec w} = \dfrac{\cos(w)\cdot 1}{\sec w} = \cos^2 w$

97. $\dfrac{(1 - \cos^2 w) - \cos^2 w}{1 - 2\cos^2 w} = \dfrac{1 - 2\cos^2 w}{1 - 2\cos^2 w} = 1$

98. $\dfrac{-\sin^3\theta}{\sin\theta(\sin^2\theta - 1)} = \dfrac{-\sin^2\theta}{(-\cos^2\theta)} = \tan^2\theta$

99. $\dfrac{\tan x(\tan^2 x - \sec^2 x)}{-\cot x} = \dfrac{\tan x(-1)}{-\cot x} = \tan^2 x$

100. $\dfrac{\sin^2 x + \cos^2 x}{\sin x} = \dfrac{1}{\sin x} = \csc x$

101.
$$\frac{1}{\sin^3 x} - \frac{\cos^2(x)/\sin^2(x)}{\sin x} = \frac{1}{\sin^3 x} - \frac{\cos^2(x)}{\sin^3 x} = \frac{1 - \cos^2 x}{\sin^3 x} = \frac{\sin^2 x}{\sin^3 x} = \frac{1}{\sin x} = \csc x$$

102.
$$1 - \frac{1/\cos^2 x}{\sin^2 x/\cos^2 x} = 1 - \frac{1}{\cos^2 x}\cdot\frac{\cos^2 x}{\sin^2 x} = 1 - \frac{1}{\sin^2 x} = 1 - \csc^2 x = -\cot^2 x$$

103. $(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) = (\sin^2 x - \cos^2 x)(1) = \sin^2 x - \cos^2 x$

104. $(\csc^2 x + \cot^2 x)(\csc^2 x - \cot^2 x) = (\csc^2 x + \cot^2 x)(1) = \csc^2 x + \cot^2 x$

105. $\cos\theta = \pm\sqrt{1 - \sin^2\theta} = \pm\sqrt{1 - (1/3)^2} = \pm\sqrt{1 - 1/9} = \pm\sqrt{\dfrac{8}{9}} = \pm\dfrac{2\sqrt{2}}{3}$

106. Note $\sec\theta = \dfrac{5}{4}$.

Then $\tan\theta = \pm\sqrt{\sec^2\theta - 1} = \pm\sqrt{(5/4)^2 - 1} = \pm\sqrt{25/16 - 1} = \pm\sqrt{\dfrac{9}{16}} = \pm\dfrac{3}{4}$.

107. $\cos\theta = \pm\sqrt{1 - \sin^2\theta} = \pm\sqrt{1 - u^2}$

108. Note $\sec\theta = \dfrac{1}{u}$.

Then $\tan\theta = \pm\sqrt{\sec^2\theta - 1} = \pm\sqrt{\left(\dfrac{1}{u}\right)^2 - 1} = \pm\sqrt{\dfrac{1 - u^2}{u^2}} = \pm\dfrac{\sqrt{1 - u^2}}{\sqrt{u^2}} =$

$$\pm\frac{\sqrt{1-u^2}}{\pm u} = \pm\frac{\sqrt{1-u^2}}{u}.$$

109. Note, $\tan x = \dfrac{\sin x}{\cos x}$ is not valid if $\cos x = 0$. Thus, the identity is not valid if $x = \dfrac{\pi}{2} + k\pi$ where k is an integer.

110. Note, $\cot x = \dfrac{\cos x}{\sin x}$ is not valid if $\sin x = 0$. Thus, the identity is not valid if $x = k\pi$ where k is an integer.

113. Let h be the height of the building. Using right triangle trigonometry, we find

$$h = 2000 \tan 30^\circ \approx 1155 \text{ ft}.$$

114. Let α be the central angle. Using the formula $s = r\alpha$, we obtain

$$\begin{aligned} 5 &= 60\alpha \\ \frac{5}{60} \text{ radians} &= \alpha \\ \frac{5}{60} \cdot \frac{180^\circ}{\pi} \text{ degrees} &= \alpha \\ 4.8^\circ &= \alpha \end{aligned}$$

115. The amplitude is 5.

Since $B = 2$, the period is $\dfrac{2\pi}{B} = \dfrac{2\pi}{2} = \pi$.

Since $2x - \pi = 2\left(x - \dfrac{\pi}{2}\right)$, phase shift is $\dfrac{\pi}{2}$.

The range is the interval $[-5+3, 5+3] = [-2, 8]$

116. The period and frequency are reciprocals of each other. Then the frequency is

$$\frac{1}{0.125} = 8 \text{ cycles/sec}$$

117. $\cos\beta = 0$, for $\cos^2\beta + \sin^2\beta = 1$

118. **a)** $\sin^{-1}\left(-\dfrac{1}{2}\right) = -30^\circ$

b) $\cos^{-1}\left(-\dfrac{1}{2}\right) = 120^\circ$

c) $\tan^{-1}(-1) = -45^\circ$

Thinking Outside the Box LII

Let r be the radius of the small circle, and let x be the distance from the center of the small circle to the point of tangency of any two circles with radius 1.

By the Pythagorean theorem, we find

$$1 + (x+r)^2 = (1+r)^2$$

and

$$1 + (1 + 2r + x)^2 = 2^2.$$

The second equation may be written as

$$1 + (r+1)^2 + 2(r+1)(r+x) + (r+x)^2 = 4.$$

Using the first equation, the above equation simplifies to

$$(r+1)^2 + 2(r+1)(r+x) + (1+r)^2 = 4$$

or

$$(r+1)^2 + (r+1)(r+x) = 2.$$

Since (from first equation, again)

$$x + r = \sqrt{(1+r)^2 - 1}$$

we obtain

$$(r+1)^2 + (r+1)\left(\sqrt{(1+r)^2 - 1}\right) = 2.$$

Solving for r, we find

$$r = \frac{2\sqrt{3} - 3}{3}.$$

6.1 Pop Quiz

1. $\dfrac{\cos x}{\sin x}\dfrac{1}{\cos x} = \dfrac{1}{\sin x} = \csc x$

2. $\cos\alpha = \sqrt{1 - \left(\dfrac{1}{3}\right)^2} = \sqrt{\dfrac{8}{9}} = \dfrac{2\sqrt{2}}{3}$

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \frac{2\sqrt{2}/3}{1/3} = 2\sqrt{2}$$

3. Even, since $\cos(3(-x)) = \cos(3x)$

4. Note, $\arcsin(w)$ is an angle in quadrant 1 or 4. Since cosine is positive in quadrant 1 or 4, we find $\cos(\arcsin(w)) = \sqrt{1-\sin^2(\arcsin(w))} = \sqrt{1-w^2}$.

5. $\dfrac{2}{1/\cos^2\alpha} + \dfrac{2}{1/\sin^2\alpha} = 2\left(\cos^2\alpha + \sin^2\alpha\right) = 2(1) = 2$

6.1 Linking Concepts

a) Since $x = 100(4)\cos 60° = 200$ and $y = -16(4)^2 + 100(4)\sin 60° = 200\sqrt{3} - 256$, after $t = 4$ the coordinates are $(200, 200\sqrt{3} - 256) \approx (200, 90.4)$.

b) Note, $y = -16t^2 + 100t\sin 60° = -16t^2 + 50\sqrt{3}t$. Set $y = 0$.

$$\begin{aligned} -t(16t - 50\sqrt{3}) &= 0 \\ t &= 0, \frac{50\sqrt{3}}{16} \\ t &= 0, \frac{25\sqrt{3}}{8} \end{aligned}$$

The projectile is in the air for $\dfrac{25\sqrt{3}}{8}$ seconds.

c) Using the answer from part b), we get

$$x = 100t\cos 60° = 50t = 50\left(\frac{25\sqrt{3}}{8}\right) \approx 270.6.$$

The projectile lands 270.6 feet from the gun.

d) The vertex of the function $y = -16t^2 + 50\sqrt{3}t$ (given by the height) can be shown to be $\left(\dfrac{25\sqrt{3}}{16}, \dfrac{1875}{16}\right) \approx (2.7, 117.2)$.

The maximum height is 117.2 feet.

e) If A is the time in the air which is the same as the number of seconds before the projectile lands, then

$$\begin{aligned} -16A^2 + v_o A\sin\theta &= 0 \\ -A(16A - v_o\sin\theta) &= 0. \end{aligned}$$

Then $A = \dfrac{v_o\sin\theta}{16}$.

The distance, d, from the gun to the point where the projectile lands is given by $d = x = v_o t\cos\theta = v_o A\cos\theta = v_o\left(\dfrac{v_o\sin\theta}{16}\right)\cos\theta = \dfrac{v_o^2}{16}\sin\theta\cos\theta$, i.e., $d = \dfrac{v_o^2}{16}\sin\theta\cos\theta$.

The t-coordinate of the vertex of $y = -16t^2 + v_o\sin(\theta)t$ (given by the height) is $\dfrac{-b}{2a} = \dfrac{-v_o\sin\theta}{-32} = \dfrac{v_o\sin\theta}{32}$. The maximum height, y_{max}, is given by

$$\begin{aligned} y_{max} &= -16\left(\frac{v_o\sin\theta}{32}\right)^2 + v_o\sin(\theta)\left(\frac{v_o\sin\theta}{32}\right) \\ &= -\frac{v_o^2\sin^2\theta}{64} + \frac{v_o^2\sin^2\theta}{32} \\ &= \frac{v_o^2\sin^2\theta}{64} \end{aligned}$$

f) Since $x = v_o t\cos\theta$, we find $t = \dfrac{x}{v_o\cos\theta}$. Then

$$\begin{aligned} y &= -16t^2 + v_o t\sin\theta \\ &= -16\left(\frac{x}{v_o\cos\theta}\right)^2 + v_o\left(\frac{x}{v_o\cos\theta}\right)\sin\theta \\ y &= -\frac{16\sec^2\theta}{v_o^2}x^2 + x\tan\theta. \end{aligned}$$

For Thought

1. True, $\dfrac{\sin x}{1/\sin x} = \sin x \cdot \dfrac{\sin x}{1} = \sin^2 x$.

2. False, if $x = \pi/3$ then $\dfrac{\cot(\pi/3)}{\tan(\pi/3)} = \dfrac{\sqrt{3}/3}{\sqrt{3}} = \dfrac{1}{3}$ and $\tan^2(\pi/3) = (\sqrt{3})^2 = 3$.

3. True, $\dfrac{1/\cos x}{1/\sin x} = \dfrac{1}{\cos x} \cdot \dfrac{\sin x}{1} = \dfrac{\sin x}{\cos x} = \tan x$.

4. True, $\sin x \cdot \dfrac{1}{\cos x} = \dfrac{\sin x}{\cos x} = \tan x$.

5. True, $\dfrac{\cos x}{\cos x} + \dfrac{\sin x}{\cos x} = 1 + \tan x$.

6. False, if $x = \pi/4$ then
$\sec(\pi/4) + \dfrac{\sin(\pi/4)}{\cos(\pi/4)} = \sqrt{2} + 1$ and
$\dfrac{1 + \sin(\pi/4)\cos(\pi/4)}{\cos(\pi/4)} = \dfrac{1 + (\sqrt{2}/2)(\sqrt{2}/2)}{\sqrt{2}/2} =$
$\dfrac{1 + (1/2)}{\sqrt{2}/2} = (3/2)(2/\sqrt{2}) = 3/\sqrt{2}.$

7. True, $\dfrac{1 + \sin x}{1 - \sin^2 x} = \dfrac{1 + \sin x}{(1 - \sin x)(1 + \sin x)} =$
$\dfrac{1}{1 - \sin x}.$

8. True, since $\tan x \cdot \cot x = \tan x \cdot \dfrac{1}{\tan x} = 1.$

9. False, if $x = \pi/3$ then $(1 - \cos(\pi/3))^2 =$
$(1 - 1/2)^2 = (1/2)^2 = 1/4$ and
$\sin^2(\pi/3) = (\sqrt{3}/2)^2 = 3/4.$

10. False, if $x = \pi/6$ then
$(1 - \csc(\pi/6))(1 + \csc(\pi/6)) = (1 - 2)(1 + 2) =$
-3 and $\cot^2(\pi/6) = (\sqrt{3})^2 = 3.$

6.2 Exercises

1. D, $\cos x \tan x = \cos x \cdot \dfrac{\sin x}{\cos x} = \sin x$.

2. I, $\sec x \cot x = \dfrac{1}{\cos x} \cdot \dfrac{\cos x}{\sin x} = \dfrac{1}{\sin x} = \csc x$.

3. A, $\csc^2 x - \cot^2 x = 1$.

4. J, $\dfrac{\sin x}{\sin x} + \dfrac{\cos x}{\sin x} = 1 + \cot x$.

5. B, $1 - \sec^2 x = -\tan^2 x$.

6. C, $\csc^2 x - 1 = \cot^2 x$.

7. H, $\dfrac{\csc x}{\csc x} - \dfrac{\sin x}{\csc x} = 1 - \sin^2 x = \cos^2 x$.

8. E, $\dfrac{\cos x}{\sec x} - \dfrac{\sec x}{\sec x} = \cos^2 x - 1 = -\sin^2 x$.

9. G, $\csc^2 x = 1 + \cot^2 x$.

10. F, $\dfrac{\sin^2 x + \cos^2 x}{\cos x \sin x} = \dfrac{1}{\cos x \sin x}$.

11. $2\cos^2 \beta - \cos\beta - 1$

12. $2\csc^2 \beta - 7\csc\beta + 3$

13. $\csc^2 x + 2\csc x \sin x + \sin^2 x = \csc^2 x + 2 + \sin^2 x$

14. $4\cos^2 x - 4\cos x \sec x + \sec^2 x =$
$4\cos^2 x - 4 + \sec^2 x$

15. $4\sin^2\theta - 1$ **16.** $9\sec^2\theta - 4$

17. $9\sin^2\theta + 12\sin\theta + 4$ **18.** $9\cos^2\theta - 12\cos\theta + 4$

19. $4\sin^4 y - 4\sin^2 y \csc^2 y + \csc^4 y =$
$4\sin^4 y - 4 + \csc^4 y$

20. $\tan^4 y + 2\tan^2 y \cot^2 y + \cot^4 y =$
$\tan^4 y + 2 + \cot^4 y$

21. Note the factorization of a difference of two squares: $(1 - \sin\alpha)(1 + \sin\alpha) = 1 - \sin^2\alpha = \cos^2\alpha$.

22. Note the factorization of a difference of two squares: $(1 - \cos\alpha)(1 + \cos\alpha) = 1 - \cos^2\alpha = \sin^2\alpha$.

23. Note the factorization of a difference of two squares: $(\csc\alpha - 1)(\csc\alpha + 1) = \csc^2\alpha - 1 = \cot^2\alpha$.

24. Note the factorization of a difference of two squares: $(\sec\alpha - 1)(\sec\alpha + 1) = \sec^2\alpha - 1 = \tan^2\alpha$.

25. Note the factorization of a difference of two squares: $(\tan\alpha - \sec\alpha)(\tan\alpha + \sec\alpha) = \tan^2\alpha - \sec^2\alpha = -1$.

26. Note the factorization of a difference of two squares: $(\cot\alpha - \csc\alpha)(\cot\alpha + \csc\alpha) = \cot^2\alpha - \csc^2\alpha = -1$.

27. $(2\sin\gamma + 1)(\sin\gamma - 3)$

28. $(\cos\gamma - 3)(\cos\gamma + 2)$

29. $(\tan\alpha - 4)(\tan\alpha - 2)$

30. $(2\cot\alpha + 3)(\cot\alpha - 1)$

31. $(2\sec\beta + 1)^2$ **32.** $(3\csc\theta - 2)^2$

33. $(\tan\alpha - \sec\beta)(\tan\alpha + \sec\beta)$

34. $(\sin^2 y - \cos^2 x)(\sin^2 y + \cos^2 x) =$
$(\sin y - \cos x)(\sin y + \cos x)(\sin^2 y + \cos^2 x)$

35. $\cos\beta\left(\sin^2\beta+\sin\beta-2\right)=$
$\cos\beta\left(\sin\beta+2\right)\left(\sin\beta-1\right)$

36. $\tan\theta\left(\cos^2\theta-2\cos\theta-3\right)=$
$\tan\theta\left(\cos\theta-3\right)\left(\cos\theta+1\right)$

37. $\left(2\sec^2 x-1\right)^2$

38. $\left(\cos^2 x-1\right)^2=$
$[(\cos x-1)(\cos x+1)]^2=$
$(\cos x-1)^2(\cos x+1)^2$

39. $\cos\alpha(\sin\alpha+1)+(\sin\alpha+1)=$
$(\sin\alpha+1)\left(\cos\alpha+1\right)$

40. $\sin\theta\left(2\sin\theta+1\right)-\cos\theta\left(2\sin\theta+1\right)=$
$\left(\sin\theta-\cos\theta\right)\left(2\sin\theta+1\right)$

41. Combining, we get $\dfrac{1-\cos^2 x}{a}=\dfrac{\sin^2 x}{a}$.

42. Combining, we get $\dfrac{1-\sin^2 x}{\cos x}=\dfrac{\cos^2 x}{\cos x}=\cos x$.

43. We obtain $\dfrac{\sin(2x)}{2}+\dfrac{2\sin(2x)}{2}=\dfrac{3\sin(2x)}{2}$.

44. We obtain $\dfrac{2\cos(2x)}{2}-\dfrac{\cos(2x)}{2}=\dfrac{\cos(2x)}{2}$.

45. Since 6 is the LCD, we get $\dfrac{2\tan x}{6}+\dfrac{3\tan x}{6}=\dfrac{5\tan x}{6}$.

46. Since $3b$ is the LCD, we get $\dfrac{3\sin x}{3b}+\dfrac{\sin x}{3b}=\dfrac{4\sin x}{3b}$.

47. Separating the fraction, we obtain
$\dfrac{\sin x}{\sin x}-\dfrac{\sin^2 x}{\sin x}=1-\sin x$.

48. Factoring, we get $\dfrac{\cos x(\cos^2 x-1)}{-\cos x}=$
$\dfrac{\cos x(-\sin^2 x)}{-\cos x}=\sin^2 x$.

49. Factoring: $\dfrac{(\sin x-\cos x)(\sin x+\cos x)}{\sin x-\cos x}$
$=\sin x+\cos x$.

50. Factoring: $\dfrac{(1-\cos x)(1+\cos x)}{1-\cos x}$
$=1+\cos x$.

51. Factoring: $\dfrac{(\sin x-2)(\sin x+1)}{(\sin x-2)(\sin x+2)}$
$=\dfrac{\sin x+1}{\sin x+2}$.

52. Note, $\tan(-x)=-\tan(x)$ and
$\dfrac{\tan x-1}{1-\tan x}=-1$. Then
$\dfrac{(\tan x-1)^2}{1-\tan x}=-(\tan x-1)=1-\tan x$.

53. Note, $\sin(-x)=-\sin(x)$. Factoring, we obtain $\dfrac{\sin^2 x+\sin x}{1+\sin x}=\dfrac{\sin x(\sin x+1)}{1+\sin x}=\sin x$.

54. Note, $\cos(-x)=\cos(x)$. Factoring, we obtain
$\dfrac{\cos^2 x-\cos x}{1-\cos x}=\dfrac{\cos x(\cos x-1)}{1-\cos x}=$
$\dfrac{\cos x(\cos x-1)}{-(\cos x-1)}=-\cos x$.

55.

$$\begin{aligned}\sin x\cot x &=\\ \sin x\frac{\cos x}{\sin x} &=\\ \cos x\end{aligned}$$

56.

$$\begin{aligned}\cos^2 x\tan^2 x &=\\ \cos^2 x\frac{\sin^2 x}{\cos^2 x} &=\\ \sin^2 x\end{aligned}$$

57.

$$\begin{aligned}1-\sec x\cos^3 x &=\\ 1-\frac{1}{\cos x}\cos^3 x &=\\ 1-\cos^2 x &=\\ \sin^2 x\end{aligned}$$

58.

$$\begin{aligned} 1-\csc x\sin^3 x &= \\ 1-\frac{1}{\sin x}\sin^3 x &= \\ 1-\sin^2 x &= \\ \cos^2 x \end{aligned}$$

59.

$$\begin{aligned} 1+\sec^2 x\sin^2 x &= \\ 1+\frac{1}{\cos^2 x}\sin^2 x &= \\ 1+\tan^2 x &= \\ \sec^2 x \end{aligned}$$

60.

$$\begin{aligned} 1+\csc^2 x\cos^2 x &= \\ 1+\frac{1}{\sin^2 x}\cos^2 x &= \\ 1+\cot^2 x &= \\ \csc^2 x \end{aligned}$$

61.

$$\begin{aligned} \frac{\sin^3 x+\sin x\cos^2 x}{\cos x} &= \\ \frac{\sin x(\sin^2 x+\cos^2 x)}{\cos x} &= \\ \frac{(\sin x)(1)}{\cos x} &= \\ \tan x \end{aligned}$$

62.

$$\begin{aligned} \frac{\cos x\sin^2 x+\cos^3 x}{\sin x} &= \\ \frac{\cos x(\sin^2 x+\cos^2 x)}{\sin x} &= \\ \frac{(\cos x)(1)}{\sin x} &= \\ \cot x \end{aligned}$$

63.

$$\begin{aligned} \frac{\sin x}{\csc x}+\frac{\cos x}{\sec x} &= \\ \frac{\sin x}{1/\sin x}+\frac{\cos x}{1/cosx} &= \\ \sin^2 x+\cos^2 x &= \\ 1 \end{aligned}$$

64.

$$\begin{aligned} \sin^3 x\csc x+\cos^3 x\sec x &= \\ \sin^3\frac{1}{\sin x}+\cos^3 x\frac{1}{\cos x} &= \\ \sin^2 x+\cos^2 x &= \\ 1 \end{aligned}$$

65.

$$\begin{aligned} \frac{1}{\csc\theta-\cot\theta}\cdot\frac{\sin\theta}{\sin\theta} &= \\ \frac{\sin\theta}{1-\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} &= \\ \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} &= \\ \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} &= \\ \frac{1+\cos\theta}{\sin\theta} \end{aligned}$$

66.

$$\begin{aligned} \frac{-1}{\tan\theta-\sec\theta}\cdot\frac{\cos\theta}{\cos\theta} &= \\ \frac{-\cos(\theta)}{\sin(\theta)-1}\cdot\frac{\sin\theta+1}{\sin\theta+1} &= \\ \frac{(-\cos\theta)(\sin\theta+1)}{\sin^2\theta-1} &= \\ \frac{(-\cos\theta)(\sin\theta+1)}{-\cos^2\theta} &= \\ \frac{1+\sin(\theta)}{\cos\theta} \end{aligned}$$

67.

$$\begin{aligned} \frac{\sec x - \cos x}{\sec x} &= \\ 1 - \frac{\cos x}{\sec x} &= \\ 1 - \cos^2 x &= \\ \sin^2 x \end{aligned}$$

68.

$$\begin{aligned} \frac{\sec x - \cos x}{\cos x} &= \\ \frac{\sec x}{\cos x} - 1 &= \\ \sec^2 x - 1 &= \\ \tan^2 x \end{aligned}$$

69.

$$\begin{aligned} &= \frac{1 - (-\sin x)^2}{1 + \sin x} \\ &= \frac{1 - \sin^2 x}{1 + \sin x} \\ &= \frac{(1 - \sin x)(1 + \sin x)}{1 + \sin x} \\ & 1 - \sin(x) \end{aligned}$$

70.

$$\begin{aligned} &= \frac{1 - 1 + \sin^2(x)}{\cos^2(x)} \\ &= \frac{\sin^2(x)}{\cos^2(x)} \\ & \tan^2(x) \end{aligned}$$

71.

$$\begin{aligned} &= \frac{1 - \cot^2 w\,(1 - \cos^2 w)}{\csc^2 w} \\ &= \frac{1 - \cot^2 w \sin^2 w}{\csc^2 w} \\ &= \frac{1 - \cos^2 w}{\csc^2 w} \\ &= \frac{\sin^2 w}{\csc^2 w} \\ & \sin^4 w \end{aligned}$$

72.

$$\begin{aligned} &= \frac{\sec^2 z - \csc^2 z\,(1 - \cos^2 z)}{\cot^2 z} \\ &= \frac{\sec^2 z - \csc^2 z \sin^2 z}{\cot^2 z} \\ &= \frac{\sec^2 z - 1}{\cot^2 z} \\ &= \frac{\tan^2 z}{\cot^2 z} \\ & \tan^4 z \end{aligned}$$

73.

$$\begin{aligned} &= \frac{\cos x + \csc x}{\cos x} \\ &= \frac{\cos x}{\cos x} + \frac{\csc x}{\cos x} \\ & 1 + \csc x \sec x \end{aligned}$$

74.

$$\begin{aligned} \tan^2(-x) - \frac{-\sin x}{\sin x} &= \\ \tan^2 x + 1 &= \\ \sec^2 x \end{aligned}$$

75. Rewrite the left side of the equation.

$$\begin{aligned} \tan(x)\cos(x) + \csc(x)\sin^2(x) &= \\ \sin x + \sin x &= \\ 2\sin x \end{aligned}$$

76.

$$\begin{aligned} \cot(x)\sin(x) - \cos^2(x)\sec(x) &= \\ \cos x - \cos x &= \\ 0 \end{aligned}$$

77.

$$\begin{aligned} (1 + \sin\alpha)^2 + \cos^2\alpha &= \\ 1 + 2\sin\alpha + \sin^2\alpha + \cos^2\alpha &= \\ 2 + 2\sin\alpha \end{aligned}$$

78.

$$\begin{aligned}\left(1+2\cot\alpha+\cot^2\alpha\right)-2\cot\alpha &= \\ 1+\cot^2\alpha &= \\ \csc^2\alpha &= \\ \frac{1}{\sin^2\alpha} &= \\ \frac{1}{1-\cos^2\alpha} &= \\ \frac{1}{(1-\cos\alpha)(1+\cos\alpha)}\end{aligned}$$

79.

$$\begin{aligned}\frac{\sin^2\beta+\sin\beta-2}{2\sin\beta-2} &= \\ \frac{(\sin\beta+2)(\sin\beta-1)}{2(\sin\beta-1)} &= \\ \frac{\sin\beta+2}{2}\end{aligned}$$

80.

$$\begin{aligned}\frac{4\sec^2\beta+4\sec\beta+1}{2\sec\beta+1} &= \\ \frac{(2\sec\beta+1)^2}{\sec\beta+1} &= \\ 2\sec\beta+1 &= \\ \frac{2}{\cos\beta}+1\end{aligned}$$

81.

$$\begin{aligned}2-\csc(\beta)\sin(\beta) &= \\ 2-1 &= \\ 1 &= \\ \sin^2(\beta)+\cos^2(\beta)\end{aligned}$$

82.

$$\begin{aligned}\left(1-\sin^2\beta\right)\left(1+\sin^2\beta\right) &= \\ \cos^2(\beta)\left(2-\cos^2(\beta)\right) &= \\ 2\cos^2\beta-\cos^4\beta\end{aligned}$$

83.

$$\begin{aligned}\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x} &= \\ \frac{\sin^2 x+\cos^2 x}{\sin(x)\cos(x)} &= \\ \frac{1}{\sin(x)\cos(x)} &= \\ \sec(x)\csc(x)\end{aligned}$$

84.

$$\begin{aligned}\frac{\csc x}{\cot x}-\frac{\cot x}{\csc x} &= \\ \frac{\csc^2 x-\cot^2 x}{\cot(x)\csc(x)} &= \\ \frac{1}{\cot(x)\csc(x)} &= \\ \frac{\tan x}{\csc x}\end{aligned}$$

85.

$$\begin{aligned}\frac{\sec(x)}{\tan(x)}-\frac{\tan(x)}{\sec(x)} &= \\ \frac{\sec^2(x)-\tan^2(x)}{\tan(x)\sec(x)} &= \\ \frac{1}{\tan(x)\sec(x)} &= \\ \cot(x)\cos(x)\end{aligned}$$

86.

$$\begin{aligned}\frac{(1-\sin x)(1+\sin x)}{1-\sin x} &= \\ 1+\sin(x) &= \\ \frac{\csc x}{\csc x}+\frac{1}{\csc x} &= \\ \frac{\csc x+1}{\csc x}\end{aligned}$$

87. Rewrite the right side of the equation.

$$\begin{aligned}&= \frac{\csc x}{\csc x-\sin x}\cdot\frac{\sin x}{\sin x} \\ &= \frac{1}{1-\sin^2 x}\end{aligned}$$

$$= \frac{1}{\cos^2 x}$$
$$\sec^2 x$$

88.

$$= \frac{\csc x - 1}{\cot^2 x}$$
$$= \frac{\csc x - 1}{\csc^2 x - 1}$$
$$= \frac{\csc x - 1}{(\csc x - 1)(\csc x + 1)}$$
$$= \frac{1}{\csc x + 1} \cdot \frac{\sin x}{\sin x}$$
$$\frac{\sin x}{\sin x + 1}$$

89.

$$= \frac{1 + \sin(y)}{1 - \sin(y)} \cdot \frac{\csc(y)}{\csc(y)}$$
$$\frac{\csc(y) + 1}{\csc(y) - 1}$$

90.

$$= \frac{\sin y + \cos y}{\sin y - \cos y} \cdot \frac{\sin y - \cos y}{\sin y - \cos y}$$
$$= \frac{\sin^2 y - \cos^2 y}{\sin^2 y - 2\sin y \cos y + \cos^2 y}$$
$$= \frac{(1 - \cos^2 y) - \cos^2 y}{1 - 2\sin y \cos y}$$
$$\frac{1 - 2\cos^2 y}{1 - 2\cos y \sin y}$$

91.

$$\ln(\sec\theta) =$$
$$\ln((\cos\theta)^{-1}) =$$
$$-\ln(\cos\theta)$$

92.

$$\ln(\tan\theta) =$$
$$\ln\left(\frac{\sin\theta}{\cos\theta}\right) =$$
$$\ln(\sin\theta) - \ln(\cos\theta) =$$
$$\ln(\sin\theta) + \ln((\cos\theta)^{-1}) =$$
$$\ln(\sin\theta) + \ln(\sec\theta)$$

93.

$$\ln\left|(\sec\alpha + \tan\alpha) \cdot \frac{\sec\alpha - \tan\alpha}{\sec\alpha - \tan\alpha}\right| =$$
$$\ln\left|\frac{\sec^2\alpha - \tan^2\alpha}{\sec\alpha - \tan\alpha}\right| =$$
$$\ln\left|\frac{1}{\sec\alpha - \tan\alpha}\right| =$$
$$-\ln|\sec\alpha - \tan\alpha|$$

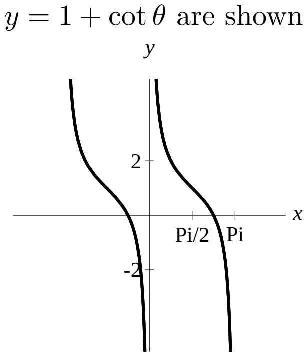
94.

$$\ln\left|(\csc\alpha + \cot\alpha) \cdot \frac{\csc\alpha - \cot\alpha}{\csc\alpha - \cot\alpha}\right| =$$
$$\ln\left|\frac{\csc^2\alpha - \cot^2\alpha}{\csc\alpha - \cot\alpha}\right| =$$
$$\ln\left|\frac{1}{\csc\alpha - \cot\alpha}\right| =$$
$$-\ln|\csc\alpha - \cot\alpha|$$

95. It is an identity since

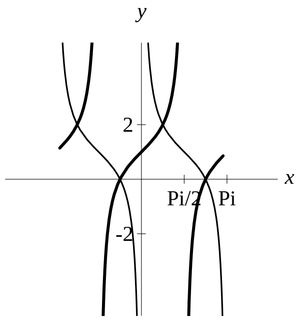
$$\frac{\sin\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta} =$$
$$1 + \cot\theta.$$

The graphs of $y = \dfrac{\sin\theta + \cos\theta}{\sin\theta}$ and $y = 1 + \cot\theta$ are shown to be identical.

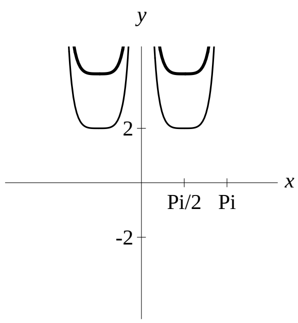


96. It is not an identity since the graphs of $y = \dfrac{\sin\theta + \cos\theta}{\cos\theta}$ and $y = 1 + \cot\theta$

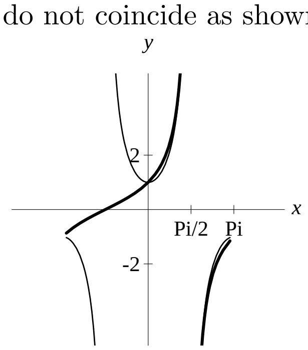
do not coincide as shown below.



97. It is not an identity since the graphs of $y = (\sin x + \csc x)^2$ and $y = \sin^2 x + \csc^2 x$ do not coincide as shown.



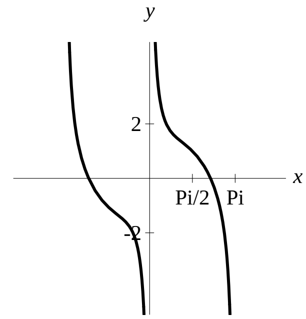
98. It is not an identity since the graphs of $y = \tan x + \sec x$ and $y = \dfrac{\sin^2 x + 1}{\cos x}$ do not coincide as shown.



99. It is an identity. Re-arranging the numerator of the right-hand side one finds

$$\begin{aligned} &= \frac{1-\cos^2 x + \cos x}{\sin x} \\ &= \frac{\sin^2 x + \cos x}{\sin x} \\ &= \frac{\sin^2 x}{\sin x} + \frac{\cos x}{\sin x} \\ &\sin x + \cot x. \end{aligned}$$

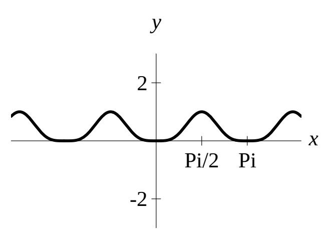
The graphs of $y = \cot x + \sin x$ and $y = \dfrac{1 + \cos x - \cos^2 x}{\sin x}$ are shown to be identical.



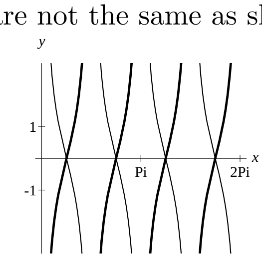
100. It is an identity. To see this, factor the left-hand side as follows

$$\begin{aligned} \left(1 - \cos^2 x\right)^2 &= \\ \left(\sin^2 x\right)^2 &= \\ \sin^4 x. \end{aligned}$$

The graphs of $y = 1 - 2\cos^2 x + \cos^4 x$ and $y = \sin^4 x$ are shown to be identical.



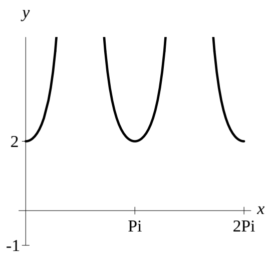
101. It is not an identity since the graphs of $y = \dfrac{\sin x}{\cos x} - \dfrac{\cos x}{\sin x}$ and $y = \dfrac{2\cos^2 x - 1}{\sin x \cos x}$ are not the same as shown.



102. It is an identity.

$$\begin{aligned}\frac{(1+\sin x)+(1-\sin x)}{(1-\sin x)(1+\sin x)} &= \\ \frac{2}{1-\sin^2(x)} &= \\ \frac{2}{\cos^2 x} &=\end{aligned}$$

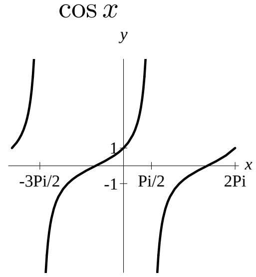
The graphs of $y = \dfrac{1}{1-\sin x} + \dfrac{1}{1+\sin x}$ and $y = \dfrac{2}{\cos^2 x}$ are shown to be identical.



103. It is an identity.

$$\begin{aligned}\frac{\cos x}{1-\sin(x)} \cdot \frac{1+\sin x}{1+\sin x} &= \\ \frac{\cos x(1+\sin x)}{1-\sin^2 x} &= \\ \frac{\cos x(1+\sin x)}{\cos^2 x} &= \\ \frac{1+\sin x}{\cos x} &= \\ \frac{1-\sin(-x)}{\cos x} &=\end{aligned}$$

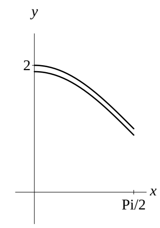
The graphs of $y = \dfrac{\cos(-x)}{1-\sin x}$ and $y = \dfrac{1-\sin(-x)}{\cos x}$ are shown to be identical.



104. It is not an identity since the graphs of

$$y = \frac{\sin^2 x}{1-\cos x} \text{ and } y = 0.99 + \cos x$$

do not coincide as shown below:



107. $\sin^2 x + \cos^2 x = 1$, $1 + \cot^2 x = \csc^2 x$, $\tan^2 x + 1 = \sec^2 x$

108. $\dfrac{1}{\cos^2 x} - \tan^2 x = \sec^2 x - \tan^2 x = -1$

109. $\dfrac{\csc x}{\sec x} = \dfrac{1/\sin x}{1/\cos x} = \dfrac{\cos x}{\sin x} = \cot x$

110. The midpoint is

$$\left(\frac{\pi/3+\pi/2}{2}, \frac{1+1}{2}\right) = \left(\frac{5\pi/6}{2}, \frac{2}{2}\right) = \left(\frac{5\pi}{12}, 1\right)$$

111. Amplitude is 4. Since $B = 2\pi/3$, the period is $\dfrac{2\pi}{B} = \dfrac{2\pi}{2\pi/3} = 3$.

Since $\dfrac{2\pi x}{3} - \dfrac{\pi}{3} = \dfrac{2\pi}{3}\left(x - \dfrac{1}{2}\right)$, the phase shift is $\dfrac{1}{2}$.

112. Since $B = 1/4$, the period is $\dfrac{2\pi}{B} = \dfrac{2\pi}{1/4} = 8\pi$.

Solve for x in $\dfrac{x}{4} = \dfrac{\pi}{2} + k\pi$. Then the asymptotes are

$$x = 2\pi + 4k\pi$$

where k is an integer.

The range is $(-\infty, -2] \cup [2, \infty)$.

113. Solve for v_0 in ft/sec:

$$\begin{aligned}\frac{1}{32} v_0^2 \sin 2(33^\circ) &= 200 \\ v_0 &= \sqrt{\frac{200(32)}{\sin 66^\circ}} \text{ ft/sec}\end{aligned}$$

$$\begin{aligned} v_0 &= \sqrt{\frac{200(32)}{\sin 66^\circ}} \cdot \frac{5280}{3600} \text{ mph} \\ v_0 &\approx 57.1 \text{ mph.} \end{aligned}$$

Thinking Outside the Box LIII

The amplitude of the sine wave is 1/2 since the height of the sine wave is 1. We use a coordinate system such that the sine wave begins at the origin and extends to the right side and the first quadrant. Note, the period of the sine wave is π, which is the diameter of the tube. Then the highest point on the sine wave is $(\pi/2, 1)$. Thus, an equation of the sine wave is

$$y = -\frac{1}{2}\cos(2x) + \frac{1}{2}.$$

6.2 Pop Quiz

1. $2\sin^2 x - \sin x - 1$

2. $(2\cos x - 1)(\cos x + 1)$

3. $\dfrac{1}{\cos x} - \dfrac{\sin^2 x}{\cos x} = \dfrac{1 - \sin^2 x}{\cos x} =$
$\dfrac{\cos^2 x}{\cos x} = \cos x$

4.

$$\begin{aligned} \frac{\cos(-x) - \sec(-x)}{\sec(x)} &= \\ \frac{\cos x - \sec x}{\sec x} &= \\ \frac{\cos x - \sec x}{\sec x} \cdot \frac{\cos x}{\cos x} &= \\ \frac{\cos^2 x - 1}{1} &= \\ \frac{-\sin^2 x}{1} &= \\ -\sin^2 x & \end{aligned}$$

6.2 Linking Concepts

a) Assume the circle is given by $x^2 + y^2 = r^2$ and $\left(\frac{w}{2}, r - h\right)$ is a point on the circle (corresponding to the upper right hand corner of the rectangular window). Substituting this point into the equation of the circle, we obtain

$$\begin{aligned} \frac{w^2}{4} + (r-h)^2 &= r^2 \\ \frac{w^2}{4} + (r^2 - 2rh + h^2) &= r^2 \\ \frac{w^2}{4} - 2rh + h^2 &= 0 \\ 2rh &= \frac{w^2 + 4h^2}{4} \\ r &= \frac{w^2 + 4h^2}{8h}. \end{aligned}$$

If $w = 36$ and $h = 10$, then the radius of the circle is $r = \dfrac{36^2 + 4(10)^2}{80} = 21.2$ inches.

b) Consider the right triangle with vertices at the point $A(0,0)$, $B\left(\frac{w}{2}, r - h\right)$, and $C(0, r - h)$. Let θ be the angle at point A. Then

$$\begin{aligned} \tan\theta &= \frac{w/2}{r-h} \\ \tan\theta &= \frac{w}{2(r-h)} \\ \theta &= \tan^{-1}\left(\frac{w}{2(r-h)}\right). \end{aligned}$$

If $w = 36$ and $h = 10$, then the length of the circular arc is $L = 2r\theta =$ $2(21.2)\tan^{-1}\left(\dfrac{36}{2(21.2 - 10)}\right) \approx 43.0$ in.

c) $r = \dfrac{w^2 + 4h^2}{8h}$ as derived in part a)

d) In part b), we obtained $\theta = \tan^{-1}\left(\dfrac{w}{2(r-h)}\right)$.

A formula for the arclength L is given by

$$\begin{aligned} L &= 2r\theta \\ &= \frac{w^2 + 4h^2}{4h}\tan^{-1}\left(\frac{w}{2(r-h)}\right). \end{aligned}$$

Equivalently, by using the fact that

$r = \dfrac{w^2 + 4h^2}{8h}$ (see part c) we can rewrite L as

$$L = \frac{w^2 + 4h^2}{4h} \sin^{-1}\left(\frac{4hw}{4h^2 + w^2}\right).$$

If we interpret arcsin in degrees, then

$$L = \frac{w^2 + 4h^2}{4h} \sin^{-1}\left(\frac{4hw}{4h^2 + w^2}\right) \cdot \frac{\pi}{180}$$

or

$$L = \frac{\pi w^2 + 4\pi h^2}{720h} \sin^{-1}\left(\frac{4hw}{4h^2 + w^2}\right).$$

For Thought

1. False, the right-hand side should be $\cos(5°)$.

2. True, by the sum identity for cosine.

3. True, $\cos(t - \pi/2) = \cos(\pi/2 - t) = \sin t$.

4. False, $\sin(\alpha - \pi/2) = -\sin(\pi/2 - \alpha) = -\cos\alpha$.

5. True, $\sec(\pi/3) = \sec(\pi/2 - \pi/6) = \csc(\pi/6)$.

6. False, since $\sin(5\pi/6) = 1/2$ and $\sin(2\pi/3) + \sin(\pi/6) = \sqrt{3}/2 + 1/2$.

7. True, since the sum identity for sine is applied to $5\pi/12 = \pi/6 + \pi/4$.

8. True, since the cofunction identity for tangent is applied to $90° - 68°29'55" = 21°30'5"$.

9. False, the equation fails when $x = \pi/2$.

10. True, since both sides of the equation (by the sum identity for tangent) are equal to $\tan(-7°)$.

6.3 Exercises

1. cosine

2. cofunction

3. $\cos(\pi/3 + \pi/4) =$
$\cos(\pi/3)\cos(\pi/4) - \sin(\pi/3)\sin(\pi/4) =$
$\dfrac{1}{2} \cdot \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{3}}{2} \cdot \dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{2} - \sqrt{6}}{4}$

4. $\cos(2\pi/3 + \pi/4) =$
$\cos(2\pi/3)\cos(\pi/4) - \sin(2\pi/3)\sin(\pi/4) =$
$-\dfrac{1}{2} \cdot \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{3}}{2} \cdot \dfrac{\sqrt{2}}{2} = \dfrac{-\sqrt{2} - \sqrt{6}}{4}$

5. $\cos(60° - 45°) =$
$\cos(60°)\cos(45°) + \sin(60°)\sin(45°) =$
$\dfrac{1}{2} \cdot \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{3}}{2} \cdot \dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{2} + \sqrt{6}}{4}$

6. $\cos(120° - 45°) =$
$\cos(120°)\cos(45°) + \sin(120°)\sin(45°) =$
$-\dfrac{1}{2} \cdot \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{3}}{2} \cdot \dfrac{\sqrt{2}}{2} = \dfrac{-\sqrt{2} + \sqrt{6}}{4}$

7. Since $\sin(20°) = \cos(90° - 20°) = \cos(70°)$, the answer is $70°$.

8. Since $\cos(15°) = \cos(90° - 150°)$ or $\cos(75°)$, the answer is $75°$.

9. Since $\tan\left(\dfrac{\pi}{6}\right) = \cot\left(\dfrac{\pi}{2} - \dfrac{\pi}{6}\right) = \cot\left(\dfrac{\pi}{3}\right)$, the answer is $\pi/3$.

10. Since $\cot\left(\dfrac{\pi}{3}\right) = \tan\left(\dfrac{\pi}{2} - \dfrac{\pi}{3}\right) = \tan\left(\dfrac{\pi}{6}\right)$, the answer is $\pi/6$.

11. Since $\sec(90° - 6°) = \csc(6°)$, the answer is $6°$.

12. Since $\csc(90° - 17°) = \sec(17°)$, the answer is $17°$.

13. $\sin(\pi/3 + \pi/4) =$
$\sin(\pi/3)\cos(\pi/4) + \cos(\pi/3)\sin(\pi/4) =$
$\dfrac{\sqrt{3}}{2} \cdot \dfrac{\sqrt{2}}{2} + \dfrac{1}{2} \cdot \dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{6} + \sqrt{2}}{4}$

14. $\sin(\pi/4 + 2\pi/3) =$
$\sin(\pi/4)\cos(2\pi/3) + \cos(\pi/4)\sin(2\pi/3) =$
$\dfrac{\sqrt{2}}{2} \cdot \dfrac{-1}{2} + \dfrac{\sqrt{2}}{2} \cdot \dfrac{\sqrt{3}}{2} = \dfrac{-\sqrt{2} + \sqrt{6}}{4}$

15. $\sin(60° - 45°) =$
$\sin(60°)\cos(45°) - \cos(60°)\sin(45°) =$
$\dfrac{\sqrt{3}}{2} \cdot \dfrac{\sqrt{2}}{2} - \dfrac{1}{2} \cdot \dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{6} - \sqrt{2}}{4}$

16. $\sin(45° - 120°) =$
$\sin(45°)\cos(120°) - \cos(45°)\sin(120°) =$
$\frac{\sqrt{2}}{2} \cdot \frac{-1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{-\sqrt{2} - \sqrt{6}}{4}$

17. $\tan\left(\frac{3\pi}{4} + \frac{\pi}{3}\right) = \frac{\tan(3\pi/4) + \tan(\pi/3)}{1 - \tan(3\pi/4)\tan(\pi/3)} =$
$\frac{-1 + \sqrt{3}}{1 - (-1)(\sqrt{3})} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} - 1}{\sqrt{3} - 1} =$
$\frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$

18. $\tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \frac{\tan(\pi/4) + \tan(\pi/3)}{1 - \tan(\pi/4)\tan(\pi/3)} =$
$\frac{1 + \sqrt{3}}{1 - (1)(\sqrt{3})} = \frac{1 + \sqrt{3}}{1 - \sqrt{3}} = \frac{1 + \sqrt{3}}{1 - \sqrt{3}} \cdot \frac{1 + \sqrt{3}}{1 + \sqrt{3}} =$
$\frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}$

19. $\tan(210° - 45°) =$
$\frac{\tan(210°) - \tan(45°)}{1 + \tan(210°)\tan(45°)} =$
$\frac{\sqrt{3}/3 - 1}{1 + (\sqrt{3}/3)(1)} = \frac{\sqrt{3}/3 - 1}{1 + (\sqrt{3}/3)(1)} \cdot \frac{3}{3} =$
$\frac{\sqrt{3} - 3}{3 + \sqrt{3}} = \frac{\sqrt{3} - 3}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}} = \frac{-12 + 6\sqrt{3}}{6} =$
$\sqrt{3} - 2$

20. $\tan(45° - 150°) =$
$\frac{\tan(45°) - \tan(150°)}{1 + \tan(45°)\tan(150°)} =$
$\frac{1 - (-\sqrt{3}/3)}{1 + (1)(-\sqrt{3}/3)} = \frac{1 + \sqrt{3}/3}{1 - \sqrt{3}/3} \cdot \frac{3}{3} =$
$\frac{3 + \sqrt{3}}{3 - \sqrt{3}} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{3 + \sqrt{3}}{3 + \sqrt{3}} = \frac{12 + 6\sqrt{3}}{6} =$
$2 + \sqrt{3}$

21. $\frac{7\pi}{12}$ **22.** $\frac{5\pi}{12}$

23. $\frac{13\pi}{12}$ **24.** $\frac{5\pi}{12}$

25. $\sin(23° + 67°) = \sin(90°) = 1$

26. $\sin(55° - 10°) = \sin(45°) = \sqrt{2}/2$

27. $\cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

28. $\cos\left(\frac{7\pi}{12} - \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{4}\right) = \sqrt{2}/2$

29. $\tan\left(\frac{\pi}{12} + \frac{\pi}{6}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

30. $\tan\left(\frac{5\pi}{12} - \frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

31. $\sin(2k + k) = \sin(3k)$

32. $\cos(3y + y) = \cos(4y)$

33. $30° + 45°$ **34.** $45° - 30°$

35. $120° + 45°$ **36.** $150° + 45°$

37. $\cos(2\pi/3 - \pi/4) =$
$\cos(2\pi/3)\cos(\pi/4) + \sin(2\pi/3)\sin(\pi/4) =$
$-\frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$

38. $\cos(\pi/3 + \pi/4) =$
$\cos(\pi/3)\cos(\pi/4) - \sin(\pi/3)\sin(\pi/4) =$
$\frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2} - \sqrt{6}}{4}$

39. $\sin(\pi/3 + \pi/4) =$
$\sin(\pi/3)\cos(\pi/4) + \cos(\pi/3)\sin(\pi/4) =$
$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$

40. $\sin(2\pi/3 - \pi/4) =$
$\sin(2\pi/3)\cos(\pi/4) - \cos(2\pi/3)\sin(\pi/4) =$
$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{-1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$

41. $\tan(45° + 30°) = \frac{\tan(45°) + \tan(30°)}{1 - \tan(45°)\tan(30°)} =$
$\frac{1 + \sqrt{3}/3}{1 - 1 \cdot \sqrt{3}/3} \cdot \frac{3}{3} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{3 + \sqrt{3}}{3 + \sqrt{3}} =$
$\frac{12 + 6\sqrt{3}}{9 - 3} = 2 + \sqrt{3}$

42. $\tan(30° - 45°) = \frac{\tan(30°) - \tan(45°)}{1 + \tan(30°)\tan(45°)} =$
$\frac{\sqrt{3}/3 - 1}{1 + \sqrt{3}/3} \cdot \frac{3}{3} = \frac{\sqrt{3} - 3}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}} =$

$\dfrac{-12+6\sqrt{3}}{9-3} = -2+\sqrt{3}$

43. $\sin(30° - 45°) =$
$\sin(30°)\cos(45°) - \cos(30°)\sin(45°) =$
$\dfrac{1}{2}\cdot\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{2}-\sqrt{6}}{4}$

44. $\sin(120° + 45°) =$
$\sin(120°)\cos(45°) + \cos(120°)\sin(45°) =$
$\dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{2}}{2} + \dfrac{-1}{2}\cdot\dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

45. $\cos(135° + 60°) =$
$\cos(135°)\cos(60°) - \sin(135°)\sin(60°) =$
$\dfrac{-\sqrt{2}}{2}\cdot\dfrac{1}{2} - \dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{3}}{2} = \dfrac{-\sqrt{2}-\sqrt{6}}{4}$

46. $\cos(-75°) = \cos(75°) = \cos(30° + 45°) =$
$\cos(30°)\cos(45°) - \sin(30°)\sin(45°) =$
$\dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{2}}{2} - \dfrac{1}{2}\cdot\dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

47. $\tan(-13\pi/12) = -\tan(13\pi/12) =$
$-\tan\left(\dfrac{3\pi}{4} + \dfrac{\pi}{3}\right) =$
$-\dfrac{\tan(3\pi/4) + \tan(\pi/3)}{1 - \tan(3\pi/4)\tan(\pi/3)} =$
$-\dfrac{-1+\sqrt{3}}{1-(-1)\sqrt{3}} = \dfrac{1-\sqrt{3}}{1+\sqrt{3}}\cdot\dfrac{1-\sqrt{3}}{1-\sqrt{3}} =$
$\dfrac{4-2\sqrt{3}}{-2} = -2+\sqrt{3}$

48. $\tan(7\pi/12) = \tan\left(\dfrac{\pi}{4} + \dfrac{\pi}{3}\right) =$
$\dfrac{\tan(\pi/4) + \tan(\pi/3)}{1 - \tan(\pi/4)\tan(\pi/3)} =$
$\dfrac{1+\sqrt{3}}{1-1\cdot\sqrt{3}} = \dfrac{1+\sqrt{3}}{1-\sqrt{3}}\cdot\dfrac{1+\sqrt{3}}{1+\sqrt{3}} =$
$\dfrac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$

49. $\sin(3°)\cos(-87°) + \cos(3°)\sin(87°) =$
$\sin(3°)\cos(87°) + \cos(3°)\sin(87°) =$
$\sin(3° + 87°) = \sin(90°) = 1$

50. $\sin(34°)\cos(13°) - \cos(34°)\sin(13°) =$
$\sin(34° - 13°) = \sin(21°)$

51. $\cos(\pi/2)\cos(\pi/5) + \sin(\pi/2)\sin(\pi/5) =$
$\cos\left(\dfrac{\pi}{2} - \dfrac{\pi}{5}\right) = \cos(3\pi/10)$

52. $\cos(12°)\cos(3°) + \sin(12°)\sin(3°) =$
$\cos(12° - 3°) = \cos(9°)$

53. $\dfrac{\tan(\pi/7) + \tan(\pi/6)}{1 - \tan(\pi/7)\tan(\pi/6)} = \tan\left(\dfrac{\pi}{7} + \dfrac{\pi}{6}\right) =$
$\tan(13\pi/42)$

54. $\dfrac{\tan(\pi/3) - \tan(\pi/6)}{1 + \tan(\pi/3)\tan(\pi/6)} = \tan\left(\dfrac{\pi}{3} - \dfrac{\pi}{6}\right) =$
$\tan(\pi/6) = \sqrt{3}/3$

55. $\sin(14°)\cos(35°) + \cos(14°)\sin(35°) =$
$\sin(14° + 35°) = \sin(49°)$

56. $\cos(10°)\cos(20°) - \cos(80°)\sin(20°) =$
$\cos(10°)\cos(20°) - \sin(10°)\sin(20°) =$
$\cos(10° + 20°) = \cos(30°) = \sqrt{3}/2$

57. G, $\cos(44°) = \sin(90° - 44°) = \sin(46°)$

58. B, $-\sin(46°) = -\cos(90° - 46°) = -\cos(44°)$

59. H, $\cos(46°) = \sin(90° - 46°) = \sin(44°)$

60. H, $\sin(136°) = \cos(90° - 136°) = \cos(-46°)$
$= \cos(46°) = \sin(90° - 46°) = \sin(44°)$

61. F, $\sec(1) = \csc\left(\dfrac{\pi}{2} - 1\right) = \csc\left(\dfrac{\pi-2}{2}\right)$

62. D, $\tan\left(\dfrac{\pi}{7}\right) = \cot\left(\dfrac{\pi}{2} - \dfrac{\pi}{7}\right) = \cot\left(\dfrac{5\pi}{14}\right)$

63. A, $\csc(\pi/2) = 1 = \cos(0)$

64. E, $-\sin(44°) = -\cos(90° - 44°) = -\cos(46°)$

65. Since α is in quadrant II and β is in quadrant I, $\cos\alpha = -\sqrt{1 - \left(\dfrac{3}{5}\right)^2} =$
$-\sqrt{1 - \dfrac{9}{25}} = -\sqrt{\dfrac{16}{25}} = -\dfrac{4}{5}$ and $\cos\beta =$
$\sqrt{1 - \left(\dfrac{5}{13}\right)^2} = \sqrt{1 - \dfrac{25}{169}} = \sqrt{\dfrac{144}{169}} = \dfrac{12}{13}.$
So $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta =$
$\dfrac{3}{5}\cdot\dfrac{12}{13} + \dfrac{-4}{5}\cdot\dfrac{5}{13} = \dfrac{16}{65}.$

66. Since α is in quadrant III and β is in quadrant IV, we obtain

$\cos\alpha = -\sqrt{1-\left(\frac{-4}{5}\right)^2} = -\sqrt{1-\frac{16}{25}} = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$ and $\sin\beta = -\sqrt{1-\left(\frac{12}{13}\right)^2} = -\sqrt{1-\frac{144}{169}} = -\sqrt{\frac{25}{169}} = -\frac{5}{13}$.

So $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{-4}{5}\cdot\frac{12}{13} - \frac{-3}{5}\cdot\frac{-5}{13} = -\frac{63}{65}$.

67. Since α is in quadrant I and β is in quadrant III, we obtain

$\cos\alpha = \sqrt{1-\left(\frac{2}{3}\right)^2} = \sqrt{1-\frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$ and $\cos\beta = -\sqrt{1-\left(\frac{-1}{2}\right)^2} = -\sqrt{1-\frac{1}{4}} = -\sqrt{\frac{3}{4}} = -\frac{\sqrt{3}}{2}$.

So $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{\sqrt{5}}{3}\cdot\frac{-\sqrt{3}}{2} - \frac{2}{3}\cdot\frac{-1}{2} = \frac{2-\sqrt{15}}{6}$.

68. Since α is in quadrant I and β is in quadrant II, we find

$\sin\alpha = \sqrt{1-\left(\frac{\sqrt{3}}{4}\right)^2} = \sqrt{1-\frac{3}{16}} = \sqrt{\frac{13}{16}} = \frac{\sqrt{13}}{4}$ and $\sin\beta = \sqrt{1-\left(\frac{-\sqrt{2}}{3}\right)^2} = \sqrt{1-\frac{2}{9}} = \sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{3}$.

So $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \frac{\sqrt{3}}{4}\cdot\frac{-\sqrt{2}}{3} + \frac{\sqrt{13}}{4}\cdot\frac{\sqrt{7}}{3} = \frac{-\sqrt{6}+\sqrt{91}}{12}$

69. Since α is in quadrant III and β is in quadrant II, we find

$\cos\alpha = -\sqrt{1-\left(\frac{-24}{25}\right)^2} = -\frac{7}{25}$ and $\sin\beta = \sqrt{1-\left(\frac{-8}{17}\right)^2} = \frac{15}{17}$.

Then $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{-24}{25}\cdot\frac{-8}{17} - \frac{-7}{25}\cdot\frac{15}{17} = \frac{297}{425}$.

70. Since α is in quadrant II and β is in quadrant III, we find

$\cos\alpha = -\sqrt{1-\left(\frac{7}{25}\right)^2} = -\frac{24}{25}$

and $\cos\beta = -\sqrt{1-\left(\frac{-8}{17}\right)^2} = -\frac{15}{17}$.

Then $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{7}{25}\cdot\frac{-15}{17} + \frac{-24}{25}\cdot\frac{-8}{17} = \frac{87}{425}$.

71. Since α is in quadrant II and β is in quadrant IV, we find

$\cos\alpha = -\sqrt{1-\left(\frac{24}{25}\right)^2} = -\frac{7}{25}$

and $\sin\beta = -\sqrt{1-\left(\frac{8}{17}\right)^2} = -\frac{15}{17}$.

Then $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \frac{-7}{25}\cdot\frac{8}{17} + \frac{24}{25}\cdot\frac{-15}{17} = -\frac{416}{425}$.

72. Since α is in quadrant IV and β is in quadrant II, we find

$\cos\alpha = \sqrt{1-\left(\frac{-7}{25}\right)^2} = \frac{24}{25}$

and $\cos\beta = -\sqrt{1-\left(\frac{8}{17}\right)^2} = -\frac{15}{17}$.

Then $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{24}{25}\cdot\frac{-15}{17} - \frac{-7}{25}\cdot\frac{8}{17} = -\frac{304}{425}$.

73. $\cos(\pi/2-(-\alpha)) = \sin(-\alpha) = -\sin\alpha$

74. $\sin\alpha\cos\pi - \cos\alpha\sin\pi =$
$\sin\alpha\cdot(-1) - \cos\alpha\cdot 0 = -\sin\alpha$

75. $\cos 180^\circ\cos\alpha + \sin 180^\circ\sin\alpha =$
$(-1)\cdot\cos\alpha + 0\cdot\sin\alpha = -\cos\alpha$

76. $\sin 180^\circ \cos\alpha - \cos 180^\circ \sin\alpha =$
$0 \cdot \cos\alpha - (-1) \cdot \sin\alpha = \sin\alpha$

77. The period is 360°, so $\sin(360^\circ - \alpha) =$
$\sin(-\alpha) = -\sin\alpha$

78. $\cos\alpha\cos\pi + \sin\alpha\sin\pi =$
$\cos\alpha \cdot (-1) + \sin\alpha \cdot 0 = -\cos\alpha$

79. $\sin(90^\circ - (-\alpha)) = \cos(-\alpha) = \cos\alpha$

80. The period is 360°, so $\cos(360^\circ - \alpha) =$
$\cos(-\alpha) = \cos\alpha$

81.

$$\begin{aligned} \sin(180^\circ - \alpha) &= \\ \sin(180^\circ)\cos\alpha - \cos(180^\circ) \quad \sin\alpha &= \\ \sin\alpha &= \\ \frac{\sin^2\alpha}{\sin\alpha} &= \\ \frac{1-\cos^2\alpha}{\sin\alpha} & \end{aligned}$$

82. We rewrite both sides:

$$\begin{aligned} \cos(x - \pi/2) &= \\ \cos x\cos(\pi/2) + \sin x\sin(\pi/2) &= \\ \sin x &= \\ \cos x \cdot \frac{\sin x}{\cos x} &= \\ \cos x\tan x & \end{aligned}$$

83.

$$\begin{aligned} \frac{\cos(x+y)}{\cos(x)\cos(y)} &= \\ \frac{\cos(x)\cos(y) - \sin(x)\sin(y)}{\cos(x)\cos(y)} &= \\ \frac{\cos(x)\cos(y)}{\cos(x)\cos(y)} - \frac{\sin(x)\sin(y)}{\cos(x)\cos(y)} &= \\ 1 - \tan(x)\tan(y) & \end{aligned}$$

84.

$$\begin{aligned} \frac{\sin(x+y)}{\sin x\cos y} &= \\ \frac{\sin x\cos y + \cos x\sin y}{\sin x\cos y} &= \\ \frac{\sin x\cos y}{\sin x\cos y} + \frac{\cos x\sin y}{\sin x\cos y} &= \\ 1 + \cot x\tan y & \end{aligned}$$

85. Substitute the sum and difference sine identities into the left-hand side to get a difference of two squares.

$$\begin{aligned} \sin(\alpha+\beta)\sin(\alpha-\beta) &= \\ (\sin\alpha\cos\beta)^2 - (\cos\alpha\sin\beta)^2 &= \\ \sin^2\alpha(1-\sin^2\beta) - (1-\sin^2\alpha)\sin^2\beta &= \\ \sin^2\alpha - \sin^2\alpha\sin^2\beta - \sin^2\beta + \sin^2\alpha\sin^2\beta &= \\ \sin^2\alpha - \sin^2\beta & \end{aligned}$$

86. Substitute the sum and difference cosine identities into the left-hand side to get a difference of two squares.

$$\begin{aligned} \cos(\alpha+\beta)\cos(\alpha-\beta) &= \\ (\cos\alpha\cos\beta)^2 - (\sin\alpha\sin\beta)^2 &= \\ (1-\sin^2\alpha)\cos^2\beta - \sin^2\alpha(1-\cos^2\beta) &= \\ \cos^2\beta - \sin^2\alpha\cos^2\beta - \sin^2\alpha + \sin^2\alpha\cos^2\beta &= \\ \cos^2\beta - \sin^2\alpha & \end{aligned}$$

87. Using the sum identity for cosine, we obtain

$$\begin{aligned} \cos(x+x) &= \\ \cos x\cos x - \sin x\sin x &= \\ \cos^2 x - \sin^2 x & \end{aligned}$$

88. Applying the sum identity for sine, we get

$$\begin{aligned} \sin(x+x) &= \\ \sin x\cos x + \cos x\sin x &= \\ 2\sin x\cos x & \end{aligned}$$

89.

$$\begin{aligned} \sin(x-y) - \sin(y-x) &= \\ \sin(x-y) + \sin(x-y) &= \\ 2\sin(x-y) &= \\ 2(\sin x\cos y - \cos x\sin y) &= \\ 2\sin x\cos y - 2\cos x\sin y & \end{aligned}$$

90.

$$\begin{aligned}\cos(x-y)+\cos(y-x) &= \\ \cos(x-y)+\cos(x-y) &= \\ 2\cos(x-y) &= \\ 2(\cos x\cos y+\sin x\sin y) &= \\ 2\cos x\cos y-2\sin x\sin y\end{aligned}$$

91.

$$\begin{aligned}\tan(s+t)\tan(s-t) &= \\ \frac{\tan s+\tan t}{1-\tan(s)\tan(t)}\cdot\frac{\tan s-\tan t}{1+\tan(s)\tan(t)} &= \\ \frac{\tan^2 s-\tan^2 t}{1+\tan^2(s)\tan^2(t)}\end{aligned}$$

92. Using the cofunction identity for tangent, we get

$$\begin{aligned}\tan(\pi/4+x) &= \\ \cot(\pi/2-(\pi/4+x)) &= \\ \cot(\pi/2-\pi/4-x) &= \\ \cot(\pi/4-x)\end{aligned}$$

93. In the proof, divide each term by $\cos\alpha\cos\beta$.

$$\begin{aligned}\frac{\cos(\alpha+\beta)}{\sin(\alpha-\beta)} &= \\ \frac{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}} &= \\ \frac{1-\tan(\alpha)\tan(\beta)}{\tan(\alpha)-\tan(\beta)}\end{aligned}$$

94. In the proof, divide each term by $\cos\alpha\cos\beta$.

$$\begin{aligned}\frac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)} &= \\ \frac{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}} &= \\ \frac{1+\tan(\alpha)\tan(\beta)}{\tan(\alpha)+\tan(\beta)}\end{aligned}$$

95. In the proof, multiply each term by $\cos(v-t)$. Also, the sum and difference identities for cosine expresses $\cos(v+t)\cos(v-t)$ as a difference of two squares.

$$\begin{aligned}\sec(v+t) &= \\ \frac{1}{\cos(v+t)} &= \\ \frac{\cos(v-t)}{\cos(v+t)\cos(v-t)} &= \\ \frac{\cos(v-t)}{\cos^2(v)\cos^2(t)-\sin^2(v)\sin^2(t)} &=\end{aligned}$$

$$\frac{\cos(v-t)}{\cos^2(v)\cos^2(t)-(1-\cos^2 v)(1-\cos^2 t)} =$$

$$\cos(v-t)\div\Big[\cos^2(v)\cos^2(t)- (1-\cos^2 v-\cos^2 t+\cos^2(v)\cos^2(t))\Big] =$$

$$\begin{aligned}\frac{\cos(v-t)}{-1+\cos^2 v+\cos^2 t} &= \\ \frac{\cos(v-t)}{\cos^2 v-\sin^2 t} &= \\ \frac{\cos(v)\cos(t)+\sin(v)\sin(t)}{\cos^2 v-\sin^2 t}\end{aligned}$$

96. In the proof, multiply each term by $\sin(v+t)$. Also, the sum and difference identities for sine expresses $\sin(v-t)\sin(v+t)$ as a difference of two squares.

$$\begin{aligned}\csc(v-t) &= \\ \frac{1}{\sin(v-t)} &= \\ \frac{\sin(v+t)}{\sin(v-t)\sin(v+t)} &= \\ \frac{\sin(v+t)}{\sin^2(v)\cos^2(t)-\cos^2(v)\sin^2(t)} &=\end{aligned}$$

$$\frac{\sin(v+t)}{\sin^2 v(1-\sin^2 t)-(1-\sin^2 v)\sin^2 t} =$$

$$\sin(v+t)\div\Big[\sin^2 v-\sin^2(v)\sin^2(t)$$

$$-\sin^2(t)+\sin^2(v)\sin^2(t)\Big] =$$

$$\frac{\sin(v+t)}{\sin^2 v-\sin^2 t} =$$

$$\frac{\sin(v)\cos(t)+\cos(v)\sin(t)}{\sin^2 v-\sin^2 t} =$$

97. In the proof, divide each term by $\cos x \sin y$.

$$\frac{\cos(x+y)}{\cos(x-y)} =$$

$$\frac{\cos(x)\cos(y)-\sin(x)\sin(y)}{\cos(x)\cos(y)+\sin(x)\sin(y)} =$$

$$\frac{\dfrac{\cos(x)\cos(y)}{\cos(x)\sin(y)}-\dfrac{\sin(x)\sin(y)}{\cos(x)\sin(y)}}{\dfrac{\cos(x)\cos(y)}{\cos(x)\sin(y)}+\dfrac{\sin(x)\sin(y)}{\cos(x)\sin(y)}} =$$

$$\frac{\cot(y)-\tan(x)}{\cot(y)+\tan(x)}$$

98. In the proof, divide each term by $\sin x \sin y$.

$$\frac{\sin(x+y)}{\sin(x-y)} =$$

$$\frac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\sin(x)\cos(y)-\cos(x)\sin(y)} =$$

$$\frac{\dfrac{\sin(x)\cos(y)}{\sin(x)\sin(y)}+\dfrac{\cos(x)\sin(y)}{\sin(x)\sin(y)}}{\dfrac{\sin(x)\cos(y)}{\sin(x)\sin(y)}-\dfrac{\cos(x)\sin(y)}{\sin(x)\sin(y)}} =$$

$$\frac{\cot(y)+\cot(x)}{\cot(y)-\cot(x)}$$

99. In the proof, multiply each term by $\sin(\alpha-\beta)$. Also, the sum and difference identities for sine expresses $\sin(\alpha+\beta)\sin(\alpha-\beta)$ as a difference of two squares.

$$\frac{\sin(\alpha+\beta)}{\sin\alpha+\sin\beta} =$$

$$\frac{\sin(\alpha+\beta)}{\sin\alpha+\sin\beta}\cdot\frac{\sin(\alpha-\beta)}{\sin(\alpha-\beta)} =$$

$$\frac{\sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta}{(\sin\alpha+\sin\beta)\sin(\alpha-\beta)} =$$

$$\frac{\sin^2\alpha(1-\sin^2\beta)-(1-\sin^2\alpha)\sin^2\beta}{(\sin\alpha+\sin\beta)\sin(\alpha-\beta)} =$$

$$\frac{\sin^2\alpha-\sin^2\alpha\sin^2\beta-\sin^2\beta+\sin^2\alpha\sin^2\beta}{(\sin\alpha+\sin\beta)\sin(\alpha-\beta)} =$$

$$\frac{\sin^2\alpha-\sin^2\beta}{(\sin\alpha+\sin\beta)\sin(\alpha-\beta)} =$$

$$\frac{(\sin\alpha-\sin\beta)(\sin\alpha+\sin\beta)}{(\sin\alpha+\sin\beta)\sin(\alpha-\beta)} =$$

$$\frac{\sin\alpha-\sin\beta}{\sin(\alpha-\beta)}$$

100. In the proof, we smultiply each term by $\cos(\alpha-\beta)$. Also, the sum and difference identities for cosine expresses $\cos(\alpha+\beta)\cos(\alpha-\beta)$ as a difference of two squares.

$$\frac{\cos(\alpha+\beta)}{\cos\alpha+\sin\beta} =$$

$$\frac{\cos(\alpha+\beta)}{\cos\alpha+\sin\beta}\cdot\frac{\cos(\alpha-\beta)}{\cos(\alpha-\beta)} =$$

$$\frac{\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta}{(\cos\alpha+\sin\beta)\cos(\alpha-\beta)} =$$

$$\frac{\cos^2\alpha(1-\sin^2\beta)-(1-\cos^2\alpha)\sin^2\beta}{(\cos\alpha+\sin\beta)\cos(\alpha-\beta)} =$$

$$\frac{\cos^2\alpha-\cos^2\alpha\sin^2\beta-\sin^2\beta+\cos^2\alpha\sin^2\beta}{(\cos\alpha+\sin\beta)\cos(\alpha-\beta)} =$$

$$\frac{\cos^2\alpha-\sin^2\beta}{(\cos\alpha+\sin\beta)\cos(\alpha-\beta)} =$$

$$\frac{(\cos\alpha-\cos\beta)(\cos\alpha+\cos\beta)}{(\cos\alpha+\sin\beta)\cos(\alpha-\beta)} =$$

$$\frac{\cos\alpha-\sin\beta}{\cos(\alpha-\beta)} =$$

$$\frac{\cos\alpha-\sin\beta}{\cos(\beta-\alpha)}$$

103. If $\alpha=\beta=\pi/6$, then $\sin(\alpha+\beta)\neq\sin\alpha+\sin\beta$

104. The following formulas will be useful

$$\sin(90^\circ-\alpha)=\cos\alpha$$

and

$$\cos(90^\circ - \alpha) = \sin\alpha.$$

In particular, $\cos(89^\circ) = \sin(1^\circ)$, $\cos(88^\circ) = \sin(2^\circ)$, $\sin(89^\circ) = \cos(1^\circ)$, and so on. Thus, for $k = 1^\circ, ..., 44^\circ$ we have

$$\sin^2(k^\circ) + \sin^2((90-k)^\circ) = 1.$$

Since $\sin^2(45^\circ) = 1/2$, we find

$$\begin{aligned}\sin^2(1^\circ) + \sin^2(2^\circ) + ... + \sin^2(90^\circ) &= \\ 44 + \sin^2(45^\circ) + \sin^2(90^\circ) &= \\ 45 + \frac{1}{2}.\end{aligned}$$

Similarly, we obtain

$$\cos^2(1^\circ) + \cos^2(2^\circ) + ... + \cos^2(90^\circ) = 44 + \frac{1}{2}.$$

Finally, we obtain

$$\frac{\sin^2(1^\circ) + ... + \sin^2(90^\circ)}{\cos^2(1^\circ) + ... + \cos^2(90^\circ)} = \frac{45 + 1/2}{44 + 1/2} = \frac{91}{89}.$$

105. $1 - \sin^2\alpha = \cos^2\alpha$.

106. $\dfrac{\sin x}{\csc x - \sin x} \cdot \dfrac{\sin x}{\sin x} = \dfrac{\sin^2 x}{1 - \sin^2 x} = \dfrac{\sin^2 x}{\cos^2 x} = \tan^2 x$

107. Since $B = 2$, the period is $\dfrac{\pi}{B} = \dfrac{\pi}{2}$.

Solve $2x = k\pi$ where k is an integer. Then the asymptotes are $x = \dfrac{k\pi}{2}$.

108. We find $g(f(h(x))) = g(f(3x)) = g(\sin 3x) = \sin(3x) + 2$.

And, $h(g(f(x))) = h(g(\sin x)) = h(\sin x + 2) = 3(\sin x + 2) = 3\sin(x) + 6$.

109. $\cos\alpha = -\sqrt{1 - (1/4)^2} = -\frac{\sqrt{15}}{4}$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{1/4}{-\sqrt{15}/4} = -\frac{1}{\sqrt{15}} = -\frac{\sqrt{15}}{15}$$

$$\csc\alpha = \frac{1}{\sin\alpha} = -4$$

$$\sec\alpha = \frac{1}{\cos\alpha} = -\frac{4}{\sqrt{15}} = -\frac{4\sqrt{15}}{15}$$

$$\cot\alpha = \frac{1}{\tan\alpha} = -\frac{15}{\sqrt{15}} = -\sqrt{15}$$

110.

a) $-\sin x$ b) $\cos x$

c) $-\tan x$ d) $-\csc x$

e) $\sec x$ f) $-\cot x$

Thinking Outside the Box LIV

The angle spanned by the first seventeen rectangles is

$$\tan^{-1}\left(\frac{1}{1}\right) + \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + ... + \tan^{-1}\left(\frac{1}{\sqrt{17}}\right) \approx 365^\circ$$

while the angle spanned by the first sixteen rectangles is

$$\tan^{-1}\left(\frac{1}{1}\right) + \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + ... + \tan^{-1}\left(\frac{1}{\sqrt{16}}\right) \approx 351^\circ.$$

Thus, the 17th rectangle is the first rectangle that overlaps with the first rectangle.

6.3 Pop Quiz

1. $\cos(135^\circ - 120^\circ) = \cos(135^\circ)\cos(120^\circ) + \sin(135^\circ)\sin(120^\circ) = \dfrac{-\sqrt{2}}{2} \cdot \dfrac{-1}{2} + \dfrac{\sqrt{2}}{2} \cdot \dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{2} + \sqrt{6}}{4}$

2. 80°, since $\sin 10^\circ = \cos(90^\circ - 10^\circ)$

3. Using the sum identity for sine, the answer is $\sin 3x$.

4. Since α and β are in quadrant I, we obtain

$$\cos\alpha = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}$$

and $\sin\beta = \sqrt{1 - \left(\dfrac{1}{2}\right)^2} = \dfrac{\sqrt{3}}{2}$.

Then $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \dfrac{4}{5} \cdot \dfrac{1}{2} - \dfrac{3}{5} \cdot \dfrac{\sqrt{3}}{2} = \dfrac{4 - 3\sqrt{3}}{10}$.

For Thought

1. True, $\dfrac{\sin(2\cdot 21^\circ)}{2} = \dfrac{2\sin(21^\circ)\cos(21^\circ)}{2}$ $= \sin(21^\circ)\cos(21^\circ)$.

2. True, by a cosine double angle identity $\cos(2\sqrt{2}) = 2\cos^2(\sqrt{2}) - 1$.

3. False, $\sin\left(\dfrac{300^\circ}{2}\right) = \sqrt{\dfrac{1-\cos(300^\circ)}{2}}$.

4. True, $\sin\left(\dfrac{400^\circ}{2}\right) = -\sqrt{\dfrac{1-\cos(400^\circ)}{2}}$ $= -\sqrt{\dfrac{1-\cos(40^\circ)}{2}}$.

5. False, $\tan\left(\dfrac{7\pi/4}{2}\right) = -\sqrt{\dfrac{1-\cos(7\pi/4)}{1+\cos(7\pi/4)}}$.

6. True, $\tan\left(\dfrac{-\pi/4}{2}\right) = \dfrac{1-\cos(-\pi/4)}{\sin(-\pi/4)} =$ $\dfrac{1-\cos(\pi/4)}{\sin(-\pi/4)}$

7. False, if $x = \pi/4$ then $\dfrac{\sin(2\cdot\pi/4)}{2} =$ $\dfrac{\sin(\pi/2)}{2} = \dfrac{1}{2}$ and $\sin(\pi/4) = \sqrt{2}/2$.

8. False, since $\cos(2\pi/3) = -1/2$ while $\sqrt{\dfrac{1+\cos(2x)}{2}}$ is a non-negative number.

9. True, since $1-\cos x \ge 0$ we find $\sqrt{(1-\cos x)^2} = |1-\cos x| = 1-\cos x$

10. True, α is in quadrant III or IV, while $\alpha/2$ is in quadrant II.

6.4 Exercises

1. $\sin(2\cdot 45^\circ) = 2\sin(45^\circ)\cos(45^\circ) =$ $2\cdot\dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{2}}{2} = 2\cdot\dfrac{2}{4} = 1.$

2. $\cos(2\cdot 30^\circ) = 2\cos^2(30^\circ) - 1 =$

$2(\sqrt{3}/2)^2 - 1 = 2\cdot\dfrac{3}{4} - 1 = \dfrac{3}{2} - 1 = \dfrac{1}{2}$

3. $\tan(2\cdot 30^\circ) = \dfrac{2\tan(30^\circ)}{1-\tan^2(30^\circ)} = \dfrac{2(\sqrt{3}/3)}{1-(\sqrt{3}/3)^2} =$ $\dfrac{2\sqrt{3}/3}{1-1/3} = \dfrac{2\sqrt{3}/3}{2/3} = \sqrt{3}$

4. $\cos(2\cdot 90^\circ) = 2\cos^2(90^\circ) - 1 = 2(0)^2 - 1 =$ $0 - 1 = -1$

5. $\sin\left(2\cdot\dfrac{3\pi}{4}\right) = 2\sin(3\pi/4)\cos(3\pi/4) =$ $2\cdot\dfrac{\sqrt{2}}{2}\cdot\dfrac{-\sqrt{2}}{2} = 2\cdot\dfrac{-2}{4} = -1$

6. $\cos\left(2\cdot\dfrac{2\pi}{3}\right) = 2\cos^2(2\pi/3) - 1 =$ $2(-1/2)^2 - 1 = 2\cdot\dfrac{1}{4} - 1 = \dfrac{1}{2} - 1 = -\dfrac{1}{2}$

7. $\tan\left(2\cdot\dfrac{2\pi}{3}\right) = \dfrac{2\tan(2\pi/3)}{1-\tan^2(2\pi/3)} = \dfrac{2(-\sqrt{3})}{1-(-\sqrt{3})^2}$ $= \dfrac{-2\sqrt{3}}{1-3} = \dfrac{-2\sqrt{3}}{-2} = \sqrt{3}$

8. $\sin\left(2\cdot\dfrac{\pi}{3}\right) = 2\sin(\pi/3)\cos(\pi/3)$ $= 2\cdot\dfrac{\sqrt{3}}{2}\cdot\dfrac{1}{2} = 2\cdot\dfrac{\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2}$

9. $\cos\left(\dfrac{30^\circ}{2}\right) = \sqrt{\dfrac{1+\cos(30^\circ)}{2}} =$ $\sqrt{\dfrac{1+\sqrt{3}/2}{2}\cdot\dfrac{2}{2}} = \sqrt{\dfrac{2+\sqrt{3}}{4}} = \dfrac{\sqrt{2+\sqrt{3}}}{2}$

10. $\cos\left(\dfrac{\pi/4}{2}\right) = \sqrt{\dfrac{1+\cos(\pi/4)}{2}} =$ $\sqrt{\dfrac{1+\sqrt{2}/2}{2}\cdot\dfrac{2}{2}} = \sqrt{\dfrac{2+\sqrt{2}}{4}} = \dfrac{\sqrt{2+\sqrt{2}}}{2}$

11. $\sin\left(\dfrac{30^\circ}{2}\right) = \sqrt{\dfrac{1-\cos(30^\circ)}{2}} =$ $\sqrt{\dfrac{1-\sqrt{3}/2}{2}\cdot\dfrac{2}{2}} = \sqrt{\dfrac{2-\sqrt{3}}{4}} = \dfrac{\sqrt{2-\sqrt{3}}}{2}$

12. $\sin\left(\dfrac{-\pi/3}{2}\right) = -\sqrt{\dfrac{1-\cos(-\pi/3)}{2}} =$ $-\sqrt{\dfrac{1-1/2}{2}\cdot\dfrac{2}{2}} = -\sqrt{\dfrac{1}{4}} = -\dfrac{1}{2}$

13. $\tan\left(\frac{30°}{2}\right) = \frac{1-\cos(30°)}{\sin(30°)} =$
$\frac{1-\sqrt{3}/2}{1/2}\cdot\frac{2}{2} = 2-\sqrt{3}$

14. $\tan\left(\frac{3\pi/4}{2}\right) = \frac{1-\cos(3\pi/4)}{\sin(3\pi/4)} =$
$\frac{1-(-\sqrt{2}/2)}{\sqrt{2}/2}\cdot\frac{2}{2} = \frac{2+\sqrt{2}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} =$
$\frac{2\sqrt{2}+2}{2} = \sqrt{2}+1$

15. $\sin\left(\frac{45°}{2}\right) = \sqrt{\frac{1-\cos(45°)}{2}} =$
$\sqrt{\frac{1-\sqrt{2}/2}{2}\cdot\frac{2}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$

16. $\tan\left(\frac{150°}{2}\right) = \frac{1-\cos(150°)}{\sin(150°)} =$
$\frac{1-(-\sqrt{3}/2)}{1/2}\cdot\frac{2}{2} = 2+\sqrt{3}$

17. Positive, $118.5°$ is in quadrant II

18. Negative, $222.5°$ is in quadrant III

19. Negative, $100°$ is in quadrant II

20. Negative, $9\pi/7$ is in quadrant III

21. Negative, $-5\pi/12$ is in quadrant IV

22. Positive, $17\pi/12$ is in quadrant III

23. $\sin(2\cdot 13°) = \sin 26°$

24. $-\cos\left(2\cdot\frac{\pi}{5}\right) = -\cos(2\pi/5)$

25. $\cos(2\cdot 22.5°) = \cos 45° = \sqrt{2}/2$

26. $\cos(2\cdot(-\pi/8)) = \cos(-\pi/4) = \sqrt{2}/2$

27. $\frac{1}{2}\cdot\frac{2\tan 15°}{1-\tan^2 15°} = \frac{1}{2}\cdot\tan(2\cdot 15°) =$
$\frac{1}{2}\cdot\tan 30° = \frac{1}{2}\cdot\frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{6}$

28. $\frac{1}{2}\cdot\frac{2\tan 30°}{1-\tan^2 30°} = \frac{1}{2}\cdot\tan(2\cdot 30°) =$
$\frac{1}{2}\cdot\tan 60° = \frac{1}{2}\cdot\sqrt{3} = \frac{\sqrt{3}}{2}$

29. $\tan\left(\frac{12°}{2}\right) = \tan 6°$

30. $\frac{1-\cos 8°}{\sin 8°} = \tan\left(\frac{8°}{2}\right) = \tan 4°$

31. $2\sin\left(\frac{\pi}{9}-\frac{\pi}{2}\right)\cos\left(\frac{\pi}{9}-\frac{\pi}{2}\right) =$
$\sin\left(2\cdot\left(\frac{\pi}{9}-\frac{\pi}{2}\right)\right) = \sin\left(\frac{2\pi}{9}-\pi\right) =$
$\sin(-7\pi/9) = -\sin(7\pi/9)$.

32. $\cos\left(2\cdot\left(\frac{\pi}{5}-\frac{\pi}{2}\right)\right) = \cos\left(\frac{2\pi}{5}-\pi\right) =$
$\cos(-3\pi/5) = \cos(3\pi/5)$

33. $\cos(2\cdot(\pi/9)) = \cos(2\pi/9)$

34. $\frac{2\tan 5}{1-\tan^2 5} = \tan(2\cdot 5) = \tan 10$

35. c, since $\sin^2 x = 1-\cos^2 x$

36. e, since $\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$

37. g, for $\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos(x)}{2}}$

38. i, for $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$ and $\cot x$ is the reciprocal of $\tan x$.

39. a, for $\sin(2x) = 2\sin x\cos x$

40. j, for $\tan^2(x) = \sec^2(x)-1$

41. h, for $\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$

42. b, since $\cos(2x) = \cos^2 x - \sin^2 x$

43. f, since $\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$

44. d, since $\cos^2 x = 1-\sin^2 x$

45. Since $\cos(2\alpha) = 2\cos^2\alpha - 1$, we get

$$\begin{aligned} 2\cos^2\alpha - 1 &= \frac{3}{5} \\ 2\cos^2\alpha &= \frac{8}{5} \\ \cos^2\alpha &= \frac{4}{5} \\ \cos\alpha &= \pm\frac{2}{\sqrt{5}}. \end{aligned}$$

But $0° < \alpha < 45°$, so $\cos\alpha = \dfrac{2}{\sqrt{5}}$ and

$\sin\alpha = \sqrt{1 - \left(\dfrac{2}{\sqrt{5}}\right)^2} = \sqrt{1 - \dfrac{4}{5}} =$

$\sqrt{\dfrac{1}{5}} = \dfrac{1}{\sqrt{5}}$.

Furthermore, $\sec\alpha = \dfrac{\sqrt{5}}{2}$, $\csc\alpha = \sqrt{5}$,

$\tan\alpha = \dfrac{1/\sqrt{5}}{2/\sqrt{5}} = \dfrac{1}{2}$, $\cot\alpha = 2$.

46. Since $\cos(2\alpha) = 2\cos^2\alpha - 1$, we obtain

$$\begin{aligned} 2\cos^2\alpha - 1 &= \frac{1}{3} \\ 2\cos^2\alpha &= \frac{4}{3} \\ \cos^2\alpha &= \frac{2}{3} \\ \cos\alpha &= \pm\frac{\sqrt{2}}{\sqrt{3}} \\ \cos\alpha &= \pm\frac{\sqrt{6}}{3}. \end{aligned}$$

But $180° < \alpha < 225°$, so $\cos\alpha = -\dfrac{\sqrt{6}}{3}$ and

$\sin\alpha = -\sqrt{1 - \left(-\dfrac{\sqrt{6}}{3}\right)^2} = -\sqrt{1 - \dfrac{6}{9}} =$

$-\sqrt{\dfrac{1}{3}} = -\dfrac{1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$.

Furthermore, $\sec\alpha = -\dfrac{3}{\sqrt{6}}$, $\csc\alpha = -\sqrt{3}$,

$\tan\alpha = \dfrac{-\sqrt{3}/3}{-\sqrt{6}/3} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$, $\cot\alpha = \sqrt{2}$.

47. Since $2\alpha = \sin^{-1}(5/13) \approx 22.6°$, we find

$\cos 2\alpha = \sqrt{1 - \left(\dfrac{5}{13}\right)^2} = \dfrac{12}{13}$. Then

$\sin\alpha = \sqrt{\dfrac{1 - \cos 2\alpha}{2}} = \sqrt{\dfrac{1 - 12/13}{2}} = \dfrac{\sqrt{26}}{26}$,

$\cos\alpha = \sqrt{\dfrac{1 + \cos 2\alpha}{2}} = \sqrt{\dfrac{1 + 12/13}{2}} = \dfrac{5\sqrt{26}}{26}$,

$\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{\sqrt{26}/26}{5\sqrt{26}/26} = \dfrac{1}{5}$

$\csc\alpha = \dfrac{1}{\sin\alpha} = \sqrt{26}$,

$\sec\alpha = \dfrac{1}{\cos\alpha} = \dfrac{\sqrt{26}}{5}$,

and $\cot\alpha = \dfrac{1}{\tan\alpha} = 5$.

48. Since 2α lies in quadrant III, we obtain

$\cos 2\alpha = -\sqrt{1 - \left(\dfrac{-8}{17}\right)^2} = -\dfrac{15}{17}$.

Then $\sin\alpha = \sqrt{\dfrac{1 - \cos 2\alpha}{2}} =$

$\sqrt{\dfrac{1 - (-15/17)}{2}} = \dfrac{4\sqrt{17}}{17}$,

$\cos\alpha = -\sqrt{\dfrac{1 + \cos 2\alpha}{2}} = -\sqrt{\dfrac{1 + (-15/17)}{2}} =$

$-\dfrac{\sqrt{17}}{17}$, $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{4\sqrt{17}/17}{-\sqrt{17}/17} = -4$

$\csc\alpha = \dfrac{1}{\sin\alpha} = \dfrac{\sqrt{17}}{4}$,

$\sec\alpha = \dfrac{1}{\cos\alpha} = -\sqrt{17}$,

and $\cot\alpha = \dfrac{1}{\tan\alpha} = -\dfrac{1}{4}$.

49. By a half-angle identity, we have

$$\begin{aligned} -\sqrt{\frac{1 + \cos\alpha}{2}} &= -\frac{1}{4} \\ \frac{1 + \cos\alpha}{2} &= \frac{1}{16} \\ 1 + \cos\alpha &= \frac{1}{8} \\ \cos\alpha &= -\frac{7}{8}. \end{aligned}$$

But $\pi \le \alpha \le 3\pi/2$,

so $\sin\alpha = -\sqrt{1 - \left(-\dfrac{7}{8}\right)^2} =$

$-\sqrt{1 - \dfrac{49}{64}} = -\sqrt{\dfrac{15}{64}} = -\dfrac{\sqrt{15}}{8}$.

Furthermore, $\sec\alpha = -\dfrac{8}{7}$, $\csc\alpha = -\dfrac{8}{\sqrt{15}}$,

$\tan\alpha = \dfrac{-\sqrt{15}/8}{-7/8} = \dfrac{\sqrt{15}}{7}$, $\cot\alpha = \dfrac{7}{\sqrt{15}}$.

50. By a half-angle identity, we obtain

$$\begin{aligned} -\sqrt{\frac{1-\cos\alpha}{2}} &= -\frac{1}{3} \\ \frac{1-\cos\alpha}{2} &= \frac{1}{9} \\ 1-\cos\alpha &= \frac{2}{9} \\ \cos\alpha &= \frac{7}{9}. \end{aligned}$$

But α is in qudrant IV,

so $\sin\alpha = -\sqrt{1-\left(\dfrac{7}{9}\right)^2} = -\sqrt{1-\dfrac{49}{81}} =$
$-\sqrt{\dfrac{32}{81}} = -\dfrac{\sqrt{32}}{9} = -\dfrac{4\sqrt{2}}{9}$.

Furthermore, $\sec\alpha = \dfrac{9}{7}$, $\csc\alpha = -\dfrac{9}{\sqrt{32}}$, or
$\csc\alpha = -\dfrac{9\sqrt{2}}{8}$, $\tan\alpha = \dfrac{-\sqrt{32}/9}{7/9} = -\dfrac{\sqrt{32}}{7}$, or
$\tan\alpha = -\dfrac{4\sqrt{2}}{7}$, and $\cot\alpha = -\dfrac{7}{\sqrt{32}} = -\dfrac{7\sqrt{2}}{8}$.

51. By a half-angle identity, we find

$$\begin{aligned} \sqrt{\frac{1-\cos\alpha}{2}} &= \frac{4}{5} \\ \frac{1-\cos\alpha}{2} &= \frac{16}{25} \\ 1-\cos\alpha &= \frac{32}{25} \\ \cos\alpha &= -\frac{7}{25}. \end{aligned}$$

Since $(\pi/2+2k\pi) \le \alpha/2 \le (\pi+2k\pi)$ for some integer k, $(\pi+4k\pi) \le \alpha \le (2\pi+4k\pi)$.
So α is in quadrant III because $\cos\alpha < 0$.

$\sin\alpha = -\sqrt{1-\left(-\dfrac{7}{25}\right)^2} = -\sqrt{1-\dfrac{49}{625}} =$
$-\sqrt{\dfrac{576}{625}} = -\dfrac{24}{25}$.

Furthermore, $\sec\alpha = -\dfrac{25}{7}$, $\csc\alpha = -\dfrac{25}{24}$,
$\tan\alpha = \dfrac{-24/25}{-7/25} = \dfrac{24}{7}$, and $\cot\alpha = \dfrac{7}{24}$.

52. By using the half-angle identity for sine, we get

$$\begin{aligned} \sqrt{\frac{1-\cos\alpha}{2}} &= \frac{1}{5} \\ \frac{1-\cos\alpha}{2} &= \frac{1}{25} \\ 1-\cos\alpha &= \frac{2}{25} \\ \cos\alpha &= \frac{23}{25}. \end{aligned}$$

Since $(\pi/2+2k\pi) \le \alpha/2 \le (\pi+2k\pi)$ for some integer k, $(\pi+4k\pi) \le \alpha \le (2\pi+4k\pi)$.
So α is in quadrant IV because $\cos\alpha > 0$.

Then $\sin\alpha = -\sqrt{1-\left(\dfrac{23}{25}\right)^2} =$
$-\sqrt{1-\dfrac{529}{625}} = -\sqrt{\dfrac{96}{625}} = -\dfrac{4\sqrt{6}}{25}$.

Furthermore, $\sec\alpha = \dfrac{25}{23}$, $\csc\alpha = -\dfrac{25}{4\sqrt{6}} =$
$-\dfrac{25\sqrt{6}}{24}$, $\tan\alpha = \dfrac{-4\sqrt{6}/25}{23/25} = -\dfrac{4\sqrt{6}}{23}$,
and $\cot\alpha = -\dfrac{23}{4\sqrt{6}} = -\dfrac{23\sqrt{6}}{24}$.

53.

$$\begin{aligned} \cos^4 s - \sin^4 s &= \\ (\cos^2 s - \sin^2 s)(\cos^2 s + \sin^2 s) &= \\ \cos(2s)\cdot(1) &= \\ \cos(2s) & \end{aligned}$$

54.

$$\begin{aligned} &= 2\sin(s)\sin(\pi/2-s) \\ &= 2\sin(s)\cos(s) \\ &\ \sin(2s) \end{aligned}$$

55.

$$\begin{aligned} \cos(2t+t) &= \\ \cos(2t)\cos(t) - \sin(2t)\sin(t) &= \\ \left[\cos^2 t - \sin^2 t\right]\cos t - [2\sin t\cos t]\sin t &= \\ \cos^3 t - \sin^2 t\cos t - 2\sin^2 t\cos t &= \\ \cos^3 t - 3\sin^2 t\cos t & \end{aligned}$$

56.

$$\begin{aligned} \frac{\sin(4t)}{4} &= \\ \frac{2\sin(2t)\cos(2t)}{4} &= \\ \frac{2\cdot 2\sin t\cos t\cdot(\cos^2 t-\sin^2 t)}{4} &= \\ \sin t\cos t(\cos^2 t-\sin^2 t) &= \\ \cos^3 t\sin t-\sin^3 t\cos t & \end{aligned}$$

57.

$$\begin{aligned} \frac{\cos(2x)+\cos(2y)}{\sin(x)+\cos(y)} &= \\ \frac{1-2\sin^2 x+2\cos^2 y-1}{\sin x+\cos y} &= \\ 2\frac{\cos^2 y-\sin^2 x}{\sin x+\cos y} &= \\ 2\frac{(\cos y-\sin x)(\cos y+\sin x)}{\sin x+\cos y} &= \\ 2\cos(y)-2\sin(x) & \end{aligned}$$

58.

$$\begin{aligned} (\sin\alpha-\cos\alpha)^2 &= \\ \sin^2\alpha-2\sin\alpha\cos\alpha+\cos^2\alpha &= \\ 1-2\sin\alpha\cos\alpha &= \\ 1-\sin(2\alpha) & \end{aligned}$$

59.

$$\begin{aligned} \frac{\cos 2x}{\sin^2 x} &= \\ \frac{1-2\sin^2 x}{\sin^2 x} &= \\ \frac{1}{\sin^2 x}-2\cdot\frac{\sin^2 x}{\sin^2 x} &= \\ \csc^2 x-2 & \end{aligned}$$

60.

$$\begin{aligned} \frac{\cos(2s)}{\cos^2 s} &= \\ \frac{1-2\sin^2 s}{\cos^2 s} &= \\ \frac{1}{\cos^2 s}-2\cdot\frac{\sin^2 s}{\cos^2 s} &= \\ \sec^2 s-2\tan^2 s & \end{aligned}$$

61.

$$\begin{aligned} &= \frac{\sin^2 u}{1+\cos u} \\ &= \frac{1-\cos^2 u}{1+\cos u} \\ &= \frac{(1-\cos u)(1+\cos u)}{1+\cos u} \\ &= (1-\cos u)\cdot\frac{2}{2} \\ &= 2\cdot\frac{1-\cos u}{2} \\ & 2\sin^2(u/2) \end{aligned}$$

62.

$$\begin{aligned} &= \frac{1-\tan^2 y}{1+\tan^2 y} \\ &= \frac{1-\tan^2 y}{\sec^2 y} \\ &= \frac{1}{\sec^2 y}-\frac{\tan^2 y}{\sec^2 y} \\ &= \cos^2(y)-\frac{\sin^2 y/\cos^2 y}{1/\cos^2 y} \\ &= \cos^2 y-\sin^2 y \\ & \cos(2y) \end{aligned}$$

63. Multiply and divide by $\cos x$.

$$\begin{aligned} &= \frac{\sec x+\cos x-2}{\sec x-\cos x}\cdot\frac{\cos x}{\cos x} \\ &= \frac{1+\cos^2 x-2\cos x}{1-\cos^2 x} \\ &= \frac{\cos^2 x-2\cos x+1}{1-\cos^2 x} \\ &= \frac{(1-\cos x)^2}{(1+\cos x)(1-\cos x)} \\ &= \frac{1-\cos x}{1+\cos x} \\ & \tan^2(x/2) \end{aligned}$$

64. Multiply and divide by $\cos x$.

$$\begin{aligned} &= \frac{2\sec x+2}{\sec x+2+\cos x}\cdot\frac{\cos x}{\cos x} \\ &= \frac{2+2\cos x}{1+2\cos x+\cos^2 x} \end{aligned}$$

$$= \frac{2(1+\cos x)}{(1+\cos x)^2}$$
$$= \frac{2}{1+\cos x}$$
$$= \left(\cos^2\left(\frac{x}{2}\right)\right)^{-1}$$
$$\sec^2\left(\frac{x}{2}\right)$$

65.

$$\frac{1-\sin^2(x/2)}{1+\sin^2(x/2)} =$$
$$\frac{1-\left(\frac{1-\cos x}{2}\right)}{1+\left(\frac{1-\cos x}{2}\right)}\cdot\frac{2}{2} =$$
$$\frac{2-(1-\cos x)}{2+(1-\cos x)} =$$
$$\frac{1+\cos x}{3-\cos x}$$

66.

$$\frac{1-\cos^2(x/2)}{1-\sin^2(x/2)} =$$
$$\frac{1-\left(\frac{1+\cos x}{2}\right)}{1-\left(\frac{1-\cos x}{2}\right)}\cdot\frac{2}{2} =$$
$$\frac{2-(1+\cos x)}{2-(1-\cos x)} =$$
$$\frac{1-\cos x}{1+\cos x}$$

67. It is not an identity. If $x = \pi/4$, then $\sin(2\cdot\pi/4) = \sin(\pi/2) = 1$ and $2\sin(\pi/4) = 2\cdot(\sqrt{2}/2) = \sqrt{2}$.

68. It is not an identity. If $x = \pi$, then $\frac{\cos(2\pi)}{2} = \frac{1}{2}$ and $\cos(\pi) = -1$.

69. It is not an identity. If $x = 2\pi/3$, then $\tan\left(\frac{2\pi/3}{2}\right) = \tan(\pi/3) = \sqrt{3}$ and $\frac{1}{2}\cdot\tan(2\pi/3) = \frac{1}{2}\cdot(-\sqrt{3})$.

70. It is not an identity. If $x = 4\pi/3$, then $\tan\left(\frac{4\pi/3}{2}\right) = \tan(2\pi/3) = -\sqrt{3}$ while $\sqrt{\frac{1-\cos(4\pi/3)}{1+\cos(4\pi 3)}}$ is a positive number.

71. It is not an identity. If $x = \pi/2$, then $\sin(2\cdot\pi/2)\sin\left(\frac{\pi/2}{2}\right) = \sin(\pi)\sin(\pi/4)$ $= 0\cdot\frac{\sqrt{2}}{2} = 0$ and $\sin^2(\pi/2) = 1$.

72. It is not an identity. If $x = \pi/4$, then $\tan(\pi/4) + \tan(\pi/4) = 1 + 1 = 2$ and $\tan(2\cdot\pi/4) = \tan(\pi/2) = 0$.

73. It is an identity. The proof below uses the double-angle identity for tangent.

$$\cot(x/2) - \tan(x/2) =$$
$$\frac{1}{\tan(x/2)} - \tan(x/2) =$$
$$\frac{1-\tan^2(x/2)}{\tan(x/2)} =$$
$$2\cdot\frac{1-\tan^2(x/2)}{2\cdot\tan(x/2)} =$$
$$2\cdot\frac{1}{\tan x} =$$
$$2\cdot\frac{\cos x}{\sin x}\cdot\frac{\sin x}{\sin x} =$$
$$\frac{2\sin x\cos x}{\sin^2 x} =$$
$$\frac{\sin(2x)}{\sin^2 x}$$

74. It is an identity. The proof uses the double-angle identity for sine.

$$\csc^2(x/2) + \sec^2(x/2) =$$
$$\frac{1}{\sin^2(x/2)} + \frac{1}{\cos^2(x/2)} =$$
$$\frac{\cos^2(x/2)+\sin^2(x/2)}{\sin^2(x/2)\cos^2(x/2)} =$$
$$\frac{1}{\sin^2(x/2)\cos^2(x/2)} =$$

$$\begin{aligned}\frac{4}{4\sin^2(x/2)\cos^2(x/2)} &= \\ \frac{4}{\left[2\sin(x/2)\cos(x/2)\right]^2} &= \\ \frac{4}{[\sin(2\cdot(x/2))]^2} &= \\ \frac{4}{\sin^2 x} &= \\ 4\csc^2 x &\end{aligned}$$

75. Note, $\cos\alpha = -\sqrt{1-\left(\frac{3}{5}\right)^2} = -\frac{4}{5}$.

Then $\sin 2\alpha = 2\sin\alpha\cos\alpha =$ $2\cdot\frac{3}{5}\cdot\frac{-4}{5} = -\frac{24}{25}$.

76. Since $\tan\alpha = -\frac{8}{15}$ and α lies in quadrant IV, we obtain $\cos\alpha = \frac{15}{\sqrt{8^2+15^2}} = \frac{15}{17}$ and $\sin\alpha = -\frac{8}{\sqrt{8^2+15^2}} = -\frac{8}{17}$.

Then $\sin 2\alpha = 2\sin\alpha\cos\alpha =$ $2\cdot\frac{-8}{17}\cdot\frac{15}{17} = -\frac{240}{289}$.

77. $\cos 2\alpha = 1 - 2\sin^2\alpha = 1 - 2\left(\frac{8}{17}\right)^2 = \frac{161}{289}$

78. Note, $\cos\alpha = -\sqrt{1-\left(\frac{-4}{5}\right)^2} = -\frac{3}{5}$.

Then $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-4/5}{-3/5} = \frac{4}{3}$ and

$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2(4/3)}{1-(4/3)^2} =$ $\frac{24/9}{-7/9} = -\frac{24}{7}$.

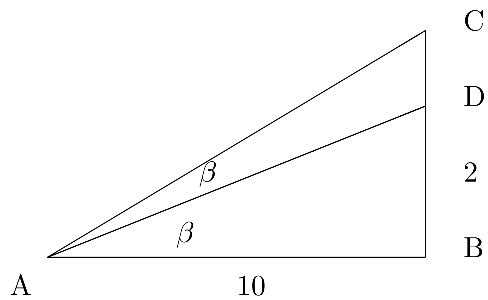
79. Since $\tan\alpha = \frac{3}{5}$, $\sin\alpha = \frac{3}{\sqrt{34}}$ and $\cos\alpha = \frac{5}{\sqrt{34}}$. By a half-angle identity, we obtain

$$\begin{aligned}\tan\frac{\alpha}{2} &= \frac{\sin\alpha}{1+\cos\alpha} \\ &= \frac{\frac{3}{\sqrt{34}}}{1+\frac{5}{\sqrt{34}}} \\ &= \frac{3}{5+\sqrt{34}}\end{aligned}$$

and since $\tan\frac{\alpha}{2} = \frac{BD}{5}$ then

$$\begin{aligned}BD &= \frac{15}{5+\sqrt{34}} \\ &= \frac{15(5-\sqrt{34})}{25-34} \\ &= \frac{15(\sqrt{34}-5)}{9} \\ BD &= \frac{5\sqrt{34}-25}{3}.\end{aligned}$$

80. Let CD be the distance between C and D.



Note, $\tan\beta = \frac{1}{5}$ and $\tan 2\beta = \frac{CD+2}{10}$.
Using the double angle identity for tangent, one finds

$$\begin{aligned}\tan 2\beta &= \frac{2\tan\beta}{1-\tan^2\beta} \\ &= \frac{2(1/5)}{1-(1/5)^2} \\ &= \frac{2/5}{24/25} \\ \frac{CD+2}{10} &= \frac{5}{12} \\ CD+2 &= \frac{50}{12}\end{aligned}$$

$$CD = \frac{13}{6}$$

81. Since the base of the TV screen is $b = d\cos\alpha$ and its height is $h = d\sin\alpha$, then the area A is given by

$$\begin{aligned} A &= bh \\ &= (d\cos\alpha)(d\sin\alpha) \\ &= d^2\cos\alpha\sin\alpha \\ A &= \frac{d^2}{2}\sin(2\alpha). \end{aligned}$$

82. The area is $A = \dfrac{32^2}{2}\sin(2\cdot 37.2^\circ) \approx 493.1$ in.^{2}

85. $\dfrac{1}{1-\sin x} + \dfrac{1}{1+\sin x} = \dfrac{1+\sin x + 1 - \sin x}{1-\sin^2 x} = \dfrac{2}{\cos^2 x} = 2\sec^2 x$

86. Factor $\cos x$ as follows:
$\dfrac{\cos x\,(\cos^2 x + \sin^2 x)}{\sin x} = \dfrac{\cos x \cdot 1}{\sin x} = \cot x$

87. **a)** $\cos x\cos y - \sin x\sin y$
b) $\cos x\cos y + \sin x\sin y$

88. Let $A = 26^\circ$, and let a and b be the sides opposite and adjacent to A, respectively. Since $\sin A = a/38.6$, we find

$$a = 38.6\sin 26^\circ \approx 16.9 \text{ in.}$$

Since $\cos A = b/38.6$, we find

$$b = 38.6\cos 26^\circ \approx 34.7 \text{ in.}$$

89. a) $\dfrac{1}{2}$ b) -1 c) undefined
d) 2 e) -1 f) 1

90. The race car will lap the track in

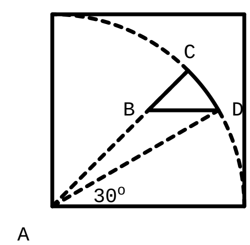
$$\frac{\text{circumference}}{\text{speed}} = \frac{2\pi\frac{1}{4}}{180} = \frac{\pi}{360}\text{ hour} = \frac{\pi}{6}\text{ min}$$

Then the angular speed is

$$\frac{2\pi}{\text{time around 1 lap}} = \frac{2\pi}{\pi/6} = 12 \text{ radians/min.}$$

Thinking Outside the Box LV

An eighth of the region that gets watered by all sprinklers is region R_a below with vertices B, C, and D.



The area of R_a is the area of the sector determined by C, A , and D minus the area of triangle ΔABD. In the figure above, we have $AB = \sqrt{2}$, $BC = 2-\sqrt{2}$, $BD = \sqrt{3}-1$, angle $\langle ABD = 135^\circ$, and $\langle CBD = 45^\circ$.

The area of the sector is

$$A_s = \frac{1}{2}\left(2^2\frac{\pi}{12}\right) = \frac{\pi}{6}$$

and the area of ΔABD is

$$\begin{aligned} A_t &= \frac{1}{2}(AB)(BD)\sin 135^\circ \\ &= \frac{1}{2}\sqrt{2}(\sqrt{3}-1)\frac{\sqrt{2}}{2} \\ A_t &= \frac{\sqrt{3}-1}{2}. \end{aligned}$$

Thus, the area watered by all sprinklers is

$$\begin{aligned} \text{Area} &= 8\,(A_s - A_t) \\ &= 8\left(\frac{\pi}{6} - \frac{\sqrt{3}-1}{2}\right) \\ \text{Area} &= \frac{4\pi}{3} + 4 - 4\sqrt{3}. \end{aligned}$$

6.4 Pop Quiz

1. Since $\sin\alpha = 1/4$ and $\cos\alpha = -\sqrt{15}/4$, we find

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\cdot\frac{1}{4}\cdot\frac{-\sqrt{15}}{4} = -\frac{\sqrt{15}}{8}.$$

2. Since $\cos\alpha = -3/5$ and $\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}$,

we obtain

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-(-3/5)}{2}} = \sqrt{\frac{4}{5}} = \frac{2\sqrt{5}}{5}$$

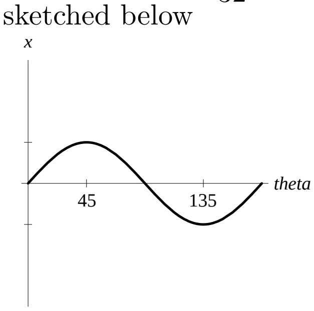
3.

$$\begin{aligned} \sin^4 x - \cos^4 x &= \\ (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) &= \\ -(\cos^2 x - \sin^2 x)(1) &= \\ -\cos 2x & \end{aligned}$$

6.4 Linking Concepts

a) $x = \dfrac{2\cdot v_0^2 \sin\theta\cos\theta}{2\cdot 16} = \dfrac{v_0^2\sin(2\theta)}{32}$.

b) A graph of $x = \dfrac{50^2\sin(2\theta)}{32}$ or $x = 78.125\sin 2\theta$ is sketched below



c) For any v_0, the maximum value of

$x = \dfrac{v_0^2\sin(2\theta)}{32}$ is attained when $\sin(2\theta) = 1$, i.e., when $\theta = 45°$. Yes, this value $\theta = 45°$ maximizes x for any velocity v_0.

d) The initial velocity v_0 is obtained by solving the following equations (note: 55 yards = 165 feet).

$$\begin{aligned} \frac{v_0^2\sin(2\cdot 45°)}{32} &= 165 \\ \frac{v_0^2\cdot 1}{32} &= 165 \\ v_0^2 &= (32)165 \\ v_0 &= \sqrt{32(165)} \text{ ft/sec} \\ v_0 &\approx 72.6636 \text{ ft/sec} \\ v_0 &\approx 72.6636\cdot\frac{3600}{5280} \text{ mph} \\ v_0 &\approx 49.5 \text{ mph} \end{aligned}$$

e) Taking air resistance into account, the actual initial velocity is larger than the answer found in part d).

For Thought

1. True, $\sin 45°\cos 15° =$
$(1/2)\left[\sin(45° + 15°) + \sin(45° - 15°)\right] =$
$0.5\left[\sin 60° + \sin 30°\right]$.

2. False, $\cos(\pi/8)\sin(\pi/4) =$
$(1/2)\left[\sin(\pi/8 + \pi/4) - \sin(\pi/8 - \pi/4)\right] =$
$0.5\left[\sin(3\pi/8) - \sin(-\pi/8)\right] =$
$0.5\left[\sin(3\pi/8) + \sin(\pi/8)\right]$.

3. True, $2\cos(6°)\cos(8°) =$
$\cos(6° - 8°) + \cos(6° + 8°) =$
$\cos(-2°) + \cos(14°) = \cos(2°) + \cos(14°)$.

4. False, $\sin(5°) - \sin(9°) =$
$2\cos\left(\dfrac{5° + 9°}{2}\right)\sin\left(\dfrac{5° - 9°}{2}\right) =$
$2\cos(7°)\sin(-2°) = -2\cos(7°)\sin(2°)$.

5. True, $\cos(4) + \cos(12) =$
$2\cos\left(\dfrac{4+12}{2}\right)\cos\left(\dfrac{4-12}{2}\right) =$
$2\cos(8)\cos(-4) = 2\cos(8)\cos(4)$.

6. False, $\cos(\pi/3) - \cos(\pi/2) =$
$-2\sin\left(\dfrac{\pi/3 + \pi/2}{2}\right)\sin\left(\dfrac{\pi/3 - \pi/2}{2}\right) =$
$-2\sin(5\pi/12)\sin(-\pi/12) =$
$2\sin(5\pi/12)\sin(\pi/12)$.

7. True, $\sqrt{2}\sin(\pi/6+\pi/4) =$
$\sqrt{2}\left[\sin(\pi/6)\cos(\pi/4)+\cos(\pi/6)\sin(\pi/4)\right] =$
$\sqrt{2}\left[\sin(\pi/6)\cdot\frac{1}{\sqrt{2}}+\cos(\pi/6)\cdot\frac{1}{\sqrt{2}}\right] =$
$\sin(\pi/6)+\cos(\pi/6)$.

8. True, $\frac{1}{2}\sin(\pi/6)+\frac{\sqrt{3}}{2}\cos(\pi/6) =$
$\frac{1}{2}\cdot\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}=\frac{1}{4}+\frac{3}{4}=1=\sin(\pi/2)$.

9. True, $y=\cos(\pi/3)\sin x+\sin(\pi/3)\cos x =$
$\sin(x+\pi/3)$.

10. True, since $y=\cos(\pi/4)\sin x+\sin(\pi/4)\cos x$ $=\sin(x+\pi/4)$ by the sum identity for sine.

6.5 Exercises

1.
$\frac{1}{2}\left[\cos(13^\circ-9^\circ)-\cos(13^\circ+9^\circ)\right] =$
$0.5\left[\cos 4^\circ-\cos 22^\circ\right]$

2.
$\frac{1}{2}\left[\cos(34^\circ-39^\circ)+\cos(34^\circ+39^\circ)\right] =$
$0.5\left[\cos(-5^\circ)+\cos 73^\circ\right]=0.5\left[\cos 5^\circ+\cos 73^\circ\right]$

3.
$\frac{1}{2}\left[\sin(16^\circ+20^\circ)+\sin(16^\circ-20^\circ)\right] =$
$0.5\left[\sin 36^\circ+\sin(-4^\circ)\right]=0.5\left[\sin 36^\circ-\sin 4^\circ\right]$

4.
$\frac{1}{2}\left[\sin(9^\circ+8^\circ)-\sin(9^\circ-8^\circ)\right] =$
$0.5\left[\sin 17^\circ-\sin 1^\circ\right]$

5.
$\frac{1}{2}\left[\sin(5^\circ+10^\circ)+\sin(5^\circ-10^\circ)\right] =$
$0.5\left[\sin 15^\circ+\sin(-5^\circ)\right]=0.5\left[\sin 15^\circ-\sin 5^\circ\right]$

6.
$\frac{1}{2}\left[\cos(6^\circ-8^\circ)+\cos(6^\circ+8^\circ)\right] =$
$0.5\left[\cos(-2^\circ)+\cos 14^\circ\right]=0.5\left[\cos 2^\circ+\cos 14^\circ\right]$

7.
$\frac{1}{2}\left[\cos\left(\frac{\pi}{6}-\frac{\pi}{5}\right)+\cos\left(\frac{\pi}{6}+\frac{\pi}{5}\right)\right] =$
$0.5\left[\cos\left(\frac{-\pi}{30}\right)+\cos\left(\frac{11\pi}{30}\right)\right] =$
$0.5\left[\cos\left(\frac{\pi}{30}\right)+\cos\left(\frac{11\pi}{30}\right)\right]$

8.
$\frac{1}{2}\left[\cos(\frac{2\pi}{9}-\frac{3\pi}{4})-\cos(\frac{2\pi}{9}+\frac{3\pi}{4})\right] =$
$0.5\left[\cos(-19\pi/36)-\cos(35\pi/36)\right] =$
$0.5\left[\cos(19\pi/36)-\cos(35\pi/36)\right]$

9.
$\frac{1}{2}\left[\cos(5y^2-7y^2)+\cos(5y^2+7y^2)\right] =$
$0.5\left[\cos(-2y^2)+\cos(12y^2)\right] =$
$0.5\left[\cos(2y^2)+\cos(12y^2)\right]$

10.
$\frac{1}{2}\left[\sin(3t+5t)-\sin(3t-5t)\right] =$
$0.5\left[\sin(8t)-\sin(-2t)\right] =$
$0.5\left[\sin(8t)+\sin(2t)\right]$

11.
$\frac{1}{2}\Big[\sin((2s-1)+(s+1))+$
$\sin((2s-1)-(s+1))\Big] =$
$0.5\Big[\sin(3s)+\sin(s-2)\Big]$

12.
$\frac{1}{2}\Big[\cos((3t-1)-(2t+3))-$
$\cos((3t-1)+(2t+3))\Big] =$
$\frac{1}{2}\Big[\cos(t-4)-\cos(5t+2)\Big]$

13.
$\frac{1}{2}\left[\cos(52.5^\circ-7.5^\circ)-\cos(52.5^\circ+7.5^\circ)\right] =$
$\frac{1}{2}\left[\cos 45^\circ-\cos 60^\circ\right] =$
$\frac{1}{2}\left[\frac{\sqrt{2}}{2}-\frac{1}{2}\right]=\frac{\sqrt{2}-1}{4}$

14.
$\frac{1}{2}\left[\cos(105^\circ-75^\circ)+\cos(105^\circ+75^\circ)\right] =$
$\frac{1}{2}\left[\cos(30^\circ)+\cos 180^\circ\right]=\frac{1}{2}\left[\frac{\sqrt{3}}{2}+(-1)\right] =$
$\frac{\sqrt{3}-2}{4}$

15.
$$\frac{1}{2}\left[\sin\left(\frac{13\pi}{24}+\frac{5\pi}{24}\right)+\sin\left(\frac{13\pi}{24}-\frac{5\pi}{24}\right)\right]=$$
$$\frac{1}{2}[\sin(18\pi/24)+\sin(8\pi/24)]=$$
$$\frac{1}{2}[\sin(3\pi/4)+\sin(\pi/3)]=$$
$$\frac{1}{2}\left[\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\right]=\frac{\sqrt{2}+\sqrt{3}}{4}$$

16.
$$\frac{1}{2}\left[\sin\left(\frac{5\pi}{24}+\frac{-\pi}{24}\right)-\sin\left(\frac{5\pi}{24}-\frac{-\pi}{24}\right)\right]=$$
$$\frac{1}{2}[\sin(4\pi/24)-\sin(6\pi/24)]=$$
$$\frac{1}{2}[\sin(\pi/6)-\sin(\pi/4)]=$$
$$\frac{1}{2}\left[\frac{1}{2}-\frac{\sqrt{2}}{2}\right]=\frac{1-\sqrt{2}}{4}$$

17.
$$2\cos\left(\frac{12^\circ+8^\circ}{2}\right)\sin\left(\frac{12^\circ-8^\circ}{2}\right)=$$
$$2\cos 10^\circ\sin 2^\circ$$

18.
$$2\sin\left(\frac{7^\circ+11^\circ}{2}\right)\cos\left(\frac{7^\circ-11^\circ}{2}\right)=$$
$$2\sin 9^\circ\cos(-2^\circ)=2\sin 9^\circ\cos 2^\circ$$

19.
$$-2\sin\left(\frac{80^\circ+87^\circ}{2}\right)\sin\left(\frac{80^\circ-87^\circ}{2}\right)=$$
$$-2\sin 83.5^\circ\sin(-3.5^\circ)=2\sin 83.5^\circ\sin 3.5^\circ$$

20.
$$2\cos\left(\frac{44^\circ+31^\circ}{2}\right)\cos\left(\frac{44^\circ-31^\circ}{2}\right)=$$
$$2\cos 37.5^\circ\cos 6.5^\circ$$

21.
$$2\cos\left(\frac{3.6+4.8}{2}\right)\sin\left(\frac{3.6-4.8}{2}\right)=$$
$$2\cos(4.2)\sin(-0.6)=-2\cos(4.2)\sin(0.6)$$

22.
$$2\sin\left(\frac{5.1+6.3}{2}\right)\cos\left(\frac{5.1-6.3}{2}\right)=$$
$$2\sin(5.7)\cos(-0.6)=2\sin(5.7)\cos(0.6)$$

23.
$$-2\sin\left(\frac{\pi/3+\pi/5}{2}\right)\sin\left(\frac{\pi/3-\pi/5}{2}\right)=$$
$$-2\sin(4\pi/15)\sin(\pi/15)=$$

24.
$$2\cos\left(\frac{1/2+2/3}{2}\right)\cos\left(\frac{1/2-2/3}{2}\right)=$$
$$2\cos(7/12)\cos(-1/12)=$$
$$2\cos(7/12)\cos(1/12)$$

25.
$$-2\sin\left(\frac{(5y-3)+(3y+9)}{2}\right)\cdot$$
$$\sin\left(\frac{(5y-3)-(3y+9)}{2}\right)=$$
$$-2\sin(4y+3)\sin(y-6)$$

26.
$$2\cos\left(\frac{(6t^2-1)+(4t^2-1)}{2}\right)\cdot$$
$$\cos\left(\frac{(6t^2-1)-(4t^2-1)}{2}\right)=$$
$$2\cos(5t^2-1)\cos(t^2)$$

27.
$$2\cos\left(\frac{5\alpha+8\alpha}{2}\right)\sin\left(\frac{5\alpha-8\alpha}{2}\right)=$$
$$2\cos(6.5\alpha)\sin(-1.5\alpha)=$$
$$-2\cos(6.5\alpha)\sin(1.5\alpha)$$

28.
$$2\sin\left(\frac{3s+5s}{2}\right)\cos\left(\frac{3s-5s}{2}\right)=$$
$$2\sin(4s)\cos(-s)=2\sin(4s)\cos s$$

29.
$$2\sin\left(\frac{75^\circ+15^\circ}{2}\right)\cos\left(\frac{75^\circ-15^\circ}{2}\right)=$$
$$2\sin 45^\circ\cos(30^\circ)=2\cdot\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}=\frac{\sqrt{6}}{2}$$

30.
$$2\cos\left(\frac{285^\circ+15^\circ}{2}\right)\sin\left(\frac{285^\circ-15^\circ}{2}\right)=$$
$$2\cos 150^\circ\sin 135^\circ=2\cdot\frac{-\sqrt{3}}{2}\frac{\sqrt{2}}{2}=-\frac{\sqrt{6}}{2}$$

31.

$$-2\sin\left(\frac{\frac{-\pi}{24}+\frac{7\pi}{24}}{2}\right)\sin\left(\frac{\frac{-\pi}{24}-\frac{7\pi}{24}}{2}\right)=$$

$-2\sin(3\pi/24)\sin(-4\pi/24)=$
$-2\sin(\pi/8)\sin(-\pi/6)=$

$$-2\sin\left(\frac{\pi/4}{2}\right)\cdot\frac{-1}{2}=-2\sqrt{\frac{1-\cos(\pi/4)}{2}}\cdot\frac{-1}{2}$$

$$=\sqrt{\frac{1-\sqrt{2}/2}{2}\cdot\frac{2}{2}}=\sqrt{\frac{2-\sqrt{2}}{4}}=\frac{\sqrt{2-\sqrt{2}}}{2}$$

32.

$$2\cos\left(\frac{5\pi/24+\pi/24}{2}\right)\cos\left(\frac{5\pi/24-\pi/24}{2}\right)=$$

$2\cos(3\pi/24)\cos(2\pi/24)=$

$2\cos(\pi/8)\cos(\pi/12)=$

$$2\cos\left(\frac{\pi/4}{2}\right)\cos\left(\frac{\pi/6}{2}\right)=$$

$$2\sqrt{\frac{1+\cos(\pi/4)}{2}}\cdot\sqrt{\frac{1+\cos(\pi/6)}{2}}=$$

$$2\sqrt{\frac{1+\sqrt{2}/2}{2}\cdot\frac{2}{2}}\cdot\sqrt{\frac{1+\sqrt{3}/2}{2}\cdot\frac{2}{2}}=$$

$$2\sqrt{\frac{2+\sqrt{2}}{4}}\cdot\sqrt{\frac{2+\sqrt{3}}{4}}=$$

$$2\sqrt{\frac{4+2\sqrt{2}+2\sqrt{3}+\sqrt{6}}{16}}=$$

$$\frac{\sqrt{4+2\sqrt{2}+2\sqrt{3}+\sqrt{6}}}{2}$$

33. Since $a=1$ and $b=-1$, we obtain

$$r=\sqrt{1^2+(-1)^2}=\sqrt{2}.$$

If the terminal side of α passes through $(1,-1)$, then $\cos\alpha=a/r=1/\sqrt{2}$ and $\sin\alpha=b/r=-1/\sqrt{2}$. Choose $\alpha=-\pi/4$.

Thus, $\sin x-\cos x=r\sin(x+\alpha)=\sqrt{2}\sin(x-\pi/4)$.

34. Since $a=2$ and $b=2$, $r=\sqrt{2^2+2^2}=2\sqrt{2}$. If the terminal side of α passes through $(2,2)$, then $\cos\alpha=a/r=\dfrac{2}{2\sqrt{2}}=\dfrac{1}{\sqrt{2}}$ and $\sin\alpha=b/r=\dfrac{2}{2\sqrt{2}}=\dfrac{1}{\sqrt{2}}$. Choose $\alpha=\pi/4$.

Thus, $2\sin x+2\cos x=r\sin(x+\alpha)=2\sqrt{2}\sin(x+\pi/4)$.

35. Since $a=-1/2$ and $b=\sqrt{3}/2$, we obtain $r=\sqrt{(-1/2)^2+(\sqrt{3}/2)^2}=1$. If the terminal side of α passes through $(-1/2,\sqrt{3}/2)$, then $\cos\alpha=a/r=a/1=a=-1/2$ and $\sin\alpha=b/r=b/1=b=\sqrt{3}/2$. Choose $\alpha=2\pi/3$. So $-\dfrac{1}{2}\sin x+\dfrac{\sqrt{3}}{2}\cos x=r\sin(x+\alpha)=\sin(x+2\pi/3)$.

36. Since $a=\sqrt{2}/2$ and $b=-\sqrt{2}/2$, we find $r=\sqrt{(\sqrt{2}/2)^2+(\sqrt{2}/2)^2}=1$. If the terminal side of α passes through $(\sqrt{2}/2,-\sqrt{2}/2)$, then $\cos\alpha=a/r=a/1=a=\sqrt{2}/2$ and $\sin\alpha=b/r=b/1=b=-\sqrt{2}/2$.
Choose $\alpha=-\pi/4$. Thus,

$\dfrac{\sqrt{2}}{2}\sin x-\dfrac{\sqrt{2}}{2}\cos x=r\sin(x+\alpha)=\sin(x-\pi/4)$.

37. Since $a=\sqrt{3}/2$ and $b=-1/2$, we have

$$r=\sqrt{(\sqrt{3}/2)^2+(-1/2)^2}=1.$$

If the terminal side of α passes through $(\sqrt{3}/2,-1/2)$, then

$$\cos\alpha=a/r=a/1=a=\sqrt{3}/2$$

and

$$\sin\alpha=b/r=b/1=b=-1/2.$$

Choose $\alpha=-\pi/6$. Thus,

$$\frac{\sqrt{3}}{2}\sin x-\frac{1}{2}\cos x=r\sin(x+\alpha)=$$

$$\sin(x-\pi/6).$$

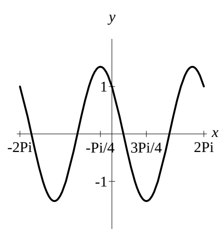
38. Since $a=-\sqrt{3}/2$ and $b=-1/2$, we get $r=\sqrt{(-\sqrt{3}/2)^2+(-1/2)^2}=1$. If the terminal side of α passes through $(-\sqrt{3}/2,-1/2)$, then $\cos\alpha=a/r=a/1=$

$a = -\sqrt{3}/2$ and $\sin\alpha = b/r = b/1 = b = -1/2$. Choose $\alpha = 7\pi/6$. Thus,

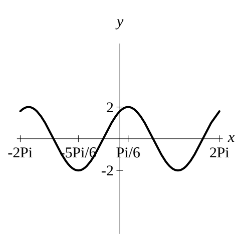
$$-\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x = r\sin(x+\alpha) =$$

$\sin(x + 7\pi/6)$.

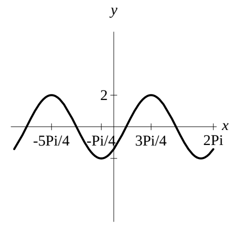
39. Since $a = -1$ and $b = 1$, we obtain $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$. If the terminal side of α passes through $(-1, 1)$, then $\cos\alpha = a/r = -1/\sqrt{2}$ and $\sin\alpha = b/r = 1/\sqrt{2}$. Choose $\alpha = 3\pi/4$. Then $y = -\sin x + \cos x = r\sin(x+\alpha) = \sqrt{2}\sin(x + 3\pi/4)$. Amplitude is $\sqrt{2}$, period is 2π, and phase shift is $-3\pi/4$.



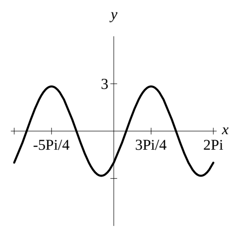
40. Since $a = 1$ and $b = \sqrt{3}$, we find $r = \sqrt{1^2 + \sqrt{3}^2} = 2$. If the terminal side of α passes through $(1, \sqrt{3})$, then $\cos\alpha = a/r = 1/2$ and $\sin\alpha = b/r = \sqrt{3}/2$. Choose $\alpha = \pi/3$. Thus, $y = \sin x + \sqrt{3}\cos x = r\sin(x+\alpha) = 2\sin(x + \pi/3)$. Amplitude is 2, period is 2π, and phase shift is $-\pi/3$.



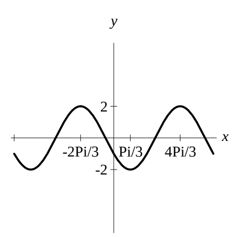
41. Since $a = \sqrt{2}$ and $b = -\sqrt{2}$, we obtain $r = \sqrt{\sqrt{2}^2 + (-\sqrt{2})^2} = 2$. If the terminal side of α passes through $(\sqrt{2}, -\sqrt{2})$, then $\cos\alpha = a/r = \sqrt{2}/2$ and $\sin\alpha = b/r = -\sqrt{2}/2$. Choose $\alpha = -\pi/4$. So $y = \sqrt{2}\sin x - \sqrt{2}\cos x = r\sin(x+\alpha) = 2\sin(x - \pi/4)$. Amplitude is 2, period is 2π, and phase shift is $\pi/4$.



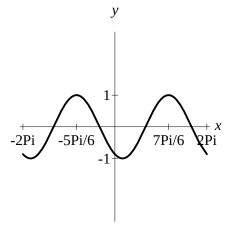
42. Since $a = 2$ and $b = -2$, we get $r = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$. If the terminal side of α passes through $(2, -2)$, then $\cos\alpha = a/r = 2/(2\sqrt{2}) = 1/\sqrt{2}$ and $\sin\alpha = b/r = -2/(2\sqrt{2}) = -1/\sqrt{2}$. Choose $\alpha = -\pi/4$. Then $y = 2\sin x - 2\cos x = r\sin(x+\alpha) = 2\sqrt{2}\sin(x - \pi/4)$. Amplitude is $2\sqrt{2}$, period is 2π, and phase shift is $\pi/4$.



43. Since $a = -\sqrt{3}$ and $b = -1$, we find $r = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$. If the terminal side of α passes through $(-\sqrt{3}, -1)$, then $\cos\alpha = a/r = -\sqrt{3}/2$ and $\sin\alpha = b/r = -1/2$. Choose $\alpha = 7\pi/6$. Then $y = -\sqrt{3}\sin x - \cos x = r\sin(x+\alpha) = 2\sin(x + 7\pi/6)$. Amplitude is 2, period is 2π, and phase shift is $-7\pi/6$.



44. Since $a = -1/2$ and $b = -\sqrt{3}/2$, we obtain $r = \sqrt{(-1/2)^2 + (-\sqrt{3}/2)^2} = 1$. If the terminal side of α passes through $(-1/2, -\sqrt{3}/2)$, then $\cos\alpha = a/r = -1/2$ and $\sin\alpha = -\sqrt{3}/2$ Choose $\alpha = 4\pi/3$. Thus, $y = -(1/2)\sin x - (\sqrt{3}/2)\cos x = r\sin(x+\alpha) = \sin(x + 4\pi/3)$. Amplitude is 1, period is 2π, and phase shift is $-4\pi/3$.



45. Since $a = 3$ and $b = 4$, the amplitude is $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$. If the terminal side of α passes through $(3, 4)$, then $\tan\alpha = 4/3$ and $\alpha = \tan^{-1}(4/3) \approx 0.9$. Phase shift is -0.9.

46. Since $a = 1$ and $b = 5$, the amplitude is $\sqrt{1^2 + 5^2} = \sqrt{26}$. If the terminal side of α passes through $(1, 5)$, then $\tan\alpha = 5/1 = 5$ and $\alpha = \tan^{-1}(5) \approx 1.4$. Phase shift is -1.4.

47. Since $a = -6$ and $b = 1$, amplitude is

$$\sqrt{(-6)^2 + 1^2} = \sqrt{37}.$$

If the terminal side of α passes through $(-6, 1)$, then $\tan\alpha = -1/6$. Using a calculator, one gets $\tan^{-1}(-1/6) \approx -0.165$ which is an angle in quadrant IV. Since $(-6, 1)$ is in quadrant II and π is the period of $\tan x$,

$$\alpha \approx -0.165 + \pi \approx 3.0.$$

The phase shift is -3.0.

48. Since $a = -\sqrt{5}$ and $b = 2$, amplitude is $\sqrt{(-\sqrt{5})^2 + 2^2} = 3$. If the terminal side of α passes through $(-\sqrt{5}, 2)$, then $\tan\alpha = -2/\sqrt{5}$. Using a calculator, one gets $\tan^{-1}(-2/\sqrt{5}) \approx -0.73$ which is an angle in quadrant IV. Since $(-\sqrt{5}, 2)$ is in quadrant II and π is the period of $\tan x$, $\alpha \approx -0.73 + \pi \approx 2.4$. The phase shift is -2.4.

49. Since $a = -3$ and $b = -5$, amplitude is $\sqrt{(-3)^2 + (-5)^2} = \sqrt{34}$. If the terminal side of α passes through $(-3, -5)$, then $\tan\alpha = 5/3$. Using a calculator, one gets $\tan^{-1}(5/3) \approx 1.03$ which is an angle in quadrant I. Since $(-3, -5)$ is in quadrant III and π is the period of $\tan x$, $\alpha \approx 1.03 + \pi \approx 4.2$. Phase shift is -4.2.

50. Since $a = -\sqrt{2}$ and $b = -\sqrt{7}$, amplitude is $\sqrt{(-\sqrt{2})^2 + (-\sqrt{7})^2} = 3$. If the terminal side of α passes through $(-\sqrt{2}, -\sqrt{7})$, then $\tan\alpha = \sqrt{7}/\sqrt{2}$. Using a calculator, one gets $\tan^{-1}(\sqrt{7}/\sqrt{2}) \approx 1.08$ which is an angle in quadrant I. Since $(-\sqrt{2}, -\sqrt{7})$ is in quadrant III and π is the period of $\tan x$, $\alpha \approx 1.08 + \pi \approx 4.2$. Phase shift is -4.2.

51. By using a sum-to-product identity, we get

$$\frac{\sin(3t) - \sin(t)}{\cos(3t) + \cos(t)} =$$

$$\frac{2\cos\left(\dfrac{3t + t}{2}\right)\sin\left(\dfrac{3t - t}{2}\right)}{2\cos\left(\dfrac{3t + t}{2}\right)\cos\left(\dfrac{3t - t}{2}\right)} =$$

$$\frac{2\cos(2t)\sin t}{2\cos(2t)\cos t} =$$

$$\tan t$$

52. By using a sum-to-product identity, we obtain

$$\frac{\sin(3x) + \sin(5x)}{\sin(3x) - \sin(5x)} =$$

$$\frac{2\sin\left(\dfrac{3x + 5x}{2}\right)\cos\left(\dfrac{3x - 5x}{2}\right)}{2\cos\left(\dfrac{3x + 5x}{2}\right)\sin\left(\dfrac{3x - 5x}{2}\right)} =$$

$$\frac{2\sin(4x)\cos(-x)}{2\cos(4x)\sin(-x)} =$$

$$-\frac{2\sin(4x)\cos x}{2\cos(4x)\sin x} =$$

$$-\frac{\tan(4x)}{\tan x}$$

53. By using a sum-to-product identity, we find

$$\frac{\cos x - \cos(3x)}{\cos x + \cos(3x)} =$$

$$\frac{-2\sin\left(\dfrac{x + 3x}{2}\right)\sin\left(\dfrac{x - 3x}{2}\right)}{2\cos\left(\dfrac{x + 3x}{2}\right)\cos\left(\dfrac{x - 3x}{2}\right)} =$$

$$\frac{-2\sin(2x)\sin(-x)}{2\cos(2x)\cos(-x)} =$$

$$\frac{2\sin(2x)\sin x}{2\cos(2x)\cos x} =$$
$$\tan(2x)\tan(x)$$

54.

$$\frac{\cos(5y)+\cos(3y)}{\cos(5y)-\cos(3y)} =$$
$$\frac{2\cos\left(\frac{5y+3y}{2}\right)\cos\left(\frac{5y-3y}{2}\right)}{-2\sin\left(\frac{5y+3y}{2}\right)\sin\left(\frac{5y-3y}{2}\right)} =$$
$$\frac{2\cos(4y)\cos y}{-2\sin(4y)\sin y} =$$
$$-\cot(4y)\cot y$$

55. By using a product-to-sum identity, we get

$$\begin{aligned}
&= -\sin(x+y)\sin(x-y)\\
&= -\frac{1}{2}\Big[\cos\left((x+y)-(x-y)\right)-\\
&\qquad \cos\left((x+y)+(x-y)\right)\Big]\\
&= -\frac{1}{2}\Big[\cos(2y)-\cos(2x)\Big]\\
&= -\frac{1}{2}\Big[(2\cos^2 y-1)-(2\cos^2 x-1)\Big]\\
&= -\frac{1}{2}\Big[2\cos^2 y-2\cos^2 x\Big]\\
&\cos^2 x-\cos^2 y
\end{aligned}$$

56. By using a product-to-sum identity, we obtain

$$\begin{aligned}
&= \sin(x+y)\sin(x-y)\\
&= \frac{1}{2}\Big[\cos\left((x+y)-(x-y)\right)-\\
&\qquad \cos\left((x+y)+(x-y)\right)\Big]\\
&= \frac{1}{2}\Big[\cos(2y)-\cos(2x)\Big]\\
&= \frac{1}{2}\Big[(1-2\sin^2 y)-(1-2\sin^2 x)\Big]\\
&= \frac{1}{2}\Big[-2\sin^2 y+2\sin^2 x)\Big]\\
&\sin^2 x-\sin^2 y.
\end{aligned}$$

57. Let $A = \frac{x+y}{2}$ and $B = \frac{x-y}{2}$.
Note, $A+B = x$ and $A-B = y$. Expand the left-hand side and use product-to-sum identities.

$$\begin{aligned}
(\sin A+\cos A)(\sin B+\cos B) &=\\
\sin A\sin B+\sin A\cos B+&\\
\cos A\sin B+\cos A\cos B &=\\
\frac{1}{2}\Big[\cos(A-B)-\cos(A+B)\Big]+&\\
\frac{1}{2}\Big[\sin(A+B)+\sin(A-B)\Big]+&\\
\frac{1}{2}\Big[\sin(A+B)-\sin(A-B)\Big]+&\\
\frac{1}{2}\Big[\cos(A-B)+\cos(A+B)\Big] &=\\
\frac{1}{2}\Big[\cos y-\cos x\Big]+\frac{1}{2}\Big[\sin x+\sin y\Big]+&\\
\frac{1}{2}\Big[\sin x-\sin y\Big]+\frac{1}{2}\Big[\cos y+\cos x\Big] &=\\
\frac{1}{2}\Big[2\cos y+2\sin x\Big] &=\\
\sin x+\cos y&
\end{aligned}$$

58. Factor the right-hand side as a difference of two squares and use sum-to-product identities.

$$\begin{aligned}
&= \sin^2(A+B)-\sin^2(A-B)\\
&= \Big[\sin(A+B)-\sin(A-B)\Big]\cdot\\
&\qquad \Big[\sin(A+B)+\sin(A-B)\Big]\\
&= 2\cos\left(\frac{(A+B)+(A-B)}{2}\right)\cdot\\
&\qquad \sin\left(\frac{(A+B)-(A-B)}{2}\right)\cdot\\
&\qquad 2\sin\left(\frac{(A+B)+(A-B)}{2}\right)\cdot\\
&\qquad \cos\left(\frac{(A+B)-(A-B)}{2}\right)\\
&= 4\cos A\sin B\sin A\cos B\\
&= (2\sin A\cos A)(2\sin B\cos B)
\end{aligned}$$

$$= \sin(2A)\sin(2B)$$

59. Use a sum-to-product identity in the 2nd line, and a product-to-sum identity in the 5th line.

$$\begin{aligned} &= \sin^2(A+B) - \sin^2(A-B) \\ &= \sin(2A)\sin(2B) \\ &= (2\sin A\cos A)(2\sin B\cos B) \\ &= [2\cos A\cos B]\cdot[2\sin A\sin B] \\ &= \Big[\cos(A-B)+\cos(A+B)\Big]\cdot \\ &\qquad \Big[\cos(A-B)-\cos(A+B)\Big] \\ &= \cos^2(A-B) - \cos^2(A+B) \end{aligned}$$

60. Expand the left-hand side. Proof uses sum and difference identities.

$$\begin{aligned} (\sin A+\cos A)(\sin B+\cos B) &= \\ \sin A\sin B+\sin A\cos B+ & \\ \cos A\sin B+\cos A\cos B &= \\ \Big[\sin A\cos B+\cos A\sin B\Big]+ & \\ \Big[\cos A\cos B+\sin A\sin B\Big] &= \\ \sin(A+B)+\cos(A-B) &= \end{aligned}$$

61. Note that x can be written in the form $x = a\sin(t+\alpha)$. The maximum displacement of $x = \sqrt{3}\sin t + \cos t$ is

$$a = \sqrt{\sqrt{3}^2 + 1^2} = 2.$$

Thus, 2 meters is the maximum distance between the block and its resting position.

Since the terminal side of α goes through $(\sqrt{3}, 1)$, we get $\tan\alpha = 1/\sqrt{3}$ and one can choose $\alpha = \pi/6$. Then $x = 2\sin(t+\pi/6)$.

62. The maximum distance of $x = -0.3\sin t + 0.5\cos t$ is its amplitude which is $\sqrt{(0.3)^2+(0.5)^2} \approx 0.58$ meters

67. a) $\dfrac{\tan x+\tan y}{1-\tan x\tan y}$ b) $\dfrac{\tan x-\tan y}{1+\tan x\tan y}$

68. $\sin\alpha\cos\beta+\cos\alpha\sin\beta =$

$$\frac{1}{3}\frac{\sqrt{3}}{2} + \left(\frac{-2\sqrt{2}}{3}\right)\frac{1}{2} = \frac{\sqrt{3}-2\sqrt{2}}{6}$$

69. $\dfrac{1-\cos x}{\sin x} = \dfrac{1-(-1/3)}{-\sqrt{8}/3} =$

$$\frac{4/3}{-2\sqrt{2}/3} = \frac{2}{-\sqrt{2}} = -\sqrt{2}$$

70. $\sqrt{\dfrac{1-\cos x}{2}} = \sqrt{\dfrac{1-(-1/4)}{2}} = \sqrt{\dfrac{5/4}{2}} =$

$$\sqrt{\frac{5}{8}} = \sqrt{\frac{10}{16}} = \frac{\sqrt{10}}{4}$$

71. Since $\sin y = -\dfrac{4}{5}$ and $\cos y = \dfrac{3}{5}$, we obtain

$$\sin 2y = 2\sin y\cos y = 2\left(-\frac{4}{5}\right)\frac{3}{5} = -\frac{24}{25}$$

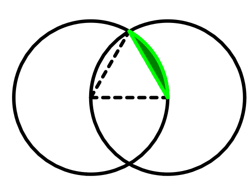
72. Let A be the angle opposite 5 miles, and B the angle opposite 12 miles. Then

$$A = \arcsin\left(\frac{5}{13}\right) \approx 22.6^\circ$$

$$B = \arcsin\left(\frac{12}{13}\right) \approx 67.4^\circ$$

Thinking Outside the Box

LVI. Given below are two circles with radii a. Consider the triangle with a vertex at the top point of intersection and with the centers of the circles as the other two vertices. This triangle is an equilateral triangle.



By subtracting the area of a triangle from the area of a sector, we find that the area of the shaded region above is

$$\frac{1}{2}a^2\frac{\pi}{3} - \frac{1}{2}a^2\frac{\sqrt{3}}{2}$$

or

$$\frac{a^2(2\pi - 3\sqrt{3})}{12}.$$

Then the area of the region inside the circle on the right that is outside the circle on the left is

$$2\left[\frac{1}{2}a^2\frac{2\pi}{3}\right] - 2\left[\frac{a^2(2\pi - 3\sqrt{3})}{12}\right]$$

or equivalently

$$\frac{a^2(2\pi + 3\sqrt{3})}{6}.$$

If we add the area of the circle on the left to the above expression we obtain the total area sprinkled, that is,

$$\pi a^2 + \frac{a^2(2\pi + 3\sqrt{3})}{6} = \frac{a^2(8\pi + 3\sqrt{3})}{6}.$$

LVII. Suppose L represents a move to the left square, R denotes a move to the right square, U represents a move to the upper square, and D represents a move to the square below.

A path from the upper left square to the lower right square may be represented as an ordered sequence using the letters L, R, U, and D. If we let l, r, u and d denote the number of L's, R's, U's, and D's, respectively, then

$$l + r + u + d = 63$$

$$r - l = 7 \quad \text{and}$$

$$d - u = 7.$$

When you add all the three equations, we obtain

$$2r + 2d = 77$$

which is impossible since the left side of the above equation is an even number but the right side is an odd number. Thus, there is no path from the upper left square to the lower right square.

6.5 Pop Quiz

1. Using the sum-to-product identity

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

we find

$$\begin{aligned} \frac{1}{2}\left[\sin(2\alpha+\beta) - \sin(2\alpha-\beta)\right] &= \\ \cos\left(\frac{2\alpha+\beta+2\alpha-\beta}{2}\right)\sin\left(\frac{2\alpha+\beta-(2\alpha-\beta)}{2}\right) &= \\ \cos(2\alpha)\sin(\beta). \end{aligned}$$

2. Using the sum-to-product identity

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

we find

$$\begin{aligned} \cos 2\alpha + \cos 4\alpha &= \\ 2\cos\left(\frac{2\alpha+4\alpha}{2}\right)\cos\left(\frac{2\alpha-4\alpha}{2}\right) &= \\ 2\cos 3\alpha\cos(-\alpha) &= \\ 2\cos 3\alpha\cos\alpha \end{aligned}$$

3. Dividing by 2, we obtain

$$\frac{y}{2} = \frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x$$

$$\frac{y}{2} = \sin\left(x - \frac{\pi}{6}\right)$$

$$y = 2\sin\left(x - \frac{\pi}{6}\right).$$

4. The amplitude is $\sqrt{3^2+5^2} = \sqrt{34}$.

6.5 Linking Concepts

a) Let P_c be the profit function (in thousands of dollars) for the Christmas Store.
Assume $P_c = A\sin(Bx + C) + D$.

Since $10 \le P_c \le 50$, $P_c = 20\sin(Bx+C)+30$. Since $(0,50)$ is a point on the graph, then

$$\begin{aligned} 20\sin(C)+30 &= 50 \\ \sin(C) &= 1 \\ C &= \frac{\pi}{2} \end{aligned}$$

Likewise, since $(6,10)$ is a point on the graph we obtain

$$\begin{aligned} 20\sin\left(6B+\frac{\pi}{2}\right)+30 &= 10 \\ \sin\left(6B+\frac{\pi}{2}\right) &= -1 \\ 6B+\frac{\pi}{2} &= \frac{3\pi}{2} \\ B &= \frac{\pi}{6} \end{aligned}$$

The profit function for the Christmas Store is

$$P_c = 20\sin\left(\frac{\pi}{6}x+\frac{\pi}{2}\right)+30$$

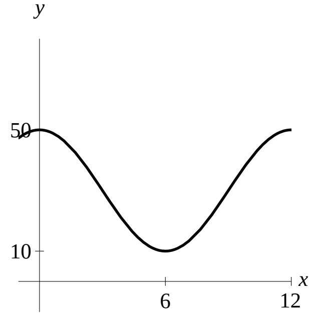
(in thousands of dollars) or equivalently

$$P_c = 20,000\sin\left(\frac{\pi}{6}x+\frac{\pi}{2}\right)+30,000$$

(in dollars) or

$$\begin{aligned} P_c &= 20,000\sin\left(\frac{\pi}{6}x+\frac{\pi}{2}-2\pi\right)+30,000 \\ &= 20,000\sin\left(\frac{\pi}{6}x-\frac{3\pi}{2}\right)+30,000 \\ P_c &= 20,000\sin\left(\frac{\pi}{6}[x-9]\right)+30,000. \end{aligned}$$

A sketch of the graph of P_c is provided.



b) Let P_p be the profit function (in thousands of dollars) for the Pool Store.

Assume $P_p = A\sin(Bx+C)+D$. Since $20 \le P_p \le 80$, $P_c = 30\sin(Bx+C)+50$. Since $(8,80)$ is a point on the graph, we obtain

$$\begin{aligned} 30\sin(8B+C)+50 &= 80 \\ \sin(8B+C) &= 1 \\ 8B+C &= \frac{\pi}{2}. \end{aligned}$$

Likewise, since $(2,20)$ is a point on the graph we get

$$\begin{aligned} 30\sin(2B+C)+50 &= 20 \\ \sin(2B+C) &= -1 \\ 2B+C &= \frac{-\pi}{2}. \end{aligned}$$

Then $(8B+C)-(2B+C) = \frac{\pi}{2} - \frac{-\pi}{2}$ or equivalently $6B = \pi$. Solving, we find $B = \frac{\pi}{6}$ and $C = \frac{-5\pi}{6}$.

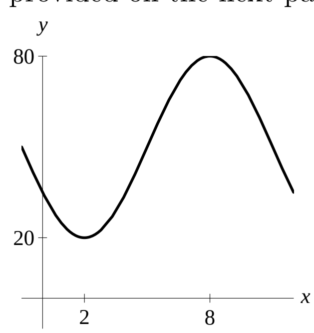
The profit function for the Pool Store is

$$P_p = 30\sin\left(\frac{\pi}{6}x-\frac{5\pi}{6}\right)+50$$

(in thousands of dollars) or equivalently

$$P_p = 30,000\sin\left(\frac{\pi}{6}(x-5)\right)+50,000$$

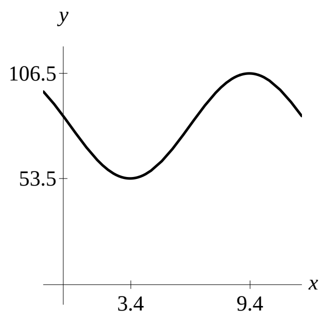
(in dollars). A sketch of the graph of P_p is provided on the next page.



c) The total profit is

$$\begin{aligned} y = {} & 20,000\sin\left(\frac{\pi}{6}[x-9]\right)+ \\ & 30,000\sin\left(\frac{\pi}{6}(x-5)\right)+80,000. \end{aligned}$$

The period is 12 and a sketch of the graph of the total profit function is given below.



d) Maximum profit is \$106,458. This occurs in September (corresponding to $x \approx 9.363$).

e) Minimum profit is \$53,542. This occurs in March (corresponding to $x \approx 3.363$).

f) One cannot express $y = \sin(x) + \sin(2x)$ in the form $y = A\sin(Bx + C) + D$ where A, B, C, D are real constants. To see this, simply sketch the graph of $y = \sin(x) + \sin(2x)$.

g) No, the sum of two periodic is not necessarily periodic. For instance, $y = \sin(x) + \sin(\sqrt{2}x)$ is not a periodic function.

For Thought

1. False, the only solutions are 45° and 315°.

2. False, there is no solution in $[0, \pi)$.

3. True, since −29° and 331° are coterminal angles.

4. True **5.** True, since the right-side is a factorization of the left-side.

6. False, $x = 0$ is a solution to the first equation and not to the second equation.

7. False, $\cos^{-1} 2$ is undefined. **8.** True

9. False, $x = 3\pi/4$ is not a solution to the first equation but is a solution to the second equation.

10 . False, rather $\left\{x|3x = \frac{\pi}{2} + 2k\pi\right\} = \left\{x|x = \frac{\pi}{6} + \frac{2k\pi}{3}\right\}$.

6.6 Exercises

1. $\{x \mid x = \pi + 2k\pi, k \text{ an integer}\}$

2. $\left\{x \mid x = \frac{\pi}{2} + k\pi, k \text{ an integer}\right\}$

3. $\{x \mid x = k\pi, k \text{ an integer}\}$

4. $\left\{x \mid x = \frac{\pi}{2} + 2k\pi, k \text{ an integer}\right\}$

5. $\left\{x \mid x = \frac{3\pi}{2} + 2k\pi, k \text{ an integer}\right\}$

6. $\{x \mid x = 2k\pi, k \text{ an integer}\}$

7. Solutions in $[0, 2\pi)$ are $x = \frac{\pi}{3}, \frac{5\pi}{3}$. So solution set is $\left\{x \mid x = \frac{\pi}{3} + 2k\pi \text{ or } x = \frac{5\pi}{3} + 2k\pi\right\}$.

8. Solutions in $[0, 2\pi)$ are $x = \frac{\pi}{4}, \frac{7\pi}{4}$. So solution set is $\left\{x \mid x = \frac{\pi}{4} + 2k\pi \text{ or } x = \frac{7\pi}{4} + 2k\pi\right\}$.

9. Solutions in $[0, 2\pi)$ are $x = \frac{\pi}{4}, \frac{3\pi}{4}$. So solution set is $\left\{x \mid x = \frac{\pi}{4} + 2k\pi \text{ or } x = \frac{3\pi}{4} + 2k\pi\right\}$.

10. Solutions in $[0, 2\pi)$ are $x = \frac{\pi}{3}, \frac{2\pi}{3}$. So solution set is $\left\{x \mid x = \frac{\pi}{3} + 2k\pi \text{ or } x = \frac{2\pi}{3} + 2k\pi\right\}$.

11. Solution in $[0, \pi)$ is $x = \frac{\pi}{4}$.

The solution set is $\left\{x \mid x = \frac{\pi}{4} + k\pi\right\}$.

12. Solution in $[0, \pi)$ is $x = \frac{\pi}{6}$.

The solution set is $\left\{x \mid x = \frac{\pi}{6} + k\pi\right\}$.

13. Solutions in $[0, 2\pi)$ are $x = \frac{5\pi}{6}, \frac{7\pi}{6}$.

Then the solution set is $\left\{x \mid x = \frac{5\pi}{6} + 2k\pi \text{ or } x = \frac{7\pi}{6} + 2k\pi\right\}$.

14. Solutions in $[0, 2\pi)$ are $x = \frac{3\pi}{4}, \frac{5\pi}{4}$.
Then the solution set is
$\left\{x \mid x = \frac{3\pi}{4} + 2k\pi \text{ or } x = \frac{5\pi}{4} + 2k\pi\right\}$.

15. Solutions in $[0, 2\pi)$ are $x = \frac{5\pi}{4}, \frac{7\pi}{4}$.
The solution set is
$\left\{x \mid x = \frac{5\pi}{4} + 2k\pi \text{ or } x = \frac{7\pi}{4} + 2k\pi\right\}$.

16. Solutions in $[0, 2\pi)$ are $x = \frac{4\pi}{3}, \frac{5\pi}{3}$.
So the solution set is
$\left\{x \mid x = \frac{4\pi}{3} + 2k\pi \text{ or } x = \frac{5\pi}{3} + 2k\pi\right\}$.

17. Solution in $[0, \pi)$ is $x = \frac{3\pi}{4}$. The solution set is $\left\{x \mid x = \frac{3\pi}{4} + k\pi\right\}$.

18. Solution in $[0, \pi)$ is $x = \frac{2\pi}{3}$. The solution set is $\left\{x \mid x = \frac{2\pi}{3} + k\pi\right\}$.

19. Solutions in $[0, 360°)$ are $\alpha = 90°, 270°$.
So solution set is $\{\alpha \mid \alpha = 90° + k \cdot 180°\}$.

20. Solution in $[0, 360°)$ is $\alpha = 180°$. So the solution set is $\{\alpha \mid \alpha = 180° + k \cdot 360°\}$.

21. Solution in $[0, 360°)$ is $\alpha = 90°$. So the solution set is $\{\alpha \mid \alpha = 90° + k \cdot 360°\}$.

22. Solution in $[0, 360°)$ is $\alpha = 270°$. So the solution set is $\{\alpha \mid \alpha = 270° + k \cdot 360°\}$.

23. Solution in $[0, 180°)$ is $\alpha = 0°$.
The solution set is $\{\alpha \mid \alpha = k \cdot 180°\}$.

24. Solution in $[0, 180°)$ is $\alpha = 135°$.
The solution set is $\{\alpha \mid \alpha = 135° + k \cdot 180°\}$.

25. One solution is $\cos^{-1}(0.873) \approx 29.2°$. Another solution is $360° - 29.2° = 330.8°$. Solution set is $\{\alpha \mid \alpha = 29.2° + k360° \text{ or } \alpha = 330.8° + k360°\}$.

26. One solution is $\cos^{-1}(-0.158) \approx 99.1°$.
Another solution is $360° - 99.1° = 260.9°$.
The solution set is
$\{\alpha \mid \alpha = 99.1° + k360° \text{ or } \alpha = 260.9° + k360°\}$.

27. One solution is $\sin^{-1}(-0.244) \approx -14.1°$. This is coterminal with $345.9°$. Another solution is $180° + 14.1° = 194.1°$. Solution set is $\{\alpha \mid \alpha = 345.9° + k360° \text{ or } \alpha = 194.1° + k360°\}$.

28. One solution is $\sin^{-1}(0.551) \approx 33.4°$. Another solution is $180° - 33.4° = 146.6°$. Solution set is $\{\alpha \mid \alpha = 33.4° + k360° \text{ or } \alpha = 146.6° + k360°\}$.

29. One solution is $\tan^{-1}(5.42) \approx 79.5°$.
Solution set is $\{\alpha \mid \alpha = 79.5° + k \cdot 180°\}$.

30. One solution is $\tan^{-1}(-2.31) \approx -66.6°$.
Since $180°$ is the period of $\tan x$, another solution is $180° - 66.6° = 113.4°$
Solution set is $\{\alpha \mid \alpha = 113.4° + k \cdot 180°\}$.

31. Values of $x/2$ in $[0, 2\pi)$ are $\pi/3$ and $5\pi/3$.
Then we get

$$\frac{x}{2} = \frac{\pi}{3} + 2k\pi \text{ or } \frac{x}{2} = \frac{5\pi}{3} + 2k\pi$$

$$x = \frac{2\pi}{3} + 4k\pi \text{ or } x = \frac{10\pi}{3} + 4k\pi.$$

The solution set is

$$\left\{x \mid x = \frac{2\pi}{3} + 4k\pi \text{ or } x = \frac{10\pi}{3} + 4k\pi\right\}.$$

32. Since $\cos(2x) = -\frac{\sqrt{2}}{2}$, values of $2x$ in $[0, 2\pi)$ are $3\pi/4$ and $5\pi/4$. Then

$$2x = \frac{3\pi}{4} + 2k\pi \text{ or } 2x = \frac{5\pi}{4} + 2k\pi$$

$$x = \frac{3\pi}{8} + k\pi \text{ or } x = \frac{5\pi}{8} + k\pi.$$

The solution set is

$$\left\{x \mid x = \frac{3\pi}{8} + k\pi \text{ or } x = \frac{5\pi}{8} + k\pi\right\}.$$

33. Value of $3x$ in $[0, 2\pi)$ is 0. Thus, $3x = 2k\pi$.
The solution set is $\left\{x \mid x = \frac{2k\pi}{3}\right\}$.

34. Values of $2x$ in $[0, 2\pi)$ are $\pi/2$ and $3\pi/2$. So

$$2x = \frac{\pi}{2} + 2k\pi \text{ or } 2x = \frac{3\pi}{2} + 2k\pi$$

$$x = \frac{\pi}{4} + k\pi \text{ or } x = \frac{3\pi}{4} + k\pi.$$

The solution set is $\left\{x \mid x = \frac{\pi}{4} + \frac{k\pi}{2}\right\}$.

35. Since $\sin(x/2) = 1/2$, values of $x/2$ in $[0, 2\pi)$ are $\pi/6$ and $5\pi/6$. Then

$$\frac{x}{2} = \frac{\pi}{6} + 2k\pi \text{ or } \frac{x}{2} = \frac{5\pi}{6} + 2k\pi$$

$$x = \frac{\pi}{3} + 4k\pi \text{ or } x = \frac{5\pi}{3} + 4k\pi.$$

The solution set is

$$\left\{x \mid x = \frac{\pi}{3} + 4k\pi \text{ or } x = \frac{5\pi}{3} + 4k\pi\right\}.$$

36. Values of $2x$ in $[0, 2\pi)$ are 0 and π. So

$$2x = 0 + 2k\pi \text{ or } 2x = \pi + 2k\pi$$

$$x = k\pi \text{ or } x = \frac{\pi}{2} + k\pi.$$

The solution set is $\left\{x \mid x = \frac{k\pi}{2}\right\}$.

37. Since $\sin(2x) = -\sqrt{2}/2$, values of $2x$ in $[0, 2\pi)$ are $5\pi/4$ and $7\pi/4$. Thus,

$$2x = \frac{5\pi}{4} + 2k\pi \text{ or } 2x = \frac{7\pi}{4} + 2k\pi$$

$$x = \frac{5\pi}{8} + k\pi \text{ or } x = \frac{7\pi}{8} + k\pi.$$

The solution set is

$$\left\{x \mid x = \frac{5\pi}{8} + k\pi \text{ or } x = \frac{7\pi}{8} + k\pi\right\}.$$

38. Since $\sin(x/3) = -1$, value of $x/3$ in $[0, 2\pi)$ is $3\pi/2$. So

$$\frac{x}{3} = \frac{3\pi}{2} + 2k\pi.$$

The solution set is $\left\{x \mid x = \frac{9\pi}{2} + 6k\pi\right\}$.

39. Value of $2x$ in $[0, \pi)$ is $\pi/3$. Then

$$2x = \frac{\pi}{3} + k\pi.$$

The solution set is $\left\{x \mid x = \frac{\pi}{6} + \frac{k\pi}{2}\right\}$.

40. Since $\tan(3x) = -1/\sqrt{3}$, value of $3x$ in $[0, \pi)$ is $5\pi/6$. Thus,

$$3x = \frac{5\pi}{6} + k\pi.$$

The solution set is $\left\{x \mid x = \frac{5\pi}{18} + \frac{k\pi}{3}\right\}$.

41. Value of $4x$ in $[0, \pi)$ is 0. Then

$$4x = k\pi.$$

The solution set is $\left\{x \mid x = \frac{k\pi}{4}\right\}$.

42. Value of $3x$ in $[0, \pi)$ is $3\pi/4$. Then

$$3x = \frac{3\pi}{4} + k\pi.$$

The solution set is $\left\{x \mid x = \frac{\pi}{4} + \frac{k\pi}{3}\right\}$.

43. The values of πx in $[0, 2\pi)$ are $\pi/6$ and $5\pi/6$. Then

$$\pi x = \frac{\pi}{6} + 2k\pi \text{ or } \pi x = \frac{5\pi}{6} + 2k\pi$$

$$x = \frac{1}{6} + 2k \text{ or } x = \frac{5}{6} + 2k.$$

The solution set is

$$\left\{x \mid x = \frac{1}{6} + 2k \text{ or } x = \frac{5}{6} + 2k\right\}.$$

44. Value of $\pi x/4$ in $[0, \pi)$ is $\pi/4$. So

$$\frac{\pi x}{4} = \frac{\pi}{4} + k\pi$$

$$\pi x = \pi + 4k\pi.$$

The solution set is $\{x \mid x = 1 + 4k\}$.

45. Values of $2\pi x$ in $[0, 2\pi)$ are $\pi/2$ and $3\pi/2$. So

$$2\pi x = \frac{\pi}{2} + 2k\pi \text{ or } 2\pi x = \frac{3\pi}{2} + 2k\pi$$

$$x = \frac{1}{4} + k \text{ or } x = \frac{3}{4} + k.$$

The solution set is

$$\left\{ x \mid x = \frac{1}{4} + \frac{k}{2} \right\}.$$

46. Value of $3\pi x$ in $[0, 2\pi)$ is $\pi/2$. Then

$$3\pi x = \frac{\pi}{2} + 2k\pi.$$

The solution set is $\left\{ x \mid x = \frac{1}{6} + \frac{2k}{3} \right\}$.

47. Since $\sin\alpha = -\sqrt{3}/2$, the solution set is $\{240°, 300°\}$. **48.** $\{120°, 300°\}$.

49. Since $\cos 2\alpha = 1/\sqrt{2}$, values of 2α in $[0, 360°)$ are $45°$ and $315°$. Thus,

$$2\alpha = 45° + k \cdot 360° \text{ or } 2\alpha = 315° + k \cdot 360°$$

$$\alpha = 22.5° + k \cdot 180° \text{ or } \alpha = 157.5° + k \cdot 180°.$$

Then let $k = 0, 1$. The solution set is

$$\{22.5°, 157.5°, 202.5°, 337.5°\}.$$

50. The value of 6α in $[0, 360°)$ is $90°$. Then

$$6\alpha = 90° + k \cdot 360°$$

$$\alpha = 15° + k \cdot 60°.$$

By choosing $k = 0, 1, ..., 5$, one obtains the solution set $\{15°, 75°, 135°, 195°, 255°, 315°\}$.

51. Values of 3α in $[0, 360°)$ are $135°$ and $225°$. Then

$$3\alpha = 135° + k \cdot 360° \text{ or } 3\alpha = 225° + k \cdot 360°$$

$$\alpha = 45° + k \cdot 120° \text{ or } \alpha = 75° + k \cdot 120°.$$

By choosing $k = 0, 1, 2$, one obtains the solution set $\{45°, 75°, 165°, 195°, 285°, 315°\}$.

52. Since $\csc 5\alpha = -2$, the values of 5α in $[0, 360°)$ are $210°$ and $330°$. Then

$$5\alpha = 210° + k \cdot 360° \text{ or } 5\alpha = 330° + k \cdot 360°$$

$$\alpha = 42° + k \cdot 72° \text{ or } \alpha = 66° + k \cdot 72°.$$

Choosing $k = 0, 1, 2, 3, 4$, the solution set is $\{42°, 66°, 114°, 138°, 186°, 210°, 258°, 282°, 330°, 354°\}$.

53. The value of $\alpha/2$ in $[0, 180°)$ is $30°$. Then

$$\frac{\alpha}{2} = 30° + k \cdot 180°$$

$$\alpha = 60° + k \cdot 360°.$$

By choosing $k = 0$, the solution set is $\{60°\}$.

54. The values of $\alpha/2$ in $[0, 360°)$ are $45°$ and $315°$. So

$$\frac{\alpha}{2} = 45° + k \cdot 360° \text{ or } \frac{\alpha}{2} = 315° + k \cdot 360°$$

$$\alpha = 90° + k \cdot 720° \text{ or } \alpha = 630° + k \cdot 720°.$$

Let $k = 0$ in the first case.
Then the solution set is $\{90°\}$.

55. A solution is $3\alpha = \sin^{-1}(0.34) \approx 19.88°$. Another solution is $3\alpha = 180° - 19.88° = 160.12°$. Then

$$3\alpha = 19.88° + k \cdot 360° \text{ or } 3\alpha = 160.12° + k \cdot 360°$$

$$\alpha \approx 6.6° + k \cdot 120° \text{ or } \alpha \approx 53.4° + k \cdot 120°.$$

Solution set is

$$\{\alpha \mid \alpha = 6.6° + k \cdot 120° \text{ or } \alpha = 53.4° + k \cdot 120°\}.$$

56. A solution is $2\alpha = \cos^{-1}(-0.22) \approx 102.71°$. Another solution is $2\alpha = 360° - 102.71° = 257.29°$. Therefore,

$$2\alpha = 102.71° + k \cdot 360° \text{ or } 2\alpha = 257.29° + k \cdot 360°$$

$$\alpha \approx 51.4° + k \cdot 180° \text{ or } \alpha \approx 128.6° + k \cdot 180°.$$

Solution set is

$$\{\alpha \mid \alpha = 51.4° + k180° \text{ or } \alpha = 128.6° + k180°\}.$$

57. A solution is $3\alpha = \sin^{-1}(-0.6) \approx -36.87°$. This is coterminal with 323.13°. Another solution is $3\alpha = 180° + 36.87° = 216.87°$. Then

$$3\alpha = 323.13° + k \cdot 360° \text{ or } 3\alpha = 216.87° + k \cdot 360°$$

$$\alpha \approx 107.7° + k \cdot 120° \text{ or } \alpha \approx 72.3° + k \cdot 120°.$$

The solution set is

$$\{\alpha \mid \alpha = 107.7° + k120° \text{ or } \alpha = 72.3° + k120°\}.$$

58. A solution is $4\alpha = \tan^{-1}(-3.2) \approx -72.65°$. Another solution is $180° - 72.65° = 107.35°$.

$$4\alpha = 107.35° + k \cdot 180°$$
$$\alpha = 26.8° + k \cdot 45°$$

Solution set is $\{\alpha \mid \alpha = 26.8° + k \cdot 45°\}$.

59. A solution is $2\alpha = \cos^{-1}(1/4.5) \approx 77.16°$. Another solution is $2\alpha = 360° - 77.16° = 282.84°$. Thus,

$$2\alpha = 77.16° + k \cdot 360° \text{ or } 2\alpha = 282.84° + k \cdot 360°$$

$$\alpha \approx 38.6° + k \cdot 180° \text{ or } \alpha \approx 141.4° + k \cdot 180°.$$

The solution set is

$$\{\alpha \mid \alpha = 38.6° + k180° \text{ or } \alpha = 141.4° + k180°\}.$$

60. A solution is $3\alpha = \sin^{-1}(-1/1.4) \approx -45.58°$. This is coterminal with 314.42°. Another solution is $3\alpha = 180° + 45.58° = 225.58°$. Thus,

$$3\alpha = 314.42° + k \cdot 360° \text{ or } 3\alpha = 225.58° + k \cdot 360°$$

$$\alpha \approx 104.8° + k \cdot 120° \text{ or } \alpha \approx 75.2° + k \cdot 120°.$$

The solution set is

$$\{\alpha \mid \alpha = 104.8° + k120° \text{ or } \alpha = 75.2° + k120°\}.$$

61. A solution is $\alpha/2 = \sin^{-1}(-1/2.3) \approx -25.77°$. This is coterminal with 334.23°. Another solution is $\alpha/2 = 180° + 25.77° = 205.77°$. Thus,

$$\frac{\alpha}{2} = 334.23° + k \cdot 360° \text{ or } \frac{\alpha}{2} = 205.77° + k \cdot 360°$$

$$\alpha \approx 668.5° + k \cdot 720° \text{ or } \alpha \approx 411.5° + k \cdot 720°.$$

The solution set is

$$\{\alpha \mid \alpha = 668.5° + k720° \text{ or } \alpha = 411.5° + k720°\}.$$

62. A solution is $\alpha/2 = \tan^{-1}(1/4.7) \approx 12.01°$. Then

$$\frac{\alpha}{2} = 12.01° + k \cdot 180°$$
$$\alpha = 24.0° + k \cdot 360°.$$

Solution set is $\{\alpha \mid \alpha = 24.0° + k \cdot 360°\}$.

63. Set the right-hand side to zero and factor.

$$\begin{aligned} 3\sin^2 x - \sin x &= 0 \\ \sin x(3\sin x - 1) &= 0 \end{aligned}$$

Set each factor to zero.

$$\begin{aligned} \sin x = 0 &\quad \text{or} \quad \sin x = 1/3 \\ x = 0, \pi &\quad \text{or} \quad x = \sin^{-1}(1/3) \approx 0.3 \end{aligned}$$

Another solution to $\sin x = 1/3$ is $x = \pi - 0.3 \approx 2.8$.
The solution set is $\{0, 0.3, 2.8, \pi\}$.

64. Set the right-hand side to zero and factor.

$$\begin{aligned} 2\tan^2 x - \tan x &= 0 \\ \tan x(2\tan x - 1) &= 0 \end{aligned}$$

Set each factor to zero.

$$\begin{aligned} \tan x = 0 &\quad \text{or} \quad \tan x = 1/2 \\ x = 0, \pi &\quad \text{or} \quad x = \tan^{-1}(1/2) \approx 0.5 \end{aligned}$$

Another solution to $\tan x = 1/2$ is $x = \pi + 0.5 \approx 3.6$. The solution set is $\{0, 0.5, \pi, 3.6\}$.

65. Set the right-hand side to zero and factor.

$$\begin{aligned} 2\cos^2 x + 3\cos x + 1 &= 0 \\ (2\cos x + 1)(\cos x + 1) &= 0 \end{aligned}$$

Set the factors to zero.

$$\begin{aligned} \cos x = -1/2 &\quad \text{or} \quad \cos x = -1 \\ x = 2\pi/3, 4\pi/3 &\quad \text{or} \quad x = \pi \end{aligned}$$

The solution set is $\{\pi, 2\pi/3, 4\pi/3\}$.

66. Set the right-hand side to zero and factor.

$$\begin{aligned} 2\sin^2 x + \sin x - 1 &= 0 \\ (2\sin x - 1)(\sin x + 1) &= 0 \end{aligned}$$

Set each factor to 0.

$$\sin x = 1/2 \quad \text{or} \quad \sin x = -1$$
$$x = \pi/6, 5\pi/6 \quad \text{or} \quad x = 3\pi/2$$

The solution set is $\{\pi/6, 5\pi/6, 3\pi/2\}$.

67. Substitute $\cos^2 x = 1 - \sin^2 x$.

$$\begin{aligned} 5\sin^2 x - 2\sin x &= 1 - \sin^2 x \\ 6\sin^2 x - 2\sin x - 1 &= 0 \end{aligned}$$

Apply the quadratic formula.

$$\begin{aligned} \sin x &= \frac{2 \pm \sqrt{28}}{12} \\ \sin x &= \frac{1 \pm \sqrt{7}}{6} \end{aligned}$$

Then

$$x = \sin^{-1}\left(\frac{1+\sqrt{7}}{6}\right) \quad \text{or} \quad x = \sin^{-1}\left(\frac{1-\sqrt{7}}{6}\right)$$
$$x \approx 0.653 \quad \text{or} \quad x \approx -0.278.$$

Another solution is $\pi - 0.653 \approx 2.5$. An angle coterminal with -0.278 is $2\pi - 0.278 \approx 6.0$. Another solution is $\pi + 0.278 \approx 3.4$. The solution set is $\{0.7, 2.5, 3.4, 6.0\}$.

68. Divide the equation by $\cos^2 x$.

$$\begin{aligned} \tan^2 x - 1 &= 0 \\ \tan^2 x &= 1 \\ \tan x &= \pm 1 \end{aligned}$$

Solution set is $\{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$.

69. Squaring both sides of the equation, we obtain

$$\begin{aligned} \tan^2 x &= \sec^2 x - 2\sqrt{3}\sec x + 3 \\ \sec^2 x - 1 &= \sec^2 x - 2\sqrt{3}\sec x + 3 \\ -4 &= -2\sqrt{3}\sec x \\ \sec x &= 2/\sqrt{3} \\ x &= \pi/6, 11\pi/6. \end{aligned}$$

Checking $x = \pi/6$, one gets $\tan(\pi/6) = 1/\sqrt{3}$ and $\sec(\pi/6) - \sqrt{3} = 2/\sqrt{3} - \sqrt{3} = -1/\sqrt{3}$. Then $x = \pi/6$ is an extraneous root and the solution set is $\{11\pi/6\}$.

70. Squaring both sides, we find

$$\begin{aligned} \csc^2 x - 2\sqrt{3}\csc x + 3 &= \cot^2 x \\ \csc^2 x - 2\sqrt{3}\csc x + 3 &= \csc^2 x - 1 \\ -2\sqrt{3}\csc x &= -4 \\ \csc x &= 2/\sqrt{3} \\ x &= \pi/3, 2\pi/3. \end{aligned}$$

Checking $x = \pi/3$, one gets $\cot(\pi/3) = 1/\sqrt{3}$ and $\csc(\pi/3) - \sqrt{3} = 2/\sqrt{3} - \sqrt{3} = -1/\sqrt{3}$. So $x = \pi/3$ is an extraneous root and the solution set is $\{2\pi/3\}$.

71. Square both sides of the equation.

$$\begin{aligned} \sin^2 x + 2\sqrt{3}\sin x + 3 &= 27\cos^2 x \\ \sin^2 x + 2\sqrt{3}\sin x + 3 &= 27(1 - \sin^2 x) \\ 28\sin^2 x + 2\sqrt{3}\sin x - 24 &= 0 \\ 14\sin^2 x + \sqrt{3}\sin x - 12 &= 0 \end{aligned}$$

By the quadratic formula, we get

$$\begin{aligned} \sin x &= \frac{-\sqrt{3} \pm \sqrt{675}}{28} \\ \sin x &= \frac{-\sqrt{3} \pm 15\sqrt{3}}{28} \\ \sin x &= \frac{\sqrt{3}}{2}, \frac{-4\sqrt{3}}{7}. \end{aligned}$$

Thus,

$$x = \frac{\pi}{3}, \frac{2\pi}{3} \quad \text{or} \quad x = \sin^{-1}\left(\frac{-4\sqrt{3}}{7}\right)$$
$$x = \frac{\pi}{3}, \frac{2\pi}{3} \quad \text{or} \quad x \approx -1.427.$$

Checking $x = 2\pi/3$, one finds $\sin(2\pi/3) + \sqrt{3} = \sqrt{3}/2 + \sqrt{3}$ and $3\sqrt{3}\cos(2\pi/3)$ is a negative number. Then $x = 2\pi/3$ is an extraneous root.

An angle coterminal with -1.427 is $2\pi - 1.427 \approx 4.9$. In a similar way, one checks that $\pi + 1.427 \approx 4.568$ is an extraneous root. Thus, the solution set is $\{\pi/3, 4.9\}$.

72. Substitute $\sin^2 x = 1 - \cos^2 x$.

$$\begin{aligned} 6(1 - \cos^2 x) - 2\cos x &= 5 \\ 6 - 6\cos^2 x - 2\cos x - 5 &= 0 \\ 6\cos^2 x + 2\cos x - 1 &= 0 \end{aligned}$$

Apply the quadratic formula.

$$\begin{aligned}\cos x &= \frac{-2 \pm \sqrt{28}}{12} \\ \cos x &= \frac{-1 \pm \sqrt{7}}{6}\end{aligned}$$

Then

$$x = \cos^{-1}\left(\frac{-1+\sqrt{7}}{6}\right) \quad \text{or} \quad x = \cos^{-1}\left(\frac{-1-\sqrt{7}}{6}\right)$$
$$x \approx 1.3 \quad \text{or} \quad x \approx 2.2.$$

Two other solutions are $2\pi - 1.3 \approx 5.0$ and $2\pi - 2.2 \approx 4.1$. The solution set is $\{1.3, 2.2, 4.1, 5.0\}$.

73. Express the equation in terms of $\sin x$ and $\cos x$.

$$\begin{aligned}\frac{\sin x}{\cos x} \cdot 2\sin x \cos x &= 0 \\ 2\sin^2 x &= 0 \\ \sin x &= 0\end{aligned}$$

Solution set is $\{0, \pi\}$.

74. Set the right-hand side to zero and factor.

$$\begin{aligned}3\sec^2 x \tan x - 4\tan x &= 0 \\ \tan x(3\sec^2 x - 4) &= 0\end{aligned}$$

Then

$$\tan x = 0 \quad \text{or} \quad \sec x = \pm\frac{2}{\sqrt{3}}$$
$$x = 0, \pi \quad \text{or} \quad x = \pi/6, 5\pi/6, 7\pi/6, 11\pi/6.$$

Solution set is $\{0, \pi/6, 5\pi/6, \pi, 7\pi/6, 11\pi/6\}$.

75. Substitute the double-angle identity for $\sin x$.

$$\begin{aligned}2\sin x \cos x - \sin x \cos x &= \cos x \\ \sin x \cos x - \cos x &= 0 \\ \cos x(\sin x - 1) &= 0 \\ \cos x = 0 \quad &\text{or} \quad \sin x = 1 \\ x = \pi/2, 3\pi/2 \quad &\text{or} \quad x = \pi/2\end{aligned}$$

Solution set is $\{\pi/2, 3\pi/2\}$.

76. Apply double-angle identities.

$$\begin{aligned}2\cos^2(2x) - 2(4\sin^2 x \cos^2 x) &= -1 \\ 2\cos^2(2x) - 2\sin^2(2x) &= -1 \\ 2(\cos^2(2x) - \sin^2(2x)) &= -1 \\ 2\cos(4x) &= -1 \\ \cos(4x) &= -1/2\end{aligned}$$

Then

$$4x = \frac{2\pi}{3} + 2k\pi \quad \text{or} \quad 4x = \frac{4\pi}{3} + 2k\pi$$
$$x = \frac{\pi}{6} + \frac{k\pi}{2} \quad \text{or} \quad x = \frac{\pi}{3} + \frac{k\pi}{2}.$$

Let $k = 0, 1, 2, 3$. Then the solution set is

$$\{\pi/6, 5\pi/6, 7\pi/6, 11\pi/6, \pi/3, 2\pi/3, 4\pi/3, 5\pi/3\}.$$

77. Use the sum identity for sine.

$$\sin(x + \pi/4) = 1/2$$
$$x + \frac{\pi}{4} = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad x + \frac{\pi}{4} = \frac{5\pi}{6} + 2k\pi$$
$$x = \frac{-\pi}{12} + 2k\pi \quad \text{or} \quad x = \frac{7\pi}{12} + 2k\pi$$

By choosing $k = 1$ in the first case and $k = 0$ in the second case, one finds the solution set is $\{23\pi/12, 7\pi/12\}$.

78. Multiply both sides by -1 and use the difference identity for sine.

$$\begin{aligned}\sin x \cos(\pi/6) - \cos x \sin(\pi/6) &= 1/2 \\ \sin(x - \pi/6) &= 1/2\end{aligned}$$

Then

$$x - \frac{\pi}{6} = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad x - \frac{\pi}{6} = \frac{5\pi}{6} + 2k\pi$$
$$x = \frac{\pi}{3} + 2k\pi \quad \text{or} \quad x = \pi + 2k\pi.$$

Choose $k = 0$, and so the solution set is $\{\pi/3, \pi\}$.

79. Apply the difference identity for sine.

$$\begin{aligned}\sin(2x - x) &= -1/2 \\ \sin x &= -1/2\end{aligned}$$

The solution set is $\{7\pi/6, 11\pi/6\}$.

80. By the sum identity for cosine, we get

$$\begin{aligned} \cos(2x + x) &= 1/2 \\ \cos(3x) &= 1/2. \end{aligned}$$

Then

$$\begin{aligned} 3x = \frac{\pi}{3} + 2k\pi \quad &\text{or} \quad 3x = \frac{5\pi}{3} + 2k\pi \\ x = \frac{\pi}{9} + \frac{2k\pi}{3} \quad &\text{or} \quad x = \frac{5\pi}{9} + \frac{2k\pi}{3}. \end{aligned}$$

By choosing $k = 0, 1, 2$, one finds the solution set is $\{\pi/9, 5\pi/9, 7\pi/9, 11\pi/9, 13\pi/9, 17\pi/9\}$.

81. Since $4 \cdot 4^{2\sin^2 x} = 4^{3\sin x}$, we set the exponents equal to each other. Then

$$\begin{aligned} 2\sin^2 x + 1 &= 3\sin x \\ 2\sin^2 x - 3\sin x + 1 &= 0 \\ (2\sin x - 1)(\sin x - 1) &= 0 \\ \sin x &= 1, \frac{1}{2}. \end{aligned}$$

Thus, the solution set is $\{\pi/6, \pi/2, 5\pi/6\}$.

82. Since $2^{-1} \cdot 2^{2\cos^2 x} = 2^{\cos x}$, we set the exponents equal to each other. Then

$$\begin{aligned} 2\cos^2 x - 1 &= \cos x \\ 2\cos^2 x - \cos x - 1 &= 0 \\ (2\cos x + 1)(\cos x - 1) &= 0 \\ \cos x &= 1, -\frac{1}{2}. \end{aligned}$$

Thus, the solution set is $\{0, 2\pi/3, 4\pi/3\}$.

83. Use a half-angle identity for cosine and express equation in terms of $\cos\theta$.

$$\begin{aligned} \frac{1 + \cos\theta}{2} &= \frac{1}{\cos\theta} \\ \cos\theta + \cos^2\theta &= 2 \\ \cos^2\theta + \cos\theta - 2 &= 0 \\ (\cos\theta + 2)(\cos\theta - 1) &= 0 \\ \cos\theta = -2 \quad &\text{or} \quad \cos\theta = 1 \\ \text{no solution} \quad &\text{or} \quad \theta = 0^\circ \end{aligned}$$

Solution set is $\{0^\circ\}$.

84. By the half-angle identity for sine, we find

$$\begin{aligned} 1 - \cos\theta &= \cos\theta \\ 1 &= 2\cos\theta \\ 1/2 &= \cos\theta. \end{aligned}$$

Solution set is $\{60^\circ, 300^\circ\}$.

85. Dividing the equation by $2\cos\theta$, we get

$$\begin{aligned} \frac{\sin\theta}{\cos\theta} &= \frac{1}{2} \\ \tan\theta &= 0.5 \\ \theta &= \tan^{-1}(0.5) \approx 26.6^\circ. \end{aligned}$$

Another solution is $180^\circ + 26.6^\circ = 206.6^\circ$. Solution set is $\{26.6^\circ, 206.6^\circ\}$.

86. Dividing equation by $3\cos 2\theta$, we obtain

$$\begin{aligned} \frac{\sin 2\theta}{\cos 2\theta} &= \frac{1}{3} \\ \tan 2\theta &= 1/3 \\ 2\theta = \tan^{-1}(1/3) &\approx 18.43^\circ + k \cdot 180^\circ \\ \theta &\approx 9.2^\circ + k \cdot 90^\circ. \end{aligned}$$

By choosing $k = 0, 1, 2, 3$, one gets that the solution set is $\{9.2^\circ, 99.2^\circ, 189.2^\circ, 279.2^\circ\}$.

87. Express equation in terms of $\sin 3\theta$.

$$\begin{aligned} \sin 3\theta &= \frac{1}{\sin 3\theta} \\ \sin^2 3\theta &= 1 \\ \sin 3\theta &= \pm 1 \end{aligned}$$

Then

$$\begin{aligned} 3\theta = 90^\circ + k \cdot 360^\circ \quad &\text{or} \quad 3\theta = 270^\circ + k \cdot 360^\circ \\ \theta = 30^\circ + k \cdot 120^\circ \quad &\text{or} \quad \theta = 90^\circ + k \cdot 120^\circ. \end{aligned}$$

By choosing $k = 0, 1, 2$, one finds that the solution set is $\{30^\circ, 90^\circ, 150^\circ, 210^\circ, 270^\circ, 330^\circ\}$.

88. Express equation in terms of $\tan\theta$.

$$\begin{aligned} \tan^2\theta - \frac{1}{\tan^2\theta} &= 0 \\ \tan^4\theta - 1 &= 0 \\ \tan\theta &= \pm 1 \end{aligned}$$

The solution set is $\{45^\circ, 135^\circ, 225^\circ, 315^\circ\}$.

89. By the method of completing the square, we get

$$\begin{aligned} \tan^2\theta - 2\tan\theta &= 1 \\ \tan^2\theta - 2\tan\theta + 1 &= 2 \\ (\tan\theta - 1)^2 &= 2 \\ \tan\theta - 1 &= \pm\sqrt{2} \end{aligned}$$

$$\theta = \tan^{-1}(1+\sqrt{2}) \quad \text{or} \quad \theta = \tan^{-1}(1-\sqrt{2})$$

$$\theta \approx 67.5^\circ \quad \text{or} \quad \theta = -22.5^\circ.$$

Other solutions are $180^\circ + 67.5^\circ = 247.5^\circ$, $180^\circ - 22.5^\circ = 157.5^\circ$, and $180^\circ + 157.5^\circ = 337.5^\circ$. The solution set is $\{67.5^\circ, 157.5^\circ, 247.5^\circ, 337.5^\circ\}$.

90. By the method of completing the square, we find

$$\begin{aligned} \cot^2\theta - 4\cot\theta &= -2 \\ \cot^2\theta - 4\cot\theta + 4 &= 2 \\ (\cot\theta - 2)^2 &= 2 \\ \cot\theta - 2 &= \pm\sqrt{2} \end{aligned}$$

$$\theta = \tan^{-1}\left(\frac{1}{2+\sqrt{2}}\right) \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{1}{2-\sqrt{2}}\right)$$

$$\theta \approx 16.3^\circ + k\cdot 180^\circ \quad \text{or} \quad \theta = 59.6^\circ + k \cdot 180^\circ.$$

By choosing $k = 0, 1$, one obtains that the solution set is $\{16.3^\circ, 59.6^\circ, 196.3^\circ, 239.6^\circ\}$.

91. Factor as a perfect square.

$$\begin{aligned} (3\sin\theta + 2)^2 &= 0 \\ \sin\theta &= -2/3 \\ \theta &= \sin^{-1}(-2/3) \approx -41.8^\circ \end{aligned}$$

An angle coterminal with -41.8° is $360^\circ - 41.8^\circ = 318.2^\circ$. Another solution is $180^\circ + 41.8^\circ = 221.8^\circ$. The solution set is $\{221.8^\circ, 318.2^\circ\}$.

92. Factoring, we obtain

$$(4\cos\theta + 3)(3\cos\theta - 2) = 0$$

$$\cos\theta = -3/4 \quad \text{or} \quad \cos\theta = 2/3$$

$$\theta = \cos^{-1}(-3/4) \quad \text{or} \quad \theta = \cos^{-1}(2/3)$$

$$\theta \approx 138.6^\circ \quad \text{or} \quad \theta \approx 48.2^\circ.$$

Other solutions are $360^\circ - 138.6^\circ = 221.4^\circ$ and $360^\circ - 48.2^\circ = 311.8^\circ$. The solution set is $\{48.2^\circ, 138.6^\circ, 221.4^\circ, 311.8^\circ\}$.

93. By using the sum identity for tangent, we get

$$\begin{aligned} \tan(3\theta - \theta) &= \sqrt{3} \\ 2\theta &= 60^\circ + k\cdot 180^\circ \\ \theta &= 30^\circ + k \cdot 90^\circ. \end{aligned}$$

By choosing $k = 1, 3$, one obtains that the solution set is $\{120^\circ, 300^\circ\}$. Note, 30° and 210° are not solutions.

94. By using the sum identity for tangent, we find

$$\begin{aligned} \tan(3\theta + 2\theta) &= 1 \\ \tan 5\theta &= 1 \\ 5\theta &= 45^\circ + k \cdot 180^\circ \\ \theta &= 9^\circ + k\cdot 36^\circ. \end{aligned}$$

The solution set is $\{9^\circ, 81^\circ, 117^\circ, 153^\circ, 189^\circ, 261^\circ, 297^\circ, 333^\circ\}$. Note, 45° and 225° are not solutions.

95. Factoring, we get

$$(4\cos^2\theta - 3)(2\cos^2\theta - 1) = 0.$$

Then

$$\cos^2\theta = 3/4 \quad \text{or} \quad \cos^2\theta = 1/2$$

$$\cos\theta = \pm\sqrt{3}/2 \quad \text{or} \quad \cos\theta = \pm 1/\sqrt{2}.$$

The solution set is

$$\{30^\circ, 45^\circ, 135^\circ, 150^\circ, 210^\circ, 225^\circ, 315^\circ, 330^\circ\}.$$

96. Factoring, we have

$$(4\sin^2\theta - 1)(\sin^2\theta - 1) = 0.$$

Then

$$\sin^2\theta = 1/4 \quad \text{or} \quad \sin^2\theta = 1$$

$$\sin\theta = \pm 1/2 \quad \text{or} \quad \sin\theta = \pm 1.$$

The solution set is $\{30^\circ, 90^\circ, 150^\circ, 210^\circ, 270^\circ, 330^\circ\}$.

97. Factoring, we obtain

$$(\sec^2\theta - 1)(\sec^2\theta - 4) = 0$$

$$\sec^2\theta = 1 \quad \text{or} \quad \sec^2\theta = 4$$

$$\sec\theta = \pm 1 \quad \text{or} \quad \sec\theta = \pm 2.$$

Solution set is $\{0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ\}$.

98. Factoring, we obtain

$$\begin{aligned}(\cot^2\theta - 1)(\cot^2\theta - 3) &= 0\\ \cot^2\theta = 1 \quad \text{or} \quad & \cot^2\theta = 3\\ \cot\theta = \pm 1 \quad \text{or} \quad & \cot\theta = \pm\sqrt{3}.\end{aligned}$$

Solution set is

$$\{30^\circ, 45^\circ, 135^\circ, 150^\circ, 210^\circ, 225^\circ, 315^\circ, 330^\circ\}.$$

99. Multiplying the equation by LCD, we get

$$\begin{aligned}13.7\sin 33.2^\circ &= a\cdot\sin 45.6^\circ\\ \frac{13.7\sin 33.2^\circ}{\sin 45.6^\circ} &= a\\ 10.5 &\approx a.\end{aligned}$$

100. Multiplying by the LCD, we find

$$\begin{aligned}b\cdot\sin 49.6^\circ &= 55.1\sin 88.2^\circ\\ b &= \frac{55.1\sin 88.2^\circ}{\sin 49.6^\circ}\\ b &\approx 72.3.\end{aligned}$$

101. Multiplying by the LCD, we get

$$\begin{aligned}25.9\sin\alpha &= 23.4\sin 67.2^\circ\\ \sin\alpha &= \frac{23.4\sin 67.2^\circ}{25.9}\\ \sin\alpha &\approx 0.833\\ \alpha &\approx \sin^{-1}(0.833)\\ \alpha &\approx 56.4^\circ.\end{aligned}$$

102. Multiplying by the LCD, we obtain

$$\begin{aligned}52.9\sin 9.7^\circ &= 15.4\sin\beta\\ \frac{52.9\sin 9.7^\circ}{15.4} &= \sin\beta\\ 0.579 &\approx \sin\beta\\ \sin^{-1}(0.579) &\approx \beta\\ 35.4^\circ &\approx \beta.\end{aligned}$$

Since $90^\circ < \beta < 180^\circ$, we find $\beta = 180^\circ - 35.4^\circ = 144.6^\circ$.

103. Isolate $\cos\alpha$ on one side.

$$\begin{aligned}2(5.4)(8.2)\cos\alpha &= 5.4^2 + 8.2^2 - 3.6^2\\ \cos\alpha &= \frac{5.4^2 + 8.2^2 - 3.6^2}{2(5.4)(8.2)}\\ \cos\alpha &\approx 0.942\\ \alpha &\approx \cos^{-1}(0.942)\\ \alpha &\approx 19.6^\circ\end{aligned}$$

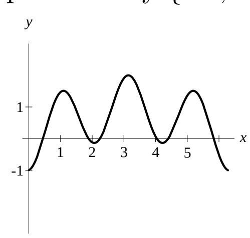
104. Isolate $\cos\alpha$ on one side.

$$\begin{aligned}2(3.2)(4.6)\cos\alpha &= 3.2^2 + 4.6^2 - 6.8^2\\ \cos\alpha &= \frac{3.2^2 + 4.6^2 - 6.8^2}{2(3.2)(4.6)}\\ \cos\alpha &\approx -0.504\\ \alpha &\approx \cos^{-1}(-0.504)\\ \alpha &\approx 120.3^\circ\end{aligned}$$

105. Given below is the graph of

$$y = \sin(x/2) - \cos(3x).$$

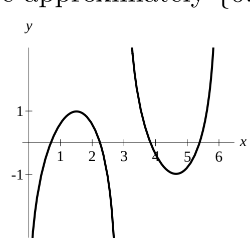
The intercepts or solutions on $[0, 2\pi)$ are approximately $\{0.4, 1.9, 2.2, 4.0, 4.4, 5.8\}$.



106. Given below is the graph of

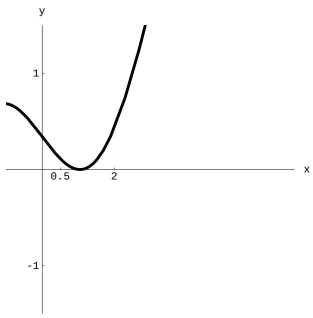
$$y = 2\sin(x) - \csc(x + 0.2).$$

The intercepts or solutions on $[0, 2\pi)$ are approximately $\{0.7, 2.2, 3.8, 5.4\}$.

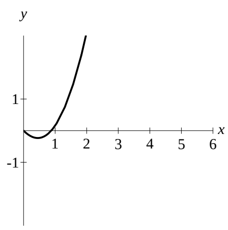


107. The graph of $y = \dfrac{x}{2} - \dfrac{\pi}{6} + \dfrac{\sqrt{3}}{2} - \sin x$

is shown. The solution set is $\{\pi/3\}$.



108. Below is the graph of $y = x^2 - \sin x$. The intercepts or solutions on $[0, 2\pi)$ are approximately $\{0, 0.9\}$.



109. Since $a = \sqrt{3}$ and $b = 1$, we obtain $r = \sqrt{\sqrt{3}^2 + 1^2} = 2$. If the terminal side of α goes through $(\sqrt{3}, 1)$, then $\tan\alpha = 1/\sqrt{3}$. Then one can choose $\alpha = \pi/6$ and $x = 2\sin(2t + \pi/6)$. The times when $x = 0$ are given by

$$\begin{aligned}
\sin\left(2t + \frac{\pi}{6}\right) &= 0 \\
2t + \frac{\pi}{6} &= k \cdot \pi \\
2t &= -\frac{\pi}{6} + k \cdot \pi \\
t &= -\frac{\pi}{12} + \frac{k \cdot \pi}{2} \\
t &= -\frac{\pi}{12} + \frac{\pi}{2} + \frac{k \cdot \pi}{2} \\
t &= \frac{5\pi}{12} + \frac{k \cdot \pi}{2}
\end{aligned}$$

where k is a nonnegative integer.

110. Since $a = -0.3$ and $b = 0.5$, we obtain $r = \sqrt{(-0.3)^2 + (0.5)^2} = \sqrt{0.34}$. If the terminal side of α goes through $(-0.3, 0.5)$, then $\tan\alpha = -0.5/0.3$. Since $\tan^{-1}(-5/3) \approx -1.03$, one can choose $\alpha = \pi - 1.03 \approx 2.11$ and $x = \sqrt{0.34}\sin(3t + 2.11)$. The times when $x = 0$ are given by

$$\begin{aligned}
\sin(3t + 2.11) &= 0 \\
3t + 2.11 &= k \cdot \pi \\
3t &= -2.11 + k \cdot \pi \\
t &= -0.703 + \frac{k \cdot \pi}{3} \\
t &= -0.703 + \frac{\pi}{3} + \frac{k \cdot \pi}{3} \\
t &= 0.34 + \frac{k \cdot \pi}{3}
\end{aligned}$$

where k is a nonnegative integer.

111. First, find the values of t when $x = \sqrt{3}$.

$$\begin{aligned}
2\sin\left(\frac{\pi t}{3}\right) &= \sqrt{3} \\
\sin\left(\frac{\pi t}{3}\right) &= \frac{\sqrt{3}}{2}
\end{aligned}$$

$$\begin{array}{ccc}
\frac{\pi t}{3} = \frac{\pi}{3} + 2k\pi & \text{or} & \frac{\pi t}{3} = \frac{2\pi}{3} + 2k\pi \\
\pi t = \pi + 6k\pi & \text{or} & \pi t = 2\pi + 6k\pi \\
t = 1 + 6k & \text{or} & t = 2 + 6k
\end{array}$$

Then the ball is $\sqrt{3}$ ft above sea level for the values of t satisfying

$$1 + 6k < t < 2 + 6k$$

where k is a nonnegative integer.

112. First, find the values of t when $x = 9.3$.

$$\begin{aligned}
6.2 + 3.1\sin\left(\frac{\pi}{6}(t - 9)\right) &= 9.3 \\
\sin\left(\frac{\pi}{6}(t - 9)\right) &= 1 \\
\frac{\pi}{6}(t - 9) &= \frac{\pi}{2} + 2k\pi \\
t - 9 &= 3 + 12k \\
t &= 12 + 12k
\end{aligned}$$

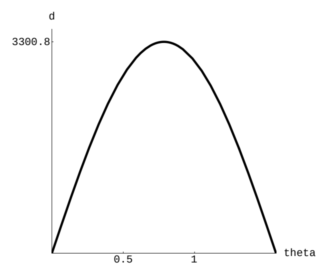
Since the values of t are limited from 1 to 12, then one must choose $k = 0$ and $t = 12$. In December, the store anticipates selling 9300 units.

113. Since $v_o = 325$ and $d = 3300$, we have

$$\begin{aligned} 325^2 \sin 2\theta &= 32(3300) \\ \sin 2\theta &= \frac{32(3300)}{325^2} \\ \sin 2\theta &\approx 0.99976 \\ 2\theta &\approx \sin^{-1}(0.99976) \\ 2\theta &\approx 88.74^\circ \\ \theta &\approx 44.4^\circ. \end{aligned}$$

Another angle is given by $2\theta = 180^\circ - 88.74^\circ = 91.26^\circ$ or $\theta = 91.26^\circ/2 \approx 45.6^\circ$.
The muzzle was aimed at 44.4° or 45.6° .

114. We will use the equation from Exercise 113 and sketch the graph $d = \dfrac{325^2 \sin(2\theta)}{32}$. We find the maximum distance is $d \approx 3300.8$ ft.



115. Note, 90 mph$= 90 \cdot \dfrac{5280}{3600}$ ft/sec$= 132$ ft/sec.
In $v_o^2 \sin 2\theta = 32d$, let $v_o = 132$ and $d = 230$.

$$\begin{aligned} 132^2 \sin 2\theta &= 32(230) \\ \sin 2\theta &= \frac{32(230)}{132^2} \\ \sin 2\theta &\approx 0.4224 \\ 2\theta = \sin^{-1}(0.4224) &\approx 25.0^\circ \text{ or } 155^\circ \\ \theta \approx 12.5^\circ &\quad \text{or} \quad 77.5^\circ \end{aligned}$$

The two possible angles are 12.5° and 77.5°. The time it takes the ball to reach home plate can be found by using $x = v_o t \cos\theta$. (See Example 11). For the angle 12.5°, it takes

$$t = \frac{230}{132\cos 12.5^\circ} \approx 1.78 \text{ sec}$$

while for 77.5° it takes

$$t = \frac{230}{132\cos 77.5^\circ} \approx 8.05 \text{ sec.}$$

The difference in time is $8.05 - 1.78 \approx 6.3$ sec.

116. In $v_o^2 \sin 2\theta = 32d$, let $d = 3(18,500) = 55,500$ ft. The muzzle velocity v_o is given by

$$\begin{aligned} v_o^2 \sin(2\cdot 45^\circ) &= 32(55,500) \\ v_o^2 \cdot 1 &= 32(55,500) \\ v_o &= \sqrt{32(55,500)} \\ v_o &\approx 1332.7 \text{ ft/sec.} \end{aligned}$$

117. Observe,

$$\begin{aligned} y &= \sqrt{2}\left((\sin x)\frac{1}{\sqrt{2}} - (\cos x)\frac{1}{\sqrt{2}}\right) \\ &= \sqrt{2}\sin\left(x + \frac{7\pi}{4}\right). \end{aligned}$$

The amplitude is $\sqrt{2}$, period is 2π, and phase shift $-\frac{7\pi}{4}$

118.

$$\begin{aligned} \sin\alpha\cos\beta + \cos\alpha\sin\beta &= \\ \frac{1}{3}\left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{2\sqrt{2}}{3}\right)\frac{1}{2} &= \\ \frac{2\sqrt{2} - \sqrt{3}}{6} & \end{aligned}$$

119. Using a cofunction relationship,

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x = \frac{3}{4}.$$

120. Using a half-angle identity, we find

$$\begin{aligned} \tan\left(\frac{x}{2}\right) &= \frac{1 - \cos x}{\sin x} \\ &= \frac{1 - (-1/3)}{2\sqrt{2}/3} \\ &= \frac{4/3}{2\sqrt{2}/3} \\ \tan\left(\frac{x}{2}\right) &= \sqrt{2} \end{aligned}$$

121. Using a half-angle identity, we obtain

$$\sin\left(\frac{x}{2}\right) = \sqrt{\frac{1 - \cos x}{2}}$$

$$= \sqrt{\frac{1-\frac{1}{4}}{2}} = \sqrt{\frac{3}{8}}$$

$$= \sqrt{\frac{6}{16}}$$

$$\sin\left(\frac{x}{2}\right) = \frac{\sqrt{6}}{4}$$

122. a) $\sin(3.5+2.1) = \sin 5.6$

b) $\sin(2x-x) = \sin x$

c) $\sin(2\cdot(4.8)) = \sin 9.6$

Thinking Outside the Box LVIII

The angles in the shaded triangle are 15°, 60°, and 105°. The side opposite the 105°-angle is one unit l long. Using the sine law (see Chapter 7), we find

$$x = \frac{\sin 60°}{\sin 105°}.$$

Recall, the area of a triangle is one-half times the product of the length of any two sides and the sine of the included angle of the two sides. Since the included angle between x and 1 is 15°, the area of the triangle is

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\frac{\sin 60°}{\sin 105°}\sin 15° \\
&= \frac{\sqrt{3}}{4}\frac{\sin 15°}{\sin 105°} \\
&= \frac{\sqrt{3}}{4}\frac{\sqrt{(2-\sqrt{3}/2)/2}}{\sqrt{(2+\sqrt{3}/2)/2}} \\
&= \frac{\sqrt{3}}{4}\frac{\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}} \\
&= \frac{\sqrt{3}}{4}(2-\sqrt{3}) \\
\text{Area} &= \frac{2\sqrt{3}-3}{4}.
\end{aligned}$$

6.6 Pop Quiz

1. $45° + k360°$ or $135° + k360°$

2. $60° + k360°$ or $300° + k360°$

3. $135° + k180°$

4. Since $\frac{x}{2} = \frac{\pi}{6} + 2k\pi$ or $\frac{x}{2} = \frac{5\pi}{6} + 2k\pi$

we obtain $x = \frac{\pi}{3} + 4k\pi$ or $x = \frac{5\pi}{3} + 4k\pi$.

Since x lies in $[0, 2\pi]$, we find

$$x = \frac{\pi}{3}, \frac{5\pi}{5}.$$

5. $0,\ 2\pi$

6. Since $2x = \frac{\pi}{4} + k\pi$, we obtain

$$x = \frac{\pi}{8} + k\frac{\pi}{2}.$$

Let $k = 0, 1, 2, 3$. The solutions in $[0, 2\pi]$ are

$$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}.$$

6.6 Linking Concepts

a) Since first base is 150 feet away, we find

$x = 130t\cos\theta = 150$, i.e., $t = \frac{150}{130\cos\theta}$.

Substituting into y we find that the angle θ satisfies

$$\begin{aligned}
-16t^2 + 130t\sin\theta + 5 &= 5 \\
-16t^2 + 130t\sin\theta &= 0 \\
-16\left(\frac{150}{130\cos\theta}\right)^2 + 130\left(\frac{150}{130\cos\theta}\right)\sin\theta &= 0 \\
-16\left(\frac{15}{13}\right)^2\frac{1}{\cos^2\theta} + 150\frac{\sin\theta}{\cos\theta} &= 0.
\end{aligned}$$

Then multiply the previous equation by $\cos^2\theta$.

$$\begin{aligned}
150\sin\theta &= 16\left(\frac{15}{13}\right)^2\frac{1}{\cos\theta} \\
150\sin\theta\cos\theta &= 16\left(\frac{15}{13}\right)^2
\end{aligned}$$

$$\begin{aligned} 75\sin(2\theta) &= 16\left(\frac{15}{13}\right)^2 \\ \sin(2\theta) &= \frac{16\left(\frac{15}{13}\right)^2}{75} \\ 2\theta &= \sin^{-1}\left(\frac{16\left(\frac{15}{13}\right)^2}{75}\right) \\ 2\theta &\approx 16.5^\circ \\ \theta &\approx 8.25^\circ \end{aligned}$$

b) It takes $t = \dfrac{150}{130\cos 8.25^\circ} \approx 1.1659$ sec for the ball to reach first base without skipping (see part a)) .

c) Setting $130t\cos\theta = 75$, we find $t = \dfrac{75}{130\cos\theta}$. Substituting into $y = 0$ and by using a calculator (such as the solver in a TI-83), we obtain the angle θ for which the ball must be thrown so that it hits the ground after 75 feet. Thus,

$$-16\left(\frac{75}{130\cos\theta}\right)^2 + 130\left(\frac{75}{130\cos\theta}\right)\sin\theta + 5 = 0$$

$$\theta \approx 0.2487^\circ.$$

d) It takes $t = \dfrac{75}{130\cos 0.2487^\circ} \approx 0.5769$ sec for the ball to reach the skip point. (See part c)).

e) Using the symmetry and the work shown in part d), the ball reaches first base in

$$2 \cdot \frac{75}{130\cos 0.2487^\circ} \approx 1.1539 \text{ sec.}$$

f) Time saved is 0.012 sec. (= 1.1659 − 1.1539).

g) If $\theta = 0$, the ball reaches first base in

$$t = \frac{150}{130\cos 0^\circ} \approx 1.1538 \text{ seconds.}$$

h) We are assuming that the ball is only subjected to gravity, and not to air resistance for example.

Review Exercises

1. $1 - \sin^2\alpha = \cos^2\alpha$

2. $\dfrac{1}{\sin x} \cdot \dfrac{\sin x}{\cos x} + \sec x = \dfrac{1}{\cos x} + \sec x = 2\sec x$

3. $(1 - \csc x)(1 + \csc x) = 1 - \csc^2 x = -\cot^2 x$

4. $\dfrac{\cos 2x}{\sin 2x} = \cot 2x$

5.
$$\frac{1}{1+\sin\alpha} + \frac{\sin\alpha}{\cos^2\alpha} = \frac{1}{1+\sin\alpha} + \frac{\sin\alpha}{1-\sin^2\alpha} = \frac{(1-\sin\alpha)+\sin\alpha}{1-\sin^2\alpha} = \frac{1}{\cos^2\alpha} = \sec^2\alpha$$

6. By using cofunction identities, the expression can be simplified to $2\cos\alpha\sin\alpha = \sin 2\alpha$.

7. $\tan(4s)$, by the double angle idenity for tangent

8. $\tan(2w - 4w) = \tan(-2w) = -\tan(2w)$

9. $\sin(3\theta - 6\theta) = \sin(-3\theta) = -\sin(3\theta)$

10. $\tan\left(\dfrac{2y}{2}\right) = \tan y$, by a double-angle identity for tangent

11. $\tan\left(\dfrac{2z}{2}\right) = \tan z$, by a double-angle identity for tangent

12. $\cos\left(2 \cdot \dfrac{x}{2}\right) = \cos x$, by a double-angle identity for cosine

13. e **14.** h **15.** c **16.** d

17. a **18.** b **19.** g **20.** f

21. Note, $\sin\alpha = \sqrt{1 - \left(\dfrac{-5}{13}\right)^2} = \sqrt{1 - \dfrac{25}{169}} = \sqrt{\dfrac{144}{169}} = \dfrac{12}{13}$. Then $\tan\alpha = \dfrac{12/13}{-5/13} = -\dfrac{12}{5}$, $\cot\alpha = -\dfrac{5}{12}$, $\csc\alpha = \dfrac{13}{12}$, $\sec\alpha = -\dfrac{13}{5}$.

22. Note, $\sec\alpha = -\sqrt{1 + \left(\dfrac{5}{12}\right)^2} = -\sqrt{1 + \dfrac{25}{144}} =$

$-\sqrt{\dfrac{169}{144}} = -\dfrac{13}{12}$. Then $\cos\alpha = -\dfrac{12}{13}$,
$\sin\alpha = -\sqrt{1-\left(\dfrac{12}{13}\right)^2} = -\sqrt{1-\dfrac{144}{169}} =$
$-\sqrt{\dfrac{25}{169}} = -\dfrac{5}{13}$. $\cot\alpha = \dfrac{12}{5}$, $\csc\alpha = -\dfrac{13}{5}$.

23. By using a cofunction identity, we get $\cos\alpha = \dfrac{-3}{5}$.

Then $\sin\alpha = -\sqrt{1-\left(\dfrac{-3}{5}\right)^2} = -\sqrt{1-\dfrac{9}{25}} =$
$-\sqrt{\dfrac{16}{25}} = -\dfrac{4}{5}$, $\sec\alpha = -\dfrac{5}{3}$, $\csc\alpha = -\dfrac{5}{4}$,
$\tan\alpha = \dfrac{-4/5}{-3/5} = \dfrac{4}{3}$, $\cot\alpha = \dfrac{3}{4}$.

24. By applying a cofunction identity, we find $\sec\alpha = 3$.

Then $\cos\alpha = \dfrac{1}{3}$ and $\sin\alpha = \sqrt{1-\left(\dfrac{1}{3}\right)^2} =$
$\sqrt{1-\dfrac{1}{9}} = \sqrt{\dfrac{8}{9}} = \dfrac{\sqrt{8}}{3} = \dfrac{2\sqrt{2}}{3}$, $\csc\alpha = \dfrac{3}{\sqrt{8}}$,
$\tan\alpha = \dfrac{2\sqrt{2}/3}{1/3} = 2\sqrt{2}$, $\cot\alpha = \dfrac{1}{\sqrt{8}}$.

25. By the half-angle identity for sine, we find

$$\begin{aligned} \sqrt{\frac{1-\cos\alpha}{2}} &= \frac{3}{5} \\ \frac{1-\cos\alpha}{2} &= \frac{9}{25} \\ 1-\cos\alpha &= \frac{18}{25} \\ \cos\alpha &= \frac{7}{25}. \end{aligned}$$

Since $\dfrac{3\pi}{2} < \alpha < 2\pi$, α is in quadrant IV

and $\sin\alpha = -\sqrt{1-\left(\dfrac{7}{25}\right)^2} = -\sqrt{1-\dfrac{49}{625}} =$
$-\sqrt{\dfrac{576}{625}} = -\dfrac{24}{25}$.

Then $\tan\alpha = \dfrac{-24/25}{7/25} = -\dfrac{24}{7}$,

$\cot\alpha = -\dfrac{7}{24}$, $\sec\alpha = \dfrac{25}{7}$, $\csc\alpha = -\dfrac{25}{24}$.

26. By the half-angle identity for cosine, we get

$$\begin{aligned} -\sqrt{\frac{1+\cos\alpha}{2}} &= -\frac{1}{3} \\ \frac{1+\cos\alpha}{2} &= \frac{1}{9} \\ 1+\cos\alpha &= \frac{2}{9} \\ \cos\alpha &= -\frac{7}{9}. \end{aligned}$$

Since $\pi < \alpha < \dfrac{3\pi}{2}$, α is in quadrant III and

$\sin\alpha = -\sqrt{1-\left(\dfrac{-7}{9}\right)^2} = -\sqrt{1-\dfrac{49}{81}} =$
$-\sqrt{\dfrac{32}{81}} = -\dfrac{\sqrt{32}}{9}$. Then $\tan\alpha = \dfrac{-\sqrt{32}/9}{-7/9} =$
$\dfrac{\sqrt{32}}{7}$, $\cot\alpha = \dfrac{7}{\sqrt{32}}$, $\sec\alpha = -\dfrac{9}{7}$,
and $\csc\alpha = -\dfrac{9}{\sqrt{32}}$.

27. It is an identity as shown below.

$$\begin{aligned} (\sin x + \cos x)^2 &= \\ \sin^2 x + 2\sin x\cos x + cos^2 x &= \\ 1 + 2\sin x\cos x &= \\ 1 + \sin(2x) \end{aligned}$$

28. It is not an identity since the right-hand side is equal to $\cos(A+B)$ and is not equal to the left-hand side.

29. It is not an identity since $\csc^2 x - \cot^2 x = 1$ and $\tan^2 - \sec^2 x = -1$.

30. It is an identity. Apply a half-angle identity and simplify the right-hand side.

$$\begin{aligned} \sin^2\frac{x}{2} &= \frac{1-\cos^2 x}{2+2\sin x\cos x\csc x} \\ \frac{1-\cos x}{2} &= \frac{1-\cos^2 x}{2+2\cos x} \\ &= \frac{(1-\cos x)(1+\cos x)}{2(1+\cos x)} \\ &= \frac{1-\cos x}{2} \end{aligned}$$

31. Odd, since $f(-x) = \dfrac{\sin(-x) - \tan(-x)}{\cos(-x)} = \dfrac{-\sin x + \tan x}{\cos x} = -\dfrac{\sin x - \tan x}{\cos x} = -f(x)$

32. Even, since $f(-x) = 1 + \sin^2(-x) = 1 + (-\sin x)^2 = 1 + \sin^2 x = f(x)$

33. It is neither even nor odd. Since $f(\pi/4) = \dfrac{\cos(\pi/4) - \sin(\pi/4)}{\sec(\pi/4)} = \dfrac{\sqrt{2}/2 - \sqrt{2}/2}{\sqrt{2}} = 0$ and $f(-\pi/4) = \dfrac{\cos(-\pi/4) - \sin(-\pi/4)}{\sec(-\pi/4)} = \dfrac{\sqrt{2}/2 + \sqrt{2}/2}{\sqrt{2}} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$, we see that $f(\pi/4) \neq \pm f(-\pi/4)$

34. Odd, since $f(-x) = \csc^3(-x) - \tan^3(-x) = (-\csc x)^3 - (-\tan x)^3 = -\csc^3 x + \tan^3 x = -f(x)$

35. Even, since $f(-x) = \dfrac{\sin(-x)\tan(-x)}{\cos(-x) + \sec(-x)} = \dfrac{(-\sin x)(-\tan x)}{\cos x + \sec x} = \dfrac{\sin x \tan x}{\cos x + \sec x} = f(x)$

36. It is neither even nor odd. Since $f(\pi/4) = \sin(\pi/4) + \cos(\pi/4) = \sqrt{2}/2 + \sqrt{2}/2 = \sqrt{2}$ and $f(-\pi/4) = \sin(-\pi/4) + \cos(-\pi/4) = -\sqrt{2}/2 + \sqrt{2}/2 = 0$, then $f(\pi/4) \neq \pm f(-\pi/4)$.

37. f, since $\sin(\pi/2 - \alpha) = \cos\alpha$

38. g, since $\sin(-x) = -\sin x$

39. e, for $\sin x + \sin y = 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

40. a, for $\sin x - \sin y = 2\cos\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

41. b, since $\sin 2x = 2\sin x\cos x$

42. d, since $\cos 2x = \cos^2 x - \sin^2 x$

43. h, for $\cos x + \cos y = 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

44. i, for $\cos x - \cos y = -2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

45. c, for $\tan\left(\dfrac{x}{2}\right) = \dfrac{\sin x}{1 + \cos x}$

46. j, for $\cos^2\dfrac{x}{2} = \dfrac{1 + \cos x}{2}$

47. Rewrite the right side.

$$\begin{aligned}
&= \frac{1 + \tan^2\theta}{1 - \tan^2\theta} \\
&= \frac{\sec^2\theta}{1 - \dfrac{\sin^2\theta}{\cos^2\theta}} \cdot \frac{\cos^2\theta}{\cos^2\theta} \\
&= \frac{1}{\cos^2\theta - \sin^2\theta} \\
&= \frac{1}{\cos 2\theta} \\
&= \sec 2\theta
\end{aligned}$$

48. Rewrite the right side.

$$\begin{aligned}
&= \frac{1 - \cos 2\theta}{1 + \cos 2\theta} \\
&= \frac{1 - (1 - 2\sin^2\theta)}{1 + (2\cos^2\theta - 1)} \\
&= \frac{2\sin^2\theta}{2\cos^2\theta} \\
&= \tan^2\theta
\end{aligned}$$

49. Rewrite the right side as follows:

$$\begin{aligned}
&= \frac{\csc^2 x - \cot^2 x}{2\csc^2 x + 2\csc x\cot x} \\
&= \frac{1}{\dfrac{2}{\sin^2 x} + 2\cdot\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x}} \\
&= \frac{1}{\dfrac{2}{\sin^2 x} + \dfrac{2\cos x}{\sin^2 x}} \cdot \frac{\sin^2 x}{\sin^2 x} \\
&= \frac{\sin^2 x}{2 + 2\cos x} \\
&= \frac{1 - \cos^2 x}{2(1 + \cos x)} \\
&= \frac{(1 - \cos x)(1 + \cos x)}{2(1 + \cos x)}
\end{aligned}$$

$$= \frac{1-\cos x}{2}$$
$$= \sin^2\left(\frac{x}{2}\right)$$

50. Rewrite the right side as follows:

$$\begin{aligned}
&= \frac{1-\sin^2 x}{\cos(-x)\sin(-x)} \\
&= \frac{\cos^2 x}{-\cos x \sin x} \\
&= \frac{\cos x}{-\sin x} \\
&= \cot(-x)
\end{aligned}$$

51. Rewrite the left side as follows:

$$\begin{aligned}
\cot(\alpha - 45^\circ) &= \\
(\tan(\alpha - 45^\circ))^{-1} &= \\
\left(\frac{\tan\alpha - \tan 45^\circ}{1+\tan\alpha\tan 45^\circ}\right)^{-1} &= \\
\left(\frac{\tan\alpha - 1}{1+\tan\alpha}\right)^{-1} &= \\
\frac{1+\tan\alpha}{\tan\alpha - 1} &=
\end{aligned}$$

52. Rewrite the left side as follows:

$$\begin{aligned}
\cos(\alpha + 45^\circ) &= \\
\cos\alpha\cos 45^\circ - \sin\alpha\sin 45^\circ &= \\
\cos\alpha \cdot \frac{1}{\sqrt{2}} - \sin\alpha \cdot \frac{1}{\sqrt{2}} &= \\
\frac{\cos\alpha - \sin\alpha}{\sqrt{2}} &=
\end{aligned}$$

53. Rewrite the left side.

$$\begin{aligned}
\frac{\sin 2\beta}{2\csc\beta} &= \\
\frac{2\sin\beta\cos\beta}{2/\sin\beta} \cdot \frac{\sin\beta}{\sin\beta} &= \\
\sin^2\beta\cos\beta &=
\end{aligned}$$

54. Rewrite the right side.

$$\begin{aligned}
&= \frac{\cos 2\beta}{\sqrt{2}(\cos\beta + \sin\beta)} \\
&= \frac{\cos^2\beta - \sin^2\beta}{\sqrt{2}(\cos\beta + \sin\beta)} \\
&= \frac{(\cos\beta - \sin\beta)(\cos\beta + \sin\beta)}{\sqrt{2}(\cos\beta + \sin\beta)} \\
&= \frac{\cos\beta - \sin\beta}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}} \cdot \cos\beta - \frac{1}{\sqrt{2}} \cdot \sin\beta \\
&= \sin 45^\circ \cos\beta - \cos 45^\circ \sin\beta \\
&= \sin(45^\circ - \beta)
\end{aligned}$$

55. Factor the numerator on the left-hand side as a difference of two cubes.
Note, $\cot w \tan w = 1$.

$$\begin{aligned}
\frac{\cot^3 y - \tan^3 y}{\sec^2 y + \cot^2 y} &= \\
\frac{(\cot y - \tan y)\left(\cot^2 y + 1 + \tan^2 y\right)}{\sec^2 y + \cot^2 y} &= \\
\frac{(\cot y - \tan y)\left(\cot^2 y + \sec^2 y\right)}{\sec^2 y + \cot^2 y} &= \\
\cot y - \tan y &= \\
\frac{1}{\tan y} - \tan y &= \\
\frac{1-\tan^2 y}{\tan y} &= \\
2 \cdot \frac{1-\tan^2 y}{2\tan y} &= \\
2 \cdot (\tan 2y)^{-1} &= \\
2\cot(2y) &=
\end{aligned}$$

56. Factor the numerator on the left-hand side as a difference of two cubes.

$$\begin{aligned}
\frac{\sin^3 y - \cos^3 y}{\sin y - \cos y} &= \\
\frac{(\sin y - \cos y)\left(\sin^2 y + \sin y\cos y + \cos^2 y\right)}{\sin y - \cos y} &=
\end{aligned}$$

$$\frac{(\sin y - \cos y)(1 + \sin y \cos y)}{\sin y - \cos y} =$$

$$(1 + \sin y \cos y) \cdot \frac{2}{2} =$$

$$\frac{2 + 2\sin y \cos y}{2} =$$

$$\frac{2 + \sin(2y)}{2} =$$

57. By using double-angle identities, we obtain

$$\begin{aligned} \cos(2 \cdot 2x) &= \\ 1 - 2\sin^2(2x) &= \\ 1 - 2(2\sin x \cos x)^2 &= \\ 1 - 8\sin^2 x \cos^2 x &= \\ 1 - 8\sin^2 x(1 - \sin^2 x) &= \\ 8\sin^4 x - 8\sin^2 x + 1. & \end{aligned}$$

58. By the sum identity for cosine, we have

$$\begin{aligned} \cos(x + 2x) &= \\ \cos x \cos 2x - \sin x \sin 2x &= \\ \cos x\left(1 - 2\sin^2 x\right) - \sin x(2\sin x \cos x) &= \\ \cos x - 2\cos x \sin^2 x - 2\sin^2 x \cos x &= \\ \cos x - 4\sin^2 x \cos x &= \\ \cos x\left(1 - 4\sin^2 x\right). & \end{aligned}$$

59. By the double-angle identity for sine, we get

$$\begin{aligned} \sin^4(2x) &= \\ (2\sin x \cos x)^4 &= \\ 16\sin^4 x \cos^4 x &= \\ 16\sin^4 x(1 - \sin^2 x)^2 &= \\ 16\sin^4 x(1 - 2\sin^2 x + \sin^4 x) &= \\ 16\sin^4 x - 32\sin^6 x + 16\sin^8 x &= \end{aligned}$$

60. By using the identity $\cos^2 x = 1 - \sin^2 x$, we get

$$\begin{aligned} 1 - \cos^6 x &= \\ 1 - \left(1 - \sin^2 x\right)^3 &= \\ 1 - \left(1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x\right) &= \\ 3\sin^2 x - 3\sin^4 x + \sin^6 x. & \end{aligned}$$

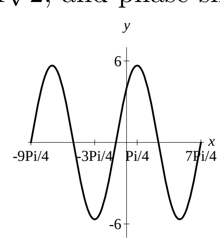
61. $\tan\left(\frac{-\pi/6}{2}\right) = \frac{1 - \cos(-\pi/6)}{\sin(-\pi/6)} =$
$\frac{1 - \sqrt{3}/2}{-1/2} \cdot \frac{2}{2} = \frac{2 - \sqrt{3}}{-1} = \sqrt{3} - 2$

62. $\sin\left(\frac{-\pi/4}{2}\right) = -\sqrt{\frac{1 - \cos(-\pi/4)}{2}} =$
$-\sqrt{\frac{1 - \sqrt{2}/2}{2} \cdot \frac{2}{2}} = -\sqrt{\frac{2 - \sqrt{2}}{4}} =$
$-\frac{\sqrt{2 - \sqrt{2}}}{2}$

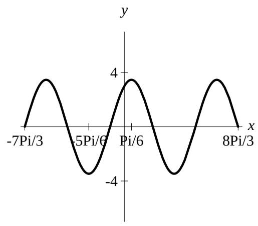
63. $\sin\left(\frac{-150^\circ}{2}\right) = -\sqrt{\frac{1 - \cos(-150^\circ)}{2}} =$
$-\sqrt{\frac{1 - (-\sqrt{3}/2)}{2} \cdot \frac{2}{2}} = -\sqrt{\frac{2 + \sqrt{3}}{4}} =$
$-\frac{\sqrt{2 + \sqrt{3}}}{2}$

64. $\cos\left(\frac{210^\circ}{2}\right) = -\sqrt{\frac{1 + \cos 210^\circ}{2}} =$
$-\sqrt{\frac{1 + (-\sqrt{3}/2)}{2} \cdot \frac{2}{2}} = -\sqrt{\frac{2 - \sqrt{3}}{4}} =$
$-\frac{\sqrt{2 - \sqrt{3}}}{2}$

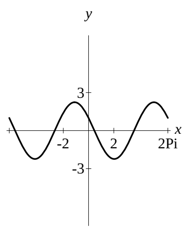
65. Let $a = 4$, $b = 4$, and $r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$. If the terminal side of α goes through $(4, 4)$, then $\tan\alpha = 4/4 = 1$ and one can choose $\alpha = \pi/4$. So $y = 4\sqrt{2}\sin(x + \pi/4)$, amplitude is $4\sqrt{2}$, and phase shift is $-\pi/4$.



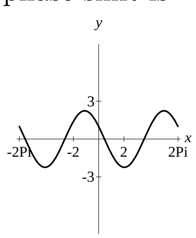
66. Let $a = \sqrt{3}$, $b = 3$, and $r = \sqrt{\sqrt{3}^2 + 3^2} = 2\sqrt{3}$. If the terminal side of α goes through $(\sqrt{3}, 3)$, then $\tan\alpha = 3/\sqrt{3} = \sqrt{3}$ and one can choose $\alpha = \pi/3$. So $y = 2\sqrt{3}\sin(x + \pi/3)$, amplitude is $2\sqrt{3}$, and phase shift is $-\pi/3$.



67. Let $a=-2$, $b=1$, and $r=\sqrt{(-2)^2+1^2}=\sqrt{5}$. If the terminal side of α goes through $(-2,1)$, then $\tan\alpha=-1/2$. Since $\tan^{-1}(-1/2)\approx -0.46$ and $(-2,1)$ is in quadrant II, one can choose $\alpha=\pi-0.46=2.68$. So $y=\sqrt{5}\sin(x+2.68)$, amplitude is $\sqrt{5}$, and phase shift is -2.68.



68. Let $a=-2$, $b=-1$, and $r=\sqrt{(-2)^2+(-1)^2}=\sqrt{5}$. If the terminal side of α goes through $(-2,-1)$, then $\tan\alpha=1/2$. Since $(-2,-1)$ is in quadrant III, one can choose $\alpha=\pi+\tan^{-1}(1/2)\approx 3.61$. Thus, $y=\sqrt{5}\sin(x+3.61)$, amplitude is $\sqrt{5}$, and phase shift is -3.61.



69. Isolate $\cos 2x$ on one side.

$$\begin{aligned} 2\cos 2x &= -1 \\ \cos 2x &= -\frac{1}{2} \end{aligned}$$

$$2x=\frac{2\pi}{3}+2k\pi \quad \text{or} \quad 2x=\frac{4\pi}{3}+2k\pi$$

$$x=\frac{\pi}{3}+k\pi \quad \text{or} \quad x=\frac{2\pi}{3}+k\pi$$

The solution set is

$$\left\{x \mid x=\frac{\pi}{3}+k\pi \text{ or } x=\frac{2\pi}{3}+k\pi\right\}.$$

70. Isolate $\sin 2x$ on one side.

$$\begin{aligned} 2\sin 2x &= -\sqrt{3} \\ \sin 2x &= -\frac{\sqrt{3}}{2} \end{aligned}$$

$$2x=\frac{4\pi}{3}+2k\pi \quad \text{or} \quad 2x=\frac{5\pi}{3}+2k\pi$$

$$x=\frac{2\pi}{3}+k\pi \quad \text{or} \quad x=\frac{5\pi}{6}+k\pi$$

The solution set is

$$\left\{x \mid x=\frac{2\pi}{3}+k\pi \text{ or } x=\frac{5\pi}{6}+k\pi\right\}.$$

71. Set each factor to zero.

$$(\sqrt{3}\csc x-2)(\csc x-2) = 0$$

$$\csc x=\frac{2}{\sqrt{3}} \quad \text{or} \quad \csc x=2$$

Thus, $x=\frac{\pi}{3},\frac{2\pi}{3},\frac{\pi}{6},\frac{5\pi}{6}$ plus multiples of 2π. The solution set is

$$\left\{x \mid x=\frac{\pi}{3}+2k\pi,\frac{2\pi}{3}+2k\pi,\frac{\pi}{6}+2k\pi,\frac{5\pi}{6}+2k\pi\right\}.$$

72. Set each factor to zero.

$$(\sec x-\sqrt{2})(\sqrt{3}\sec x+2) = 0$$

$$\sec x=\sqrt{2} \quad \text{or} \quad \sec x=-\frac{2}{\sqrt{3}}$$

Then $x=\frac{\pi}{4},\frac{7\pi}{4},\frac{5\pi}{6},\frac{7\pi}{6}$ plus multiples of 2π. The solution set is

$$\left\{x \mid x=\frac{\pi}{4}+2k\pi,\frac{7\pi}{4}+2k\pi,\frac{5\pi}{6}+2k\pi,\frac{7\pi}{6}+2k\pi\right\}.$$

73. Set the right-hand side to zero and factor.

$$\begin{aligned} 2\sin^2 x-3\sin x+1 &= 0 \\ (2\sin x-1)(\sin x-1) &= 0 \end{aligned}$$

$$\sin x=\frac{1}{2} \quad \text{or} \quad \sin x=1$$

The $x=\frac{\pi}{6},\frac{5\pi}{6},\frac{\pi}{2}$ plus multiples of 2π. The solution set is

$$\left\{x \mid x=\frac{\pi}{6}+2k\pi,\frac{5\pi}{6}+2k\pi,\frac{\pi}{2}+2k\pi\right\}.$$

74. Set the right-hand side to zero and factor.

$$\begin{aligned} 4\sin^2 x - \sin x - 3 &= 0 \\ (4\sin x + 3)(\sin x - 1) &= 0 \\ \sin x = -\frac{3}{4} \quad \text{or} \quad \sin x &= 1 \end{aligned}$$

Since $\sin^{-1}\left(-\frac{3}{4}\right) \approx -0.848$ and $2\pi - 0.848 \approx 5.44$ and $\pi + 0.848 \approx 3.99$, the solution set is

$$\left\{x \mid x = 5.44 + 2k\pi, 3.99 + 2k\pi, \frac{\pi}{2} + 2k\pi\right\}.$$

75. Isolate $\sin\frac{x}{2}$ on one side.

$$\begin{aligned} \sin\frac{x}{2} &= \frac{12}{8\sqrt{3}} \\ \sin\frac{x}{2} &= \frac{3}{2\sqrt{3}} \\ \sin\frac{x}{2} &= \frac{\sqrt{3}}{2} \end{aligned}$$

$$\frac{x}{2} = \frac{\pi}{3} + 2k\pi \quad \text{or} \quad \frac{x}{2} = \frac{2\pi}{3} + 2k\pi$$

$$x = \frac{2\pi}{3} + 4k\pi \quad \text{or} \quad x = \frac{4\pi}{3} + 4k\pi$$

The solution set is

$$\left\{x \mid x = \frac{2\pi}{3} + 4k\pi \text{ or } x = \frac{4\pi}{3} + 4k\pi\right\}.$$

76. Isolate $\cos\frac{x}{2}$ on one side.

$$\begin{aligned} -2\cos\frac{x}{2} &= \sqrt{2} \\ \cos\frac{x}{2} &= -\frac{\sqrt{2}}{2} \end{aligned}$$

$$\frac{x}{2} = \frac{3\pi}{4} + 2k\pi \quad \text{or} \quad \frac{x}{2} = \frac{5\pi}{4} + 2k\pi$$

$$x = \frac{3\pi}{2} + 4k\pi \quad \text{or} \quad x = \frac{5\pi}{2} + 4k\pi$$

The solution set is

$$\left\{x \mid x = \frac{3\pi}{2} + 4k\pi \text{ or } x = \frac{5\pi}{2} + 4k\pi\right\}.$$

77. By using the double-angle identity for sine, we get

$$\begin{aligned} \cos\frac{x}{2} - \sin\left(2\cdot\frac{x}{2}\right) &= 0 \\ \cos\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2} &= 0 \\ \cos\frac{x}{2}(1 - 2\sin\frac{x}{2}) &= 0 \\ \cos\frac{x}{2} = 0 \quad \text{or} \quad \sin\frac{x}{2} &= \frac{1}{2}. \end{aligned}$$

Then $\frac{x}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ plus mutiples of 2π.

Or $x = \pi, 3\pi, \frac{\pi}{3}, \frac{5\pi}{3}$ plus multiples of 4π.

The solution set is

$$\left\{x \mid x = \pi + 2k\pi, \frac{\pi}{3} + 4k\pi, \frac{5\pi}{3} + 4k\pi\right\}.$$

78. Apply the double-angle identity for sine and set the right-hand side to zero.

$$\begin{aligned} \sin 2x &= \frac{\sin x}{\cos x} \\ 2\sin x\cos x - \frac{\sin x}{\cos x} &= 0 \\ 2\sin x\cos^2 x - \sin x &= 0 \\ \sin x(2\cos^2 x - 1) &= 0 \end{aligned}$$

$$\sin x = 0 \quad \text{or} \quad \cos x = \pm\frac{1}{\sqrt{2}}$$

$$x = k\pi \quad \text{or} \quad x = \frac{\pi}{4} + \frac{k\pi}{2}$$

The solution set is

$$\left\{x \mid x = k\pi \text{ or } x = \frac{\pi}{4} + \frac{k\pi}{2}\right\}.$$

79. By the double-angle identity for cosine, we find

$$\begin{aligned} \cos 2x + \sin^2 x &= 0 \\ \cos^2 x - \sin^2 x + \sin^2 x &= 0 \\ \cos^2 x &= 0 \\ x &= \frac{\pi}{2} + k\pi. \end{aligned}$$

The solution set is $\left\{x \mid x = \frac{\pi}{2} + k\pi\right\}$.

80. By the double-angle identity for sine, we have

$$\begin{aligned}
\tan \frac{x}{2} &= \sin\left(2 \cdot \frac{x}{2}\right) \\
\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} &= 2 \sin \frac{x}{2} \cos \frac{x}{2} \\
\sin \frac{x}{2} &= 2 \sin \frac{x}{2} \cos^2 \frac{x}{2} \\
\sin \frac{x}{2}\left(1 - 2\cos^2 \frac{x}{2}\right) &= 0 \\
\sin \frac{x}{2} = 0 \quad &\text{or} \quad \cos \frac{x}{2} = \pm \frac{1}{\sqrt{2}} \\
\frac{x}{2} = k\pi \quad &\text{or} \quad \frac{x}{2} = \frac{\pi}{4} + \frac{k\pi}{2} \\
x = 2k\pi \quad &\text{or} \quad x = \frac{\pi}{2} + k\pi.
\end{aligned}$$

The solution set is

$$\left\{x \mid x = 2k\pi \text{ or } x = \frac{\pi}{2} + k\pi\right\}.$$

81. By factoring, we obtain

$$\begin{aligned}
\sin x(\cos x + 1) + (\cos x + 1) &= 0 \\
(\sin x + 1)(\cos x + 1) &= 0.
\end{aligned}$$

Then

$$\begin{aligned}
\sin x = -1 \quad &\text{or} \quad \cos x = -1 \\
x = \frac{3\pi}{2} + 2k\pi \quad &\text{or} \quad x = \pi + 2k\pi.
\end{aligned}$$

The solution set is

$$\left\{x \mid x = \frac{3\pi}{2} + 2k\pi \text{ or } x = \pi + 2k\pi\right\}.$$

82. By factoring, we find

$$\begin{aligned}
(\sin 2x \cos 2x - \cos 2x) + (\sin 2x - 1) &= 0 \\
\cos 2x(\sin 2x - 1) + (\sin 2x - 1) &= 0 \\
(\cos 2x + 1)(\sin 2x - 1) &= 0 \\
\cos 2x = -1 \text{ or } \sin 2x &= 1.
\end{aligned}$$

Then $2x = \pi + 2k\pi$ or $2x = \frac{\pi}{2} + 2k\pi$. Solution set is $\left\{x \mid x = \frac{\pi}{2} + k\pi \text{ or } x = \frac{\pi}{4} + k\pi\right\}$.

83. By multiplying the equation by 2, we obtain

$$\begin{aligned}
2 \sin \alpha \cos \alpha &= 1 \\
\sin 2\alpha &= 1 \\
2\alpha &= 90^\circ + k360^\circ \\
\alpha &= 45^\circ + k180^\circ.
\end{aligned}$$

By choosing $k = 0, 1$, one gets the solution set $\{45^\circ, 225^\circ\}$.

84. By using a double-angle identity for cosine, we find

$$\begin{aligned}
2\cos^2 \alpha - 1 &= \cos \alpha \\
2\cos^2 \alpha - \cos \alpha - 1 &= 0 \\
(2\cos \alpha + 1)(\cos \alpha - 1) &= 0 \\
\cos \alpha = -\frac{1}{2} \quad &\text{or} \quad \cos \alpha = 1.
\end{aligned}$$

The solution set is $\{0^\circ, 120^\circ, 240^\circ\}$.

85. Suppose $1 + \cos \alpha \neq 0$. Dividing the equation by $1 + \cos \alpha$, we get

$$\begin{aligned}
\frac{\sin \alpha}{1 + \cos \alpha} &= 1 \\
\tan \frac{\alpha}{2} &= 1 \\
\frac{\alpha}{2} &= 45^\circ + k180^\circ \\
\alpha &= 90^\circ + k360^\circ.
\end{aligned}$$

One solution is 90°. On the other hand if $1 + \cos \alpha = 0$, then $\cos \alpha = -1$ and $\alpha = 180^\circ$. Note $\alpha = 180^\circ$ satisfies the given equation. The solution set is $\{90^\circ, 180^\circ\}$.

86.

$$\begin{aligned}
\cos \alpha \cdot \frac{1}{\sin \alpha} &= \cot^2 \alpha \\
\cot \alpha &= \cot^2 \alpha \\
\cot \alpha - \cot^2 \alpha &= 0 \\
\cot \alpha(1 - \cot \alpha) &= 0 \\
\cot \alpha &= 0, 1
\end{aligned}$$

The solution set is $\{45^\circ, 90^\circ, 225^\circ, 270^\circ\}$.

87. No solution since the left-hand side is equal to 1 by an identity. The solution set is $\emptyset$.

88. No solution since the left-hand side is equal to 1 by an identity. The solution set is $\emptyset$.

89. Isolate $\sin 2\alpha$ on one side.

$$\begin{aligned}\sin^4 2\alpha &= \frac{1}{4}\\ \sin 2\alpha &= \pm\sqrt[4]{\frac{1}{4}}\\ \sin 2\alpha &= \pm\frac{1}{\sqrt{2}}\\ 2\alpha &= 45^\circ + k90^\circ\\ \alpha &= 22.5^\circ + k45^\circ\end{aligned}$$

By choosing $k = 0, 1, ..., 7$, one gets the solution set $\{22.5^\circ, 67.5^\circ, 112.5^\circ, 157.5^\circ, 202.5^\circ, 247.5^\circ, 292.5^\circ, 337.5^\circ\}$.

90. By using the double angle identity for sine, we find

$$\begin{aligned}2\sin\alpha\cos\alpha &= \frac{\sin\alpha}{\cos\alpha}\\ 2\sin\alpha\cos^2\alpha &= \sin\alpha\\ \sin\alpha(2\cos^2\alpha - 1) &= 0\\ \sin\alpha = 0 \quad &\text{or} \quad \cos\alpha = \pm\frac{1}{\sqrt{2}}\\ \alpha = 0^\circ, 180^\circ \quad &\text{or} \quad \alpha = 45^\circ + k90^\circ.\end{aligned}$$

By choosing $k = 0, 1, 2, 3$, one gets the solution set $\{0^\circ, 45^\circ, 135^\circ, 180^\circ, 225^\circ, 315^\circ\}$.

91. Suppose $\tan\alpha \neq 0$. Divide the equation by $\tan\alpha$.

$$\begin{aligned}\frac{2\tan\alpha}{1-\tan^2\alpha} &= \tan\alpha\\ \frac{2}{1-\tan^2\alpha} &= 1\\ 2 &= 1 - \tan^2\alpha\\ \tan^2\alpha &= -1\end{aligned}$$

The last equation is inconsistent since $\tan^2\alpha$ is nonnegative. But if $\tan\alpha = 0$, then $\alpha = 0^\circ, 180^\circ$ and these two values of α satisfy the given equation.
The solution set is $\{0^\circ, 180^\circ\}$.

92. Multiply the equation by $\tan\alpha$ and note that $\tan\alpha \cdot \cot\alpha = 1$. Then

$$\begin{aligned}\tan^2\alpha &= 1\\ \tan\alpha &= \pm 1\\ \alpha &= 45^\circ + k90^\circ.\end{aligned}$$

By choosing $k = 0, 1, 2, 3$, one gets the solution set $\{45^\circ, 135^\circ, 225^\circ, 315^\circ\}$.

93. By using the sum identity for sine, we obtain

$$\begin{aligned}\sin(2\alpha + \alpha) &= \cos 3\alpha\\ \sin 3\alpha &= \cos 3\alpha\\ \tan 3\alpha &= 1\\ 3\alpha &= 45^\circ + k180^\circ\\ \alpha &= 15^\circ + k60^\circ.\end{aligned}$$

By choosing $k = 0, 1, ..., 5$, one gets the solution set $\{15^\circ, 75^\circ, 135^\circ, 195^\circ, 255^\circ, 315^\circ\}$.

94. By appplying the sum identity for cosine, we get

$$\begin{aligned}\cos(2\alpha + \alpha) &= \frac{\cos 3\alpha}{\sin 3\alpha}\\ \cos 3\alpha &= \frac{\cos 3\alpha}{\sin 3\alpha}.\end{aligned}$$

Suppose $\cos 3\alpha \neq 0$. Divide the equation by $\cos 3\alpha$.

$$\begin{aligned}1 &= \frac{1}{\sin 3\alpha}\\ \sin 3\alpha &= 1\\ 3\alpha &= 90^\circ + k360^\circ\\ \alpha &= 30^\circ + k120^\circ\end{aligned}$$

On the other hand if $\cos 3\alpha = 0$, then $3\alpha = 90^\circ + k180^\circ$ and $\alpha = 30^\circ + k60^\circ$. Note the values for $\alpha = 30^\circ + k60^\circ$ include the values from $\alpha = 30^\circ + k120^\circ$. By choosing $k = 0, 1, ..., 5$ in $\alpha = 30^\circ + k60^\circ$, one obtains that the solution set is $\{30^\circ, 90^\circ, 150^\circ, 210^\circ, 270^\circ, 330^\circ\}$.

95. $\cos 15^\circ + \cos 19^\circ =$
$2\cos\left(\frac{15^\circ + 19^\circ}{2}\right)\cos\left(\frac{15^\circ - 19^\circ}{2}\right) =$
$2\cos 17^\circ \cos(-2^\circ) = 2\cos 17^\circ \cos 2^\circ$

96. $\cos 4^\circ - \cos 6^\circ =$
$-2\sin\left(\frac{4^\circ+6^\circ}{2}\right)\sin\left(\frac{4^\circ-6^\circ}{2}\right) =$
$-2\sin 5^\circ \sin(-1^\circ) = 2\sin 5^\circ \sin 1^\circ$

97. $\sin(\pi/4) - \sin(-\pi/8) = \sin(\pi/4) + \sin(\pi/8) =$
$2\sin\left(\frac{\pi/4+\pi/8}{2}\right)\cos\left(\frac{\pi/4-\pi/8}{2}\right) =$
$2\sin\left(\frac{3\pi/8}{2}\right)\cos\left(\frac{\pi/8}{2}\right) =$
$2\sin(3\pi/16)\cos(\pi/16)$

98. $\sin(\pi/12) - \sin(\pi/6) =$
$2\cos\left(\frac{\pi/12+\pi/6}{2}\right)\sin\left(\frac{\pi/12-\pi/6}{2}\right) =$
$2\cos\left(\frac{3\pi/12}{2}\right)\sin\left(\frac{-\pi/12}{2}\right) =$
$2\cos(3\pi/24)\sin(-\pi/24) =$
$-2\cos(\pi/8)\sin(\pi/24)$

99. $2\sin 11^\circ \cos 13^\circ =$
$\sin(11^\circ + 13^\circ) + \sin(11^\circ - 13^\circ) =$
$\sin 24^\circ + \sin(-2^\circ) = \sin 24^\circ - \sin 2^\circ$

100. $2\sin 8^\circ \sin 12^\circ =$
$\cos(8^\circ - 12^\circ) - \cos(8^\circ + 12^\circ) =$
$\cos(-4^\circ) - \cos 20^\circ = \cos 4^\circ - \cos 20^\circ$

101. $2\cos\frac{x}{4}\cos\frac{x}{3} =$
$\cos\left(\frac{x}{4} - \frac{x}{3}\right) + \cos\left(\frac{x}{4} + \frac{x}{3}\right) =$
$\cos\left(-\frac{x}{12}\right) + \cos\left(\frac{7x}{12}\right) =$
$\cos\left(\frac{x}{12}\right) + \cos\left(\frac{7x}{12}\right)$

102. $2\cos(s)\sin(3s) = \sin(s+3s) - \sin(s-3s) =$
$\sin 4s - \sin(-2s) = \sin 4s + \sin 2s$

103. Since α is in quadrant II and β is in quadrant I, $\cos\alpha = -\sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2} =$
$-\frac{1}{2}$ and $\sin\beta = \sqrt{1 - \left(\frac{\sqrt{2}}{2}\right)^2} = \frac{\sqrt{2}}{2}$.
So $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta =$
$\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{-1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$.

104. Since α is in quadrant IV and β is in quadrant II, $\cos\alpha = \sqrt{1 - \left(\frac{-\sqrt{2}}{2}\right)^2} =$
$\frac{\sqrt{2}}{2}$ and $\cos\beta = -\sqrt{1 - \left(\frac{1}{2}\right)^2} = -\frac{\sqrt{3}}{2}$.
So $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta =$
$\frac{-\sqrt{2}}{2}\cdot\frac{-\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$.

105. Since α is in quadrant I and β is in quadrant II, $\cos\alpha = \sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2} =$
$\frac{1}{2}$ and $\sin\beta = \sqrt{1 - \left(-\frac{\sqrt{2}}{2}\right)^2} = \frac{\sqrt{2}}{2}$.
So $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta =$
$\frac{1}{2}\cdot\frac{-\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$.

106. Since α is in quadrant II and β is in quadrant I, $\cos\alpha = -\sqrt{1 - \left(\frac{\sqrt{2}}{2}\right)^2} =$
$-\frac{\sqrt{2}}{2}$ and $\cos\beta = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$.
So $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta =$
$\frac{-\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = -\frac{\sqrt{6}+\sqrt{2}}{4}$.

107. Let $a = 0.6$, $b = 0.4$, and $r = \sqrt{0.6^2 + 0.4^2} \approx 0.72$. If the terminal side of α goes through $(0.6, 0.4)$, then $\tan\alpha = 0.4/0.6$ and one can choose $\alpha = \tan^{-1}(2/3) \approx 0.588$.
Thus, $x = 0.72\sin(2t + 0.588)$.
The values of t when $x = 0$ are given by

$$\begin{aligned} \sin(2t + 0.588) &= 0 \\ 2t + 0.588 &= k\pi \\ 2t &= -0.588 + k\pi \\ t &= -0.294 + \frac{k\pi}{2} \end{aligned}$$

When $k = 1, 2$, one gets $t \approx 1.28, 2.85$.

108. Let $v_0 = 400$ and $d = 3000$.

$$\begin{aligned} 400^2 \sin 2\theta &= 32(3000) \\ \sin 2\theta &= \frac{32(3000)}{400^2} \\ \sin 2\theta &= 0.6 \\ 2\theta = \sin^{-1}(0.6) &\approx 36.869° \\ \theta &\approx 18.4° \end{aligned}$$

Another solution is $2\theta = 180° - 36.869°$ $= 143.131°$. Then $\theta \approx \frac{143.13°}{2} \approx 71.6°$.
The angles are 18.4° and 71.6°.

Thinking Outside the Box LIX

WRONG = 25938 and RIGHT = 51876

Chapter 6 Test

1. $\frac{1}{\cos x} \cdot \frac{\cos x}{\sin x} \cdot 2 \sin x \cos x = 2 \cos x$

2. $\sin(2t + 5t) = \sin 7t$

3. $\frac{1}{1 - \cos y} + \frac{1}{1 + \cos y} = \frac{1 + \cos y + 1 - \cos y}{1 - cos^2 y} =$
$\frac{2}{\sin^2 y} = 2 \csc^2 y$

4. $\tan(\pi/5 + \pi/10) = \tan(3\pi/10)$

5.

$$\begin{aligned} \frac{\sin \beta \cos \beta}{\sin \beta / \cos \beta} &= \\ \sin \beta \cos \beta \cdot \frac{\cos \beta}{\sin \beta} &= \\ \cos^2 \beta &= \\ 1 - \sin^2 \beta &= \end{aligned}$$

6.

$$\begin{aligned} \frac{1}{\sec \theta - 1} - \frac{1}{\sec \theta + 1} &= \\ \frac{\sec \theta + 1 - (\sec \theta - 1)}{\sec^2 \theta - 1} &= \\ \frac{2}{\tan^2 \theta} &= \\ 2 \cot^2 \theta &= \end{aligned}$$

7. Using the cofunction identity for cosine, we get

$$\begin{aligned} \cos(\pi/2 - x) \cos(-x) &= \\ \sin x \cos x &= \\ \frac{2 \sin x \cos x}{2} &= \\ \frac{\sin(2x)}{2}. \end{aligned}$$

8. Factor the left-hand side and use a half-angle identity for tangent. Then

$$\begin{aligned} \tan(t/2) \cdot (\cos^2 t - 1) &= \\ \frac{1 - \cos t}{\sin t} \cdot (-\sin^2 t) &= \\ (1 - \cos t) \cdot (-\sin t) &= \\ (\cos t - 1) \sin t &= \\ \cos t \sin t - \sin t &= \\ \frac{\sin t}{\sec t} - \sin t. \end{aligned}$$

9. Since $-\sin \theta = 1$, we get $\sin \theta = -1$ and the solution set is $\left\{\theta \mid \theta = \frac{3\pi}{2} + 2k\pi\right\}$.

10. Since $\cos 3s = \frac{1}{2}$, we obtain

$$\begin{aligned} 3s = \frac{\pi}{3} + 2k\pi \quad &\text{or} \quad 3s = \frac{5\pi}{3} + 2k\pi \\ s = \frac{\pi}{9} + \frac{2k\pi}{3} \quad &\text{or} \quad s = \frac{5\pi}{9} + \frac{2k\pi}{3}. \end{aligned}$$

The solution set is

$$\left\{s \mid s = \frac{\pi}{9} + \frac{2k\pi}{3} \text{ or } s = \frac{5\pi}{9} + \frac{2k\pi}{3}\right\}.$$

11. Since $\tan 2t = -\sqrt{3}$, we have

$$\begin{aligned} 2t &= \frac{2\pi}{3} + k\pi \\ t &= \frac{\pi}{3} + \frac{k\pi}{2}. \end{aligned}$$

The solution set is $\left\{t \mid t = \frac{\pi}{3} + \frac{k\pi}{2}\right\}$.

12.

$$\begin{aligned} 2\sin\theta\cos\theta &= \cos\theta \\ \cos\theta(2\sin\theta - 1) &= 0 \\ \cos\theta = 0 \quad \text{or} & \quad \sin\theta = 1/2 \end{aligned}$$

The solution set is

$$\left\{\theta \mid \theta = \frac{\pi}{2} + k\pi, \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi\right\}.$$

13. By factoring, we obtain

$$\begin{aligned} (3\sin\alpha - 1)(\sin\alpha - 1) &= 0 \\ \sin\alpha = 1/3 \quad \text{or} & \quad \sin\alpha = 1 \\ \alpha = \sin^{-1}(1/3) \approx 19.5^\circ \quad \text{or} & \quad \alpha = 90^\circ. \end{aligned}$$

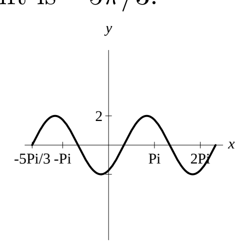
Another solution is $\alpha = 180^\circ - 19.5^\circ = 160.5^\circ$. The solution set is $\{19.5^\circ, 90^\circ, 160.5^\circ\}$.

14.

$$\begin{aligned} \tan(2\alpha - 7\alpha) &= 1 \\ \tan(-5\alpha) &= 1 \\ -\tan 5\alpha &= 1 \\ \tan 5\alpha &= -1 \\ 5\alpha &= 135^\circ + k180^\circ \\ \alpha &= 27^\circ + k36^\circ \end{aligned}$$

The solution set is $\{27^\circ, 63^\circ, 99^\circ, 171^\circ, 207^\circ, 243^\circ, 279^\circ, 351^\circ\}$. Note, 135° and 315° are not solutions.

15. Let $a = 1$, $b = -\sqrt{3}$, $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$. If the terminal side of α goes through $(1, -\sqrt{3})$, then $\tan\alpha = -\sqrt{3}$ and one can choose $\alpha = 5\pi/3$. Then $y = 2\sin(x + 5\pi/3)$, the period is 2π, amplitude is 2, and phase shift is $-5\pi/3$.



16. If $\csc\alpha = 2$, then $\sin\alpha = 1/2$. Since α is in quadrant II, we obtain $\cos\alpha = -\sqrt{1 - (1/2)^2} = -\sqrt{1 - 1/4} = -\sqrt{3/4} = -\sqrt{3}/2$, $\sec\alpha = -2/\sqrt{3}$, $\tan\alpha = \dfrac{1/2}{-\sqrt{3}/2} = -1/\sqrt{3}$, and $\cot\alpha = -\sqrt{3}$.

17. Even, $f(-x) = (-x)\sin(-x) = (-x)(-\sin x) = x\sin x = f(x)$.

18. By using a half-angle identity, we obtain

$$\sin\left(\frac{-\pi/6}{2}\right) = -\sqrt{\frac{1 - \cos(-\pi/6)}{2}} = -\sqrt{\frac{1 - \sqrt{3}/2}{2} \cdot \frac{2}{2}} = -\sqrt{\frac{2 - \sqrt{3}}{4}} = -\frac{\sqrt{2 - \sqrt{3}}}{2}.$$

19. If $x = y = \pi/6$, then $\tan x + \tan y = 2\tan(\pi/6) = 2 \cdot \dfrac{\sqrt{3}}{3} = \dfrac{2\sqrt{3}}{3}$ and $\tan(x+y) = \tan(\pi/6 + \pi/6) = \tan(\pi/3) = \sqrt{3}$. Thus, it is not an identity.

20. Let $a = 2$, $b = -4$, $r = \sqrt{2^2 + (-4)^2} = \sqrt{20}$. If the terminal side of α goes through $(2, -4)$, then one can choose $\alpha = \tan^{-1}(-4/2) \approx -1.107$. Then $d = \sqrt{20}\sin(3t - 1.107)$. The values of t when $d = 0$ are given by

$$\begin{aligned} \sin(3t - 1.107) &= 0 \\ 3t - 1.107 &= k\pi \\ 3t &= 1.107 + k\pi \\ t &\approx 0.4 + \frac{k\pi}{3}. \end{aligned}$$

By choosing $k = 0, 1, 2, 3$, one obtains the values of t in $[0, 4]$, namely, 0.4 sec, 1.4 sec, 2.5 sec, and 3.5 sec.

Tying It All Together

1. Odd, since $f(-x) = 3(-x)^3 - 2(-x) = -3x^3 + 2x = -f(x)$

2. Even, $f(-x) = 2|-x| = 2|x| = f(x)$

3. Odd, $f(-x) = (-x)^3 + \sin(-x) = -x^3 - \sin x = -f(x)$

4. Even, $f(-x) = (-x)^3 \sin(-x) = (-x^3)(-\sin x) = x^3 \sin x = f(x)$

5. Even, $f(-x) = (-x)^4 - (-x)^2 + 1 = x^4 - x^2 + 1 = f(x)$

6. Odd, $f(-x) = \dfrac{1}{-x} = -\dfrac{1}{x} = -f(x)$

7. Even, $f(-x) = \dfrac{\sin(-x)}{-x} = \dfrac{-\sin x}{-x} = \dfrac{\sin x}{x} = f(x)$

8. Even, $f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x)$

9. It is not an identity. If $\alpha = \beta = \pi/6$, then $\sin(\alpha + \beta) = \sin(\pi/3) = \sqrt{3}/2$ and $\sin(\pi/6) + \sin(\pi/6) = 2 \cdot (1/2) = 1$.

10. It is not an identity. If $\alpha = \beta = 1$, then $(\alpha + \beta)^2 = 2^2 = 4$ and $\alpha^2 + \beta^2 = 1^2 + 1^2 = 2$.

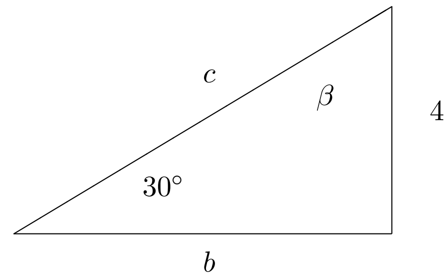
11. Identity

12. Identity

13. It is not an identity; for if $x = 1$, then $\sin^{-1}(1) = \pi/2 \approx 1.57$ and $1/\sin 1 \approx 1.19$.

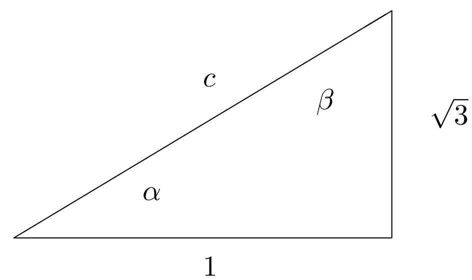
14. It is not an identity; for if $x = \sqrt{7\pi/6}$, then $\sin^2 \sqrt{7\pi/6}$ is a positive number and $\sin\left(\left(\sqrt{7\pi/6}\right)^2\right) = \sin(7\pi/6) = -1/2$.

15. Form a right triangle with $\alpha = 30°$, $a = 4$.



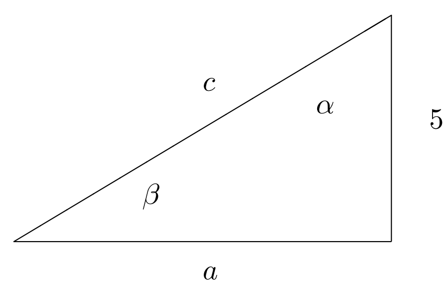
Since $\tan 30° = 4/b$ and $\sin 30° = 4/c$, we get $b = 4/\tan 30° = 4\sqrt{3}$ and $c = 4/\sin 30° = 8$. Also $\beta = 90° - 30° = 60°$.

16. Form a right triangle with $a = \sqrt{3}$, $b = 1$.



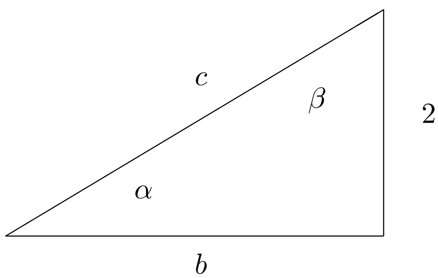
Then $c = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ by the Pythagorean Theorem.
Since $\tan \alpha = \sqrt{3}/1 = \sqrt{3}$, we get $\alpha = 60°$ and $\beta = 30°$.

17. Form a right triangle with $b = 5$ as shown below.



Since $\cos \beta = 0.3$, we find $\beta = \cos^{-1}(0.3) \approx 72.5°$ and $\alpha = 17.5°$. Since $\sin 72.5° = 5/c$ and $\tan 72.5° = 5/a$, we obtain $c = 5/\sin 72.5° \approx 5.2$ and $a = 5/\tan 72.5° \approx 1.6$.

18. Form a right triangle with $a = 2$.



Since $\sin\alpha = 0.6$, we obtain
$\alpha = \sin^{-1}(0.6) \approx 36.9^\circ$ and $\beta = 53.1^\circ$.
Since $\sin 36.9^\circ = 2/c$ and $\tan 36.9^\circ = 2/b$, we get $c = 2/\sin 36.9^\circ \approx 3.3$ and $b = 2/\tan 36.9^\circ \approx 2.7$.

19. $\frac{1}{2} + i\frac{\sqrt{3}}{2}$

20. $2\left(-\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right) = -1 + i\sqrt{3}$

21. $\cos^2 225^\circ + 2i\cos 225^\circ \sin 225^\circ - \sin^2 225^\circ =$
$\cos^2 225^\circ - \sin^2 225^\circ + 2i\cos 225^\circ \sin 225^\circ =$
$\cos(2\cdot 225^\circ) + i\cdot\sin(2\cdot 225^\circ) =$
$\cos 450^\circ + i\cdot\sin 450^\circ =$
$\cos 90^\circ + i\cdot\sin 90^\circ = i$

22. $\cos^2 3^\circ - (i\sin 3^\circ)^2 = \cos^2 3^\circ + \sin^2 3^\circ = 1$

23. $(2+i)^2(2+i) = (4+4i-1)(2+i) =$
$(3+4i)(2+i) = 6+3i+8i-4 = 2+11i$

24. $(\sqrt{2}(1-i))^4 = \sqrt{2}^4(1-i)^4 = 4\left((1-i)^2\right)^2 =$
$4(1-2i-1)^2 = 4(-2i)^2 = 4(-4) = -16$

25. minutes **26.** seconds

27. unit

28. π

29. αr

30. $r\omega$

31. y, x

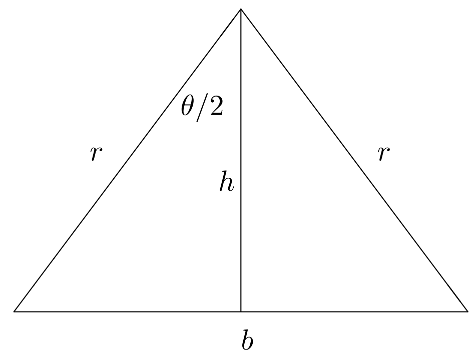
32. fundamental

33. amplitude

34. phase shift

Concepts of Calculus

1. Consider the isosceles triangle below.



A pentagon inscribed in a circle of radius r consists of 5 triangles each one like the one shown above with radius r, $\theta = 72^\circ$ or $\theta/2 = 36^\circ$, and where b is the base.
Since $h = r\cos(36^\circ)$ and $\sin(36^\circ) = \frac{b/2}{r}$, we get $b = 2r\sin(36^\circ)$ and the area of the triangle is

$$\begin{aligned}\frac{1}{2}bh &= \frac{1}{2}(2r\sin(36^\circ))(r\cos(36^\circ)) \\ &= r^2\sin(36^\circ)\cos(36^\circ) \\ \frac{1}{2}bh &= \frac{r^2}{2}\sin(72^\circ).\end{aligned}$$

Hence, the area of the pentagon is

$$\frac{5\sin(72^\circ)}{2}r^2.$$

2. An n-gon consists of n triangles like the one shown in Exercise 1 where $\theta/2 = \frac{360^\circ/n}{2}$.
For this n-gon, $h = r\cos\left(\frac{360^\circ}{2n}\right)$,
$\sin\left(\frac{360^\circ}{2n}\right) = \frac{b/2}{r}$ or $b = 2r\sin\left(\frac{360^\circ}{2n}\right)$,
and the area of the triangle is given by

$$\begin{aligned}\frac{1}{2}bh &= \frac{1}{2}\left(2r\sin\left(\frac{360^\circ}{2n}\right)\right)\left(r\cos\left(\frac{360^\circ}{2n}\right)\right)\\ &= r^2\sin\left(\frac{360^\circ}{2n}\right)\cos\left(\frac{360^\circ}{2n}\right)\\ &= r^2\sin\left(\frac{180^\circ}{n}\right)\cos\left(\frac{180^\circ}{n}\right)\\ \frac{1}{2}bh &= \frac{r^2}{2}\sin\left(\frac{360^\circ}{n}\right).\end{aligned}$$

Thus, the area A_n of an n-gon inscribed in a circle of radius r is

$$A_n = \frac{nr^2}{2}\sin\left(\frac{360^\circ}{n}\right).$$

Another formula for A_n can be obtained by using the above calculations and the fact that

$$\sin(90^\circ - \alpha) = \cos\alpha$$

and

$$\cos(90^\circ - \alpha) = \sin\alpha.$$

Thus, an equivalent formula for the area of the n-gon is

$$\begin{aligned}A_n &= nr^2\sin\left(\frac{180^\circ}{n}\right)\cos\left(\frac{180^\circ}{n}\right)\\ A_n &= nr^2\sin\left(90^\circ - \frac{180^\circ}{n}\right)\cos\left(90^\circ - \frac{180^\circ}{n}\right).\end{aligned}$$

3. For an n-gon, the constant of proportionality is

$$\frac{n}{2}\sin\left(\frac{360^\circ}{n}\right).$$

For a decagon ($n = 10$), kilogon $n = 1000$, and megagon ($n = 10^6$), the constants of proportionality are

2.938926261,

3.141571983, and

3.141592654, respectively.

4. The shape of the n-gon as n increases approaches the shape of a circle.

5. Note, when $n = 10^6$, the megagon is almost a circle. We will use the constant for the megagon calculated in Exercise 3. Thus, the area of a circle of radius r could be approximated by the area of the magagon which

$$3.141592654r^2.$$

6. As derived in part Exercise 2, the base of the triangle is

$$b = 2r\sin\left(\frac{180^\circ}{n}\right).$$

Thus, the perimeter P of an n-gon is $P = nb$, or

$$P = 2nr\sin\left(\frac{180^\circ}{n}\right)$$

or equivalently

$$P = 2nr\cos\left(90^\circ - \frac{180^\circ}{n}\right).$$

7. When n is a large number, the shape of an n-gon approximates the shape of a circle. Consequently, the circumference C of a circle of radius r is approximately

$$C \approx 2nr\sin\left(\frac{180^\circ}{n}\right).$$

Note, if $n = 10^6$ then

$$n\sin\left(\frac{180^\circ}{n}\right) \approx 3.141592654.$$

Thus, the circumference is

$$C \approx 2r(3.141592654).$$

8. Note, π is the ratio

$$\pi = \frac{\text{circumference}}{\text{diameter}}.$$

From Exercise 6, the circumference of an n-gon is

$$P = 2nr\cos\left(90^\circ - \frac{180^\circ}{n}\right)$$

which is approximately the circumference of a circle. Thus, when $n = 10^6$ we obtain

$$\begin{aligned}
\pi &= \frac{\text{circumference}}{\text{diameter}} \\
\pi &\approx \frac{P}{2r} \\
\pi &\approx \frac{2nr\cos\left(90^\circ - \frac{180^\circ}{n}\right)}{2r} \\
\pi &\approx n\cos\left(90^\circ - \frac{180^\circ}{n}\right).
\end{aligned}$$

In particular, when $n = 10^6$ we find

$$\begin{aligned}
\pi &\approx 10^6\cos\left(90^\circ - \frac{180^\circ}{10^6}\right) \\
\pi &\approx 3.141592654
\end{aligned}$$

Alternatively, we can find π by using the formula

$$\pi = \frac{\text{area of a circle with radius } r}{r^2}.$$

From Exercise 2, the area of an n-gon is

$$A_n = nr^2\sin\left(90^\circ - \frac{180^\circ}{n}\right)\cos\left(90^\circ - \frac{180^\circ}{n}\right)$$

which is approximately the area of the circle. Thus, when $n = 10^6$ we obtain

$$\begin{aligned}
\pi &= \frac{\text{area of circle with radius } r}{r^2} \\
\pi &\approx \frac{nr^2\sin\left(90^\circ - \frac{180^\circ}{n}\right)\cos\left(90^\circ - \frac{180^\circ}{n}\right)}{r^2} \\
\pi &\approx n\sin\left(90^\circ - \frac{180^\circ}{n}\right)\cos\left(90^\circ - \frac{180^\circ}{n}\right) \\
\pi &\approx 10^6\sin\left(90^\circ - \frac{180^\circ}{10^6}\right)\cos\left(90^\circ - \frac{180^\circ}{10^6}\right) \\
\pi &\approx 3.141592654.
\end{aligned}$$