For Thought

1. True

2. False, the domain of a finite sequence is $\{1, 2, ..., n\}$ for some positive integer n.

3. True

4. False, n is the independent variable.

5. False, the first four terms are $1, -8, 27, -64$.

6. False, $a_5 = -3 + 4 \cdot 6 = 21$.

7. False, the common difference is $d = 4 - 7 = -3$.

8. False, there is no common difference.

9. False, since if $14 = 4 + 2d$ then $d = 5$ and $a_4 = 4 + 3 \cdot 5 = 19$. **10.** True

11.1 Exercises

1. finite sequence

2. infinite sequence

3. terms

4. recursion

5. arithmetic

6. n factorial

7. $a_1 = 1^2 = 1$, $a_2 = 2^2 = 4$, $a_3 = 3^2 = 9$, $a_4 = 4^2 = 16$, $a_5 = 5^2 = 25$, $a_6 = 6^2 = 36$, $a_7 = 7^2 = 49$.

First seven terms are $1, 4, 9, 16, 25, 36, 49$.

8. $a_1 = 0^2 = 0$, $a_2 = 1^2 = 1$, $a_3 = 2^2 = 4$, $a_4 = 3^2 = 9$, $a_5 = 4^2 = 16$

First five terms are $0, 1, 4, 9, 16$

9. $b_1 = \dfrac{(-1)^2}{2} = \dfrac{1}{2}$, $b_2 = \dfrac{(-1)^3}{3} = -\dfrac{1}{3}$,

$b_3 = \dfrac{(-1)^4}{4} = \dfrac{1}{4}$, $b_4 = \dfrac{(-1)^5}{5} = -\dfrac{1}{5}$,

$b_5 = \dfrac{(-1)^6}{6} = \dfrac{1}{6}$, $b_6 = \dfrac{(-1)^7}{7} = -\dfrac{1}{7}$,

$b_7 = \dfrac{(-1)^8}{8} = \dfrac{1}{8}$, $b_8 = \dfrac{(-1)^9}{9} = -\dfrac{1}{9}$.

First eight terms are

$\dfrac{1}{2}, -\dfrac{1}{3}, \dfrac{1}{4}, -\dfrac{1}{5}, \dfrac{1}{6}, -\dfrac{1}{7}, \dfrac{1}{8}, -\dfrac{1}{9}$.

10. $b_1 = (-1)^1 3 = -3$, $b_2 = (-1)^2 6 = 6$, $b_3 = (-1)^3 9 = -9$, $b_4 = (-1)^4 12 = 12$,

First four terms are $-3, 6, -9, 12$.

11. $c_1 = (-2)^0 = 1$, $c_2 = (-2)^1 = -2$, $c_3 = (-2)^2 = 4$, $c_4 = (-2)^3 = -8$, $c_5 = (-2)^4 = 16$, $c_6 = (-2)^5 = -32$.
First six terms are $1, -2, 4, -8, 16, -32$

12. $c_1 = (-3)^{-1} = -\dfrac{1}{3}$, $c_2 = (-3)^0 = 1$,

$c_3 = (-3)^1 = -3$, $c_4 = (-3)^2 = 9$, $c_5 = (-3)^3 = -27$, $c_6 = (-3)^4 = 81$.

First six terms are $-\dfrac{1}{3}, 1, -3, 9, -27, 81$

13. $a_1 = 2^1 = 2$, $a_2 = 2^0 = 1$,

$a_3 = 2^{-1} = \dfrac{1}{2}$, $a_4 = 2^{-2} = \dfrac{1}{4}$, $a_5 = 2^{-3} = \dfrac{1}{8}$.

First five terms are $2, 1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}$.

14. $a_1 = \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$, $a_2 = \left(\dfrac{1}{2}\right)^1 = \dfrac{1}{2}$,

$a_3 = \left(\dfrac{1}{2}\right)^0 = 1$, $a_4 = \left(\dfrac{1}{2}\right)^{-1} = 2$,

$a_5 = \left(\dfrac{1}{2}\right)^{-2} = 4$, $a_6 = \left(\dfrac{1}{2}\right)^{-3} = 8$,

$a_7 = \left(\dfrac{1}{2}\right)^{-4} = 16$. First seven

terms are $\dfrac{1}{4}, \dfrac{1}{2}, 1, 2, 4, 8, 16$.

15. $a_1 = -6 + 0 = -6$, $a_2 = -6 - 4 = -10$,

$a_3 = -6 - 8 = -14$, $a_4 = -6 - 12 = -18$,

$a_5 = -6 - 16 = -22$. First five

terms are $-6, -10, -14, -18, -22$.

16. $a_1 = -2 + 0 = -2$, $a_2 = -2 + 4 = 2$,

$a_3 = -2 + 8 = 6$, $a_4 = -2 + 12 = 10$,

$a_5 = -2 + 16 = 14$, $a_6 = -2 + 20 = 18$,

$a_7 = -2 + 24 = 22$, $a_8 = -2 + 28 = 26$.

First eight terms are $-2, 2, 6, 10, 14, 18, 22, 26$

17. $b_1 = 5+0 = 5$, $b_2 = 5+0.5 = 5.5$, $b_3 = 5+1 = 6$, $b_4 = 5+1.5 = 6.5$, $b_5 = 5+2 = 7$, $b_6 = 5+2.5 = 7.5$, $b_7 = 5+3 = 8$. First seven terms are $5, 5.5, 6, 6.5, 7, 7.5, 8$.

18. $b_1 = \frac{1}{4}+0 = \frac{1}{4}$, $b_2 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$, $b_3 = \frac{1}{4} - 1 = -\frac{3}{4}$, $b_4 = \frac{1}{4} - \frac{3}{2} = -\frac{5}{4}$, $b_5 = \frac{1}{4} - 2 = -\frac{7}{4}$. The first five terms are $\frac{1}{4}, -\frac{1}{4}, -\frac{3}{4}, -\frac{5}{4}, -\frac{7}{4}$.

19. $a_1 = -0.1+9 = 8.9$, $a_2 = -0.2+9 = 8.8$, $a_3 = -0.3+9 = 8.7$, $a_4 = -0.4+9 = 8.6$, $a_{10} = -1+9 = 8$. First four terms are $8.9, 8.8, 8.7, 8.6$ and and 10th term is 8.

20. $a_1 = 0.3 - 0.4 = -0.1$, $a_2 = 0.6 - 0.4 = 0.2$, $a_3 = 0.9 - 0.4 = 0.5$, $a_4 = 1.2 - 0.4 = 0.8$, $a_{10} = 3 - 0.4 = 2.6$. First four terms are $-0.1, 0.2, 0.5, 0.8$ and the 10th term is 2.6 .

21. $a_1 = 8+0 = 8$, $a_2 = 8-3 = 5$, $a_3 = 5-3 = 2$, $a_4 = 2-3 = -1$, and $a_{10} = 8-27 = -19$. First four terms are $8, 5, 2, -1$ and 10th term is -19.

22. $a_1 = -7+0 = -7$, $a_2 = -7+0.5 = -6.5$, $a_3 = -6.5+0.5 = -6$, $a_4 = -6+0.5 = -5.5$, $a_{10} = -7+4.5 = -2.5$. The first four terms are $-7, -6.5, -6, -5.5$ and 10th term is -2.5 .

23. One gets $a_1 = \frac{4}{2+1} = \frac{4}{3}$, $a_2 = \frac{4}{4+1} = \frac{4}{5}$, $a_3 = \frac{4}{6+1} = \frac{4}{7}$, $a_4 = \frac{4}{8+1} = \frac{4}{9}$, $a_{10} = \frac{4}{20+1} = \frac{4}{21}$ The first four terms are $\frac{4}{3}, \frac{4}{5}, \frac{4}{7}, \frac{4}{9}$ and 10th term is $\frac{4}{21}$.

24. One has $a_1 = \frac{2}{1^2+1} = 1$, $a_2 = \frac{2}{2^2+1} = \frac{2}{5}$, $a_3 = \frac{2}{3^2+1} = \frac{1}{5}$, $a_4 = \frac{2}{4^2+1} = \frac{2}{17}$, $a_{10} = \frac{2}{10^2+1} = \frac{2}{101}$. The first four terms are $1, \frac{2}{5}, \frac{1}{5}, \frac{2}{17}$; and 10th term is $\frac{2}{101}$.

25. One gets $a_1 = \frac{(-1)^1}{2\cdot 3} = -\frac{1}{6}$, $a_2 = \frac{(-1)^2}{3\cdot 4} = \frac{1}{12}$, $a_3 = \frac{(-1)^3}{4\cdot 5} = -\frac{1}{20}$, $a_4 = \frac{(-1)^4}{5\cdot 6} = \frac{1}{30}$, $a_{10} = \frac{(-1)^{10}}{11\cdot 12} = \frac{1}{132}$. The first four terms are $-\frac{1}{6}, \frac{1}{12}, -\frac{1}{20}, \frac{1}{30}$ and 10th term is $\frac{1}{132}$.

26. $a_1 = \frac{(-1)^2}{2^2} = \frac{1}{4}$, $a_2 = \frac{(-1)^3}{3^2} = -\frac{1}{9}$, $a_3 = \frac{(-1)^4}{4^2} = \frac{1}{16}$, $a_4 = \frac{(-1)^5}{5^2} = -\frac{1}{25}$, $a_{10} = \frac{(-1)^{11}}{11^2} = -\frac{1}{121}$. The first four terms are $\frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}$, and 10th term is $-\frac{1}{121}$.

27. $a_1 = 2! = 2$, $a_2 = 4! = 24$, $a_3 = 6! = 720$, $a_4 = 8! = 40,320$, and $a_5 = 10! = 3,628,800$

28. $a_1 = 0! = 1$, $a_2 = 1! = 1$, $a_3 = 2! = 2$, $a_4 = 3! = 6$, and $a_5 = 4! = 24$

29. $a_1 = \frac{2^1}{1!} = 2$, $a_2 = \frac{2^2}{2!} = 2$, $a_3 = \frac{2^3}{3!} = \frac{4}{3}$, $a_4 = \frac{2^4}{4!} = \frac{2}{3}$, and $a_5 = \frac{2^5}{5!} = \frac{4}{15}$

30. $a_1 = \frac{1^2}{1!} = 1$, $a_2 = \frac{2^2}{2!} = 2$, $a_3 = \frac{3^2}{3!} = \frac{3}{2}$, $a_4 = \frac{4^2}{4!} = \frac{2}{3}$, and $a_5 = \frac{5^2}{5!} = \frac{5}{24}$

31. $b_1 = \frac{1!}{0!} = 1$, $b_2 = \frac{2!}{1!} = 2$, $b_3 = \frac{3!}{2!} = 3$, $b_4 = \frac{4!}{3!} = 4$, and $b_5 = \frac{5!}{4!} = \frac{5\cdot 4!}{4!} = 5$

32. $b_1 = \frac{3!}{1!} = 6$, $b_2 = \frac{4!}{2!} = 12$, $b_3 = \frac{5!}{3!} = 20$,
$b_4 = \frac{6!}{4!} = 30$, and $b_5 = \frac{7!}{5!} = \frac{7(6)\cdot 5!}{5!} = 42$

33. $c_1 = \frac{(-1)^1}{2!} = -\frac{1}{2}$, $c_2 = \frac{(-1)^2}{3!} = \frac{1}{6}$,
$c_3 = \frac{(-1)^3}{4!} = -\frac{1}{24}$, $c_4 = \frac{(-1)^4}{5!} = \frac{1}{120}$,
and $c_5 = \frac{(-1)^5}{6!} = -\frac{1}{720}$.

34. $c_1 = \frac{(-2)^1}{0!} = -2$, $c_2 = \frac{(-2)^3}{1!} = -8$,
$c_3 = \frac{(-2)^5}{2!} = -16$, $c_4 = \frac{(-2)^7}{3!} = -\frac{64}{3}$,
and $c_5 = \frac{(-2)^9}{4!} = -\frac{64}{3}$.

35. Note $t_n = \frac{1}{(n+2)!e^n}$. Then we obtain
$t_1 = \frac{1}{3!e} = \frac{1}{6e}$, $t_2 = \frac{1}{4!e^2} = \frac{1}{24e^2}$,
$t_3 = \frac{1}{5!e^3} = \frac{1}{120e^3}$, $t_4 = \frac{1}{6!e^4} = \frac{1}{720e^4}$,
and $t_5 = \frac{1}{7!e^5} = \frac{1}{5040e^5}$.

36. $t_1 = \frac{e^1}{1!} = e$, $t_2 = \frac{e^2}{2!} = \frac{e^2}{2}$,
$t_3 = \frac{e^3}{3!} = \frac{e^3}{6}$, $t_4 = \frac{e^4}{4!} = \frac{e^4}{24}$,
and $t_5 = \frac{e^5}{5!} = \frac{e^5}{120}$.

37. $a_n = 2n$ **38.** $a_n = 2n - 1$

39. $a_n = 2n + 7$ **40.** $a_n = 2n + 12$

41. $a_n = (-1)^{n+1}$ **42.** $a_n = \frac{(-1)^n}{2}$

43. $a_n = n^3$ **44.** $a_n = \frac{(-1)^{n+1}}{n^3}$

45. $a_n = e^n$ **46.** $a_n = \pi n^2$

47. $a_n = \frac{1}{2^{n-1}}$ **48.** $a_n = (-3)^{n-1}$

49. $a_2 = 3a_1 + 2 = 3(-4) + 2 = -10$,
$a_3 = 3a_2 + 2 = 3(-10) + 2 = -28$,
$a_4 = 3a_3 + 2 = 3(-28) + 2 = -82$,
$a_5 = 3a_4 + 2 = 3(-82) + 2 = -244$,
$a_6 = 3a_5 + 2 = 3(-244) + 2 = -730$,
$a_7 = 3a_6 + 2 = 3(-730) + 2 = -2188$,
$a_8 = 3a_7 + 2 = 3(-2188) + 2 = -6562$.
First four terms $-4, -10, -28, -82$ and 8th term is -6562.

50. $a_2 = 1 - \frac{1}{a_1} = 1 - \frac{1}{2} = \frac{1}{2}$,
$a_3 = 1 - \frac{1}{a_2} = 1 - \frac{1}{1/2} = -1$,
$a_4 = 1 - \frac{1}{a_3} = 1 - \frac{1}{-1} = 2$,
$a_5 = 1 - \frac{1}{a_4} = 1 - \frac{1}{2} = \frac{1}{2}$.
By a recurring pattern, $a_8 = \frac{1}{2}$. First four terms are $2, \frac{1}{2}, -1, 2$ and 8th term is $\frac{1}{2}$.

51. One finds $a_2 = a_1^2 - 3 = 2^2 - 3 = 1$,
$a_3 = a_2^2 - 3 = 1^2 - 3 = -2$,
$a_4 = a_3^2 - 3 = (-2)^2 - 3 = 1$.
By a repeating pattern, $a_8 = 1$. First four terms are $2, 1, -2, 1$ and 8th term is 1.

52. One finds $a_2 = a_1^2 - 2 = (-2)^2 - 2 = 2$,
$a_3 = a_2^2 - 2 = 2^2 - 2 = 2$,
$a_4 = a_3^2 - 2 = (2)^2 - 2 = 2$.
By a repeating pattern, $a_8 = 2$. First four terms are $-2, 2, 2, 2$ and 8th term is 2.

53. $a_2 = a_1 + 7 = (-15) + 7 = -8$,
$a_3 = a_2 + 7 = (-8) + 7 = -1$,
$a_4 = a_3 + 7 = (-1) + 7 = 6$.
There is a common difference of 7.
So $a_8 = 34$. First four terms are $-15, -8, -1, 6$ and 8th term is 34.

54. $a_2 = \frac{1}{2}a_1 = \frac{1}{2}8 = 4$,
$a_3 = \frac{1}{2}a_2 = \frac{1}{2}4 = 2$,

$a_4 = \frac{1}{2}a_3 = \frac{1}{2}2 = 1.$

Since the next term is $\frac{1}{2}$ of the previous term, then $a_8 = \frac{1}{16}$. First four terms are $8, 4, 2, 1$ and 8th term is $\frac{1}{16}$.

55. First four terms are $6, 3, 0, -3$ and 10th term is $a_{10} = 6 + 9(-3) = -21$.

56. First four terms are $-12, -8, -4, 0$ and 10th term is $b_{10} = -12 + 9(4) = 24$.

57. First four terms are $1, 0.9, 0.8, 0.7$ and 10th term is $c_{10} = 1 + 9(-0.1) = 0.1$.

58. First four terms are $5, 0, -5, -10$ and 10th term is $q_{10} = 10 - 5(10) = -40$.

59. One finds $w_1 = -\frac{1}{3}(1) + 5 = \frac{14}{3}$,

$w_2 = -\frac{1}{3}(2) + 5 = \frac{13}{3}$,

$w_3 = -\frac{1}{3}(3) + 5 = \frac{12}{3}$,

$w_4 = -\frac{1}{3}(4) + 5 = \frac{11}{3}$, and

$w_{10} = -\frac{1}{3}(10) + 5 = \frac{5}{3}$.

First four terms are $\frac{14}{3}, \frac{13}{3}, \frac{12}{3}, \frac{11}{3}$ and 10th term is $\frac{5}{3}$.

60. $t_1 = \frac{1}{2}(1) + \frac{1}{2} = 1$,

$t_2 = \frac{1}{2}(2) + \frac{1}{2} = \frac{3}{2}$,

$t_3 = \frac{1}{2}(3) + \frac{1}{2} = 2$,

$t_4 = \frac{1}{2}(4) + \frac{1}{2} = \frac{5}{2}$, and

$t_{10} = \frac{1}{2}(10) + \frac{1}{2} = \frac{11}{2}$.

First four terms are $1, \frac{3}{2}, 2, \frac{5}{2}$ and 10th term is $\frac{11}{2}$.

61. Yes, $d = 1$ is the common difference.

62. No, there is no common difference.

63. No, there is no common difference.

64. Yes, $d = -3$ is the common difference.

65. No, there is no common difference.

66. Yes, $d = \frac{1}{4}$ is the common difference.

67. Yes, $d = \frac{\pi}{4}$ is the common difference.

68. No, there is no common difference.

69. Since $d = 5$ and $a_1 = 1$, $a_n = a_1 + (n-1)d = 1 + (n-1)5$. Then $a_n = 5n - 4$.

70. Since $d = 3$ and $a_1 = 2$, $a_n = a_1 + (n-1)d = 2 + (n-1)3$. Then $a_n = 3n - 1$.

71. Since $d = 2$ and $a_1 = 0$, $a_n = a_1 + (n-1)d = 0 + (n-1)2$. So $a_n = 2n - 2$.

72. Since $d = 6$ and $a_1 = -3$, $a_n = a_1 + (n-1)d = -3 + (n-1)6$. So $a_n = 6n - 9$.

73. Since $d = -4$ and $a_1 = 5$, $a_n = a_1 + (n-1)d = 5 + (n-1)(-4)$. So $a_n = -4n + 9$.

74. Since $d = -2$ and $a_1 = 1$, $a_n = a_1 + (n-1)d = 1 + (n-1)(-2)$. So $a_n = -2n + 3$.

75. Since $d = 0.1$ and $a_1 = 1$, $a_n = a_1 + (n-1)d = 1 + (n-1)(0.1)$. So $a_n = 0.1n + 0.9$.

76. Since $d = 0.75$ and $a_1 = 2$, we get $a_n = a_1 + (n-1)d = 2 + (n-1)(0.75)$. Thus, $a_n = 0.75n + 1.25$.

77. Since $d = \frac{\pi}{6}$ and $a_1 = \frac{\pi}{6}$, $a_n = a_1 + (n-1)d = \frac{\pi}{6} + (n-1)\frac{\pi}{6}$. So $a_n = \frac{\pi}{6}n$.

78. Since $d = \frac{\pi}{12}$ and $a_1 = \frac{\pi}{12}$, we obtain $a_n = a_1 + (n-1)d = \frac{\pi}{12} + (n-1)\frac{\pi}{12}$. Thus, $a_n = \frac{\pi}{12}n$.

79. Since $d = 15$ and $a_1 = 20$, we have

$a_n = a_1 + (n-1)d = 20 + (n-1)15$.

Then $a_n = 15n + 5$.

80. Since $d = -10$ and $a_1 = 70$, we obtain

$a_n = a_1 + (n-1)d = 70 + (n-1)(-10)$.

Then $a_n = -10n + 80$.

81. Since $a_n = a_1 + (n-1)d$, we get $a_8 = (-3) + (7)5 = 32$.

82. Since $a_n = a_1 + (n-1)d$, we obtain $a_{11} = 4 + (10)(-0.8) = -4$.

83. Since $a_n = a_1 + (n-1)d$, we find

$$\begin{aligned} a_1 + 2d &= 6 \\ a_1 + 6d &= 18 \end{aligned}$$

Subtracting the first equation from the second, we get $4d = 12$ and $d = 3$. From the first equation, $a_1 + 6 = 6$ and $a_1 = 0$. Then $a_{10} = a_1 + 9d = 0 + 9 \cdot 3 = 27$. So $a_{10} = 27$.

84. Since $a_n = a_1 + (n-1)d$, we obtain

$$\begin{aligned} a_1 + d &= 20 \\ a_1 + 4d &= 10 \end{aligned}$$

Subtracting the first equation from the second, we get $3d = -10$ and $d = -\dfrac{10}{3}$.

From the first equation, we find $a_1 - \dfrac{10}{3} = 20$ and $a_1 = \dfrac{70}{3}$.

Then $a_8 = a_1 + 7d = \dfrac{70}{3} + 7\left(-\dfrac{10}{3}\right) = 0$.

So $a_8 = 0$.

85. Since $a_n = a_1 + (n-1)d$, we get $a_{21} = 12 + 20d = 96$. So $20d = 84$ and the common difference is $d = 4.2$.

86. Since $a_n = a_1 + (n-1)d$, we obtain $a_{11} = 5 + 10d = -10$. So $10d = -15$ and the common difference is $d = -1.5$.

87. Since $a_n = a_1 + (n-1)d$, we find

$$\begin{aligned} a_1 + 2d &= 10 \\ a_1 + 6d &= 20. \end{aligned}$$

Subtracting the first equation from the second, we obtain $4d = 10$ and $d = 2.5$. From the first equation, we get $a_1 + 5 = 10$ and $a_1 = 5$. Then $a_n = 5 + (n-1)(2.5) = 2.5n + 2.5$. So $a_n = 2.5n + 2.5$.

88. Since $a_n = a_1 + (n-1)d$, we have

$$\begin{aligned} a_1 + 4d &= 30 \\ a_1 + 9d &= -5. \end{aligned}$$

By subtracting the first equation from the second, one gets $5d = -35$ and $d = -7$. From the first equation, we get $a_1 - 28 = 30$ and $a_1 = 58$. Then $a_n = 58 + (n-1)(-7) = -7n + 65$. Thus, $a_n = -7n + 65$.

89. $a_n = a_{n-1} + 9$, $a_1 = 3$

90. $a_n = a_{n-1} - 5$, $a_1 = 30$

91. $a_n = 3a_{n-1}$, $a_1 = \dfrac{1}{3}$

92. $a_n = -\dfrac{1}{4}a_{n-1}$, $a_1 = 4$

93. $a_n = \sqrt{a_{n-1}}$, $a_1 = 16$

94. $a_n = (a_{n-1})^2$, $a_1 = t^2$

95. A formula for the MSRP is

$$a_n = 43,440(1.06)^n$$

where n is the number of years since 2008. For the years 2009-2013, the MSRP are $a_1 = \$46,046$, $a_2 = \$48,809$, $a_3 = \$51,738$, $a_4 = \$54,842$, and $a_5 = \$58,133$.

96. A formula for the salary is

$$a_n = 43,440 + 2606n$$

where n is the number of years since 2008. For the years 2009-2013, the salaries are $a_1 =$

\$46,406, $a_2 = \$48,652$, $a_3 = \$51,258$, $a_4 = \$53,864$, and $a_5 = \$56,470$. No, the 2013 salary will not equal (but be less than) the price of the 2013 Jeep Grand Cherokee from the previous exercise.

97. The number of pages read on the nth day of November is $a_n = 5+3(n-1)$ or $a_n = 3n+2$. On November 30, they will read $a_{30} = 92$ pages.

98. On the nth day the penalty is
$a_n = 500 + 200(n-1)$ or $a_n = 200n + 300$.
The 10th day penalty is $a_{10} = 500 + 200(9) = \$2300$.

99. In year n, the annual cost is

$$a_n = 70,810 + 4000(n - 2008).$$

In 2017, the cost is projected to be $a_{2017} = 70,810 + 4000(2017 - 2008) = \$106,810$.

100. If a_1 is her retirement pay for the first year, then

$$\begin{aligned} a_1 + 4d &= 24,500 \\ a_1 + 8d &= 25,700. \end{aligned}$$

For the solution, one finds $d = 300$ and
$a_1 = \$23,300$ - her income in the first year.
On the 13th year, her income
is $a_{13} = 23,300 + 12(300) = \$26,900$.

101. Since there are four corners, $C_n = 4$.
If the corner tiles are in place, the number of edge tiles needed for each side is $2(n-1)$. Since there are four sides, $E_n = 8(n-1)$.

If the corner tiles and edge tiles are in place, the interior tiles will occupy an $(n-1)$-by-$(n-1)$ square.

Note, the area of a tile is $\dfrac{1}{4}$ ft^2. Then the number of interior tiles is $I_n = \dfrac{(n-1)^2}{\frac{1}{4}}$ or equivalently $I_n = 4n^2-8n+4$ for $n = 1, 2, 3, ...$

102. $K_n = 4(0.89)+0.79[8(n-1)]+0.69[4(n-1)^2]$ or equivalently $K_n = 2.76n^2 + 0.8n$

105. Since $a = 5$ and $b = 6$, we find $c^2 = a^2+b^2 = 25 + 36 = 61$. Then the foci are $(\pm\sqrt{61}, 0)$, asymptotes are $y = \pm\frac{b}{a}x$ or $y = \pm\frac{6}{5}x$, and the intercepts are $(\pm a, 0) = (\pm 5, 0)$.

106. Rewrite equation as

$$x^2 + \frac{y^2}{9} = 1.$$

Since $a = 3$ and $b = 1$, we find $c^2 = a^2 - b^2 = 9 - 1 = 8$ or $c = 2\sqrt{2}$. Then the foci are $(0, \pm 2\sqrt{2})$, y-intercepts are $(0, \pm a) = (0, \pm 3)$, and the eccentricity is $e = \frac{c}{a} = \frac{2\sqrt{2}}{3}$.

107. Use the method of completing the square:

$$\begin{aligned} y &= -2(x^2 - 2x) \\ &= -2(x^2 - 2x + 1) + 2 \\ &= -2(x-1)^2 + 2 \end{aligned}$$

Since $\frac{1}{4p} = -2$, we solve $p = -\frac{1}{8}$. Since the vertex is $(h, k) = (1, 2)$, the focus is $(h, k+p) = \left(1, \frac{15}{8}\right)$, the directrix is $y = k - p$ or $y = \frac{17}{8}$, and the axis of symmetry is $x = h$ or $x = 1$.

If $y = 0$, then

$$\begin{aligned} 0 &= -2(x-1)^2 + 2 \\ (x-1)^2 &= 1 \\ x &= 0, 2. \end{aligned}$$

Thus, the x-intercepts are $(0, 0)$ and $(2, 0)$.

108. Use the method of completing the square:

$$\begin{aligned} (x-5)^2 + (y+1)^2 &= 25 + 1 \\ (x-5)^2 + (y+1)^2 &= 26. \end{aligned}$$

The center is $(5, -1)$ and the radius is $\sqrt{26}$.

109.
In general, the inverse of a 2-by-2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

if $ad - bc \neq 0$. In particular,

$$\begin{aligned} \begin{bmatrix} 6 & 7 \\ 5 & 6 \end{bmatrix}^{-1} &= \frac{1}{36-35}\begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix} \end{aligned}$$

110. Let A be the present amount.

$$A\left(1+\frac{0.054}{365}\right)^{365(3)} = 1000$$

$$A = 1000\left(1+\frac{0.054}{365}\right)^{-365(3)}$$

$$A = \$850.45.$$

Thinking Outside the Box

XCIV Since 5! is a multiple of 10 and

$$0!+1!+2!+3!+4! = 34$$

the ones digit of $0!+1!+...+10000!$ is the same as the ones digit of 34. Thus, the ones digit is 4.

Note, $n!$ is a multiple of 100 for $n \geq 10$. Then the tens digit in $0!+1!+...+10000!$ is the same as the tens digit in

$$0!+1!+...+9! = 409,114.$$

Thus, the tens digit 1.

XCV Consider the positive integer

$$n = 6+10a_1+10^2a_2+10^3a_3+...+10^pa_p$$

that satisfies

$$4n = a_1+10a_2+10^2a_3+...+10^p(6) \quad (*)$$

where each $0 \leq a_i \leq 9$ is an integer.

Then

$$\begin{aligned} 4n &= 24+10(4a_1)+...+10^p(4a_p) \\ &= 4+10(4a_1+2)+10^2(4a_2)+...+10^p(4a_p) \end{aligned}$$

Applying $(*)$, we find $a_1 = 4$ and

$$\begin{aligned} 4n &= 4+10(18)+10^2(4a_2)+...+10^p(4a_p) \\ &= 4+10(8)+10^2(4a_2+1)+...+10^p(4a_p) \end{aligned}$$

Applying $(*)$, we find $a_2 = 8$ and

$$\begin{aligned} 4n &= 4+10(8)+10^2(33)+...+10^p(4a_p) \\ &= 4+10(8)+10^2(3)+10^3(4a_3+3)+ \\ &\quad ...+10^p(4a_p) \end{aligned}$$

Applying $(*)$, we obtain $a_3 = 3$ and

$$\begin{aligned} 4n &= 4+10(8)+10^2(3)+10^3(15)+ \\ &\quad ...+10^p(4a_p) \\ 4n &= 4+10(8)+10^2(3)+10^3(5)+ \\ &\quad 10^4(4a_4+1)+...+10^p(4a_p) \end{aligned}$$

Applying $(*)$, we obtain $a_4 = 5$ and

$$\begin{aligned} 4n &= 4+10(8)+10^2(3)+10^3(5)+ \\ &\quad 10^4(21)+... \\ &= 4+10(8)+10^2(3)+10^3(5)+ \\ &\quad 10^4(1)+10^5(4a_5+2) \end{aligned}$$

Applying $(*)$, we obtain $a_5 = 1$ and

$$\begin{aligned} 4n &= 4+10(8)+10^2(3)+10^3(5)+ \\ &\quad 10^4(1)+10^5(6) \end{aligned}$$

Since 6 is now the first digit, the smallest such number n occurs when $a_1 = 4$, $a_2 = 8$, $a_3 = 3$, $a_4 = 5$, $a_5 = 1$, and $p = 5$. Hence, the smallest such integer is $n = 153,846$.

11.1 Pop Quiz

1. $a_1 = (1-1)^2 = 0$, $a_2 = (2-1)^2 = 1$, $a_3 = (3-1)^2 = 4$, and $a_4 = (4-1)^2 = 9$.

The first four terms are 0, 1, 4, and 9.

2. $a_n = (-1)^{n+1}5n$

3. $a_1 = -1$, $a_2 = -2(-1) = 2$, $a_3 = -2(2) = -4$, and $a_4 = -2(-4) = 8$.

4. Since $a_1 = 2$, $a_5 = 18$, and $a_n = a_1+(n-1)d$, we obtain

$$\begin{aligned} 18 &= 2+4d \\ 4 &= d. \end{aligned}$$

Thus, the 9th term is

$$a_5 = 2+(9-1)4 = 34.$$

11.1 Linking Concepts

a) $a_{100} = 0.9990$, $a_{1000} = 0.9900$, $a_{1,000,000} = 0.000045$

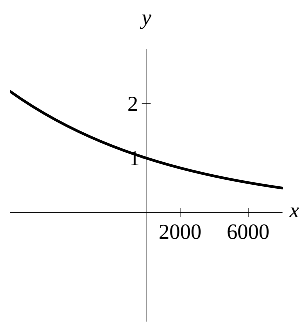
The limit is 0.

b) $a_{100} \approx 1.001$, $a_{1000} \approx 1.0101$, $a_{1,000,000} \approx 22,025$

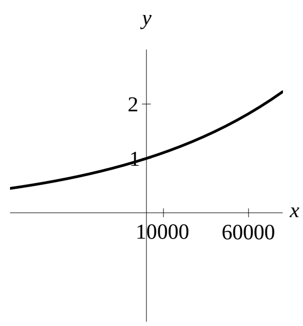
There is no limit since a_n increases without bound as n increases without bound.

c) $a_{1000} \approx 2.7169$, $a_{10,000} \approx 2.7181$, $a_{100,000} \approx 2.7183$. The limit is e. Note, the base is fixed in part b) and the base varies in c).

d) $y = 0$ is the horizontal asymptote of $f(x) = (0.99999)^x$.

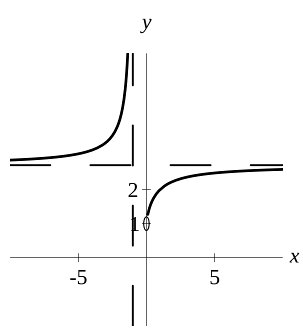


$y = 0$ is the horizontal asymptote of $g(x) = (1.00001)^x$.



$y = e$ is the horizontal asymptote and $x = -1$ is the vertical asymptote of $h(x) = \left(1 + \dfrac{1}{x}\right)^x$.

Note, $h(x)$ is not a real number for $-1 \le x \le 0$.



If the curve approaches a horizontal asymptote $y = k$ as $x \to \infty$, then the limit of the sequence is k.

e) $a_n = k^n$ converges precisely when $-1 < k \le 1$.

For Thought

1. True, $\sum_{i=1}^{3} (-2)^i = (-2)^1 + (-2)^2 + (-2)^3 = -6$

2. True, $\sum_{i=1}^{6} (0 \cdot i + 5) = (0 \cdot 1 + 5) + (0 \cdot 2 + 5) + (0 \cdot 3 + 5) + (0 \cdot 4 + 5) + (0 \cdot 5 + 5) + (0 \cdot 6 + 5) = 30.$

3. True, since $\sum_{i=1}^{k} (5i) = (5 \cdot 1) + ... + (5 \cdot k) = 5(1 + 2 + ... + k) = 5 \sum_{i=1}^{k} (i).$

4. True, $\sum_{i=1}^{k} (i^2 + 1) = (1^2 + 1) + ... + (k^2 + 1) = (1^2 + 2^2 + ... + k^2) + \underbrace{(1 + ... + 1)}_{k \text{ times}} = \sum_{i=1}^{k} i^2 + k.$

5. False, there are ten terms.

6. True, if $j = i - 1$ and $i = 2$ then $j = 1$.

7. True

8. True, $1 + 2 + ... + n = \dfrac{n}{2}(1 + n)$ represents an arithmetic series.

9. False, for if $a_1 = 8$, $a_n = 68$, $d = 2$ then $68 = 8 + (n - 1)2$ and $n = 31$. The sum of the arithmetic series is $S_{31} = \dfrac{31}{2}(8 + 68)$.

10. False, $\sum_{i=1}^{10} i^2$ is not an arithmetic series.

11.2 Exercises

1. series, sequence

2. summation notation

3. mean

4. arithmetic series

5. $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$

6. $0^2 + 1^2 + 2^2 + 3^2 + 4^2 = 30$

7. $2^0 + 2^1 + 2^2 + 2^3 = 15$

8. $0 + 2 + 4 + 6 + 8 = 20$

9. $1 + 1 + 2 + 6 + 24 = 34$

10. $1 + 1 + \frac{1}{2} + \frac{1}{6} = \frac{8}{3}$

11. $\underbrace{3 + ... + 3}_{10 \ times} = 30$

12. $\underbrace{4 + ... + 4}_{6 \ times} = 24$

13. $(2+5) + (4+5) + (6+5) + (8+5) + (10+5) + (12+5) + (14+5) = 91$

14. $(2-1) + (4-1) + (6-1) + (8-1) + (10-1) + (12-1) = 36$

15. $[(-1)^7 + (-1)^8] + ... + [(-1)^{43} + (-1)^{44}] = 0 + ... + 0 = 0$

16. $[(-1)^4 + (-1)^5] + ... + [(-1)^{46} + (-1)^{47}] + (-1)^{48} = 0 + ... + 0 + 1 = 1$

17. $0 + 0 + 0 + 3(2)(1) + 4(3)(2) + 5(4)(3) = 90$

18. $(-1)(-2) + 0(-1) + 1(0) + 2(1) + (3)(2) + (4)(3) = 22$

19. $\sum_{i=1}^{6} i$ **20.** $\sum_{i=1}^{6} 2i$

21. $\sum_{i=1}^{5} (-1)^i(2i-1)$ **22.** $\sum_{i=1}^{6} (-1)^{i+1}3i$

23. $\sum_{i=1}^{5} i^2$ **24.** $\sum_{i=1}^{5} 3^{i-1}$

25. $\sum_{i=1}^{\infty} \left(-\frac{1}{2}\right)^{i-1}$ **26.** $\sum_{i=1}^{\infty} \frac{(-1)^i}{i}$

27. $\sum_{i=1}^{\infty} \ln(x_i)$ **28.** $\sum_{i=1}^{\infty} x^{i+2}$

29. $\sum_{i=1}^{11} ar^{i-1}$ **30.** $\sum_{i=1}^{11} b^{i+1}$

31. Let $i = j - 1$. Since $1 \leq i = j - 1 \leq 3$, we find $2 \leq j \leq 4$. Then we find

$$\sum_{i=1}^{3} i^2 = \sum_{i=2}^{4} (j-1).$$

The equation is true.

32. Let $i = j - 2$. Since $0 \leq i = j - 2 \leq 4$, we find $2 \leq j \leq 6$. Then we find

$$\sum_{i=0}^{4} i! = \sum_{i=2}^{6} (j-2)!.$$

The equation is true.

33. We find $\sum_{x=1}^{5} (2x-1) = 25$ and $\sum_{y=3}^{7} (2y-1) = 45$. The equation is false.

34. We find $\sum_{x=1}^{8} (2x) = 72$ and $\sum_{y=3}^{9} (2y-4) = 56$. The equation is false.

35. Let $x + 5 = j + 6$. Since $1 \leq x = j + 1 \leq 5$, we find $0 \leq j \leq 4$. Then we obtain

$$\sum_{i=1}^{5} (x+5) = \sum_{j=0}^{4} (j+6).$$

The equation is true.

36. Let $i = j - 1$. Since $0 \leq i = j - 1 \leq 3$, we find $1 \leq j \leq 4$. Then we obtain

$$\sum_{i=0}^{3} x^i = \sum_{j=1}^{4} x^{j-1}.$$

The equation is true.

37. Let $j = i - 1$. New series is $\sum_{j=0}^{31} (-1)^{j+1}$

38. Let $j = i - 1$. New series is $\sum_{j=0}^{9} 2^{j+1}$

39. Let $j = i - 3$. New series is $\sum_{j=1}^{10} (2j+7)$

40. Let $j = i - 6$. New series is $\sum_{j=1}^{6} (3j+14)$

41. Let $j = x-2$. New series is $\sum_{j=0}^{8} \frac{10!}{(j+2)!(8-j)!}$

42. Let $j = i-2$. New series is $\sum_{j=0}^{7} \frac{x^{j+2}}{(j+2)!}$

43. Let $j = n+3$. New series is $\sum_{j=5}^{\infty} \frac{5^{j-3} \cdot e^{-5}}{(j-3)!}$

44. Let $j = i+3$. New series is $\sum_{j=3}^{\infty} 3^{2j-7}$

45. $0.5 + 0.5r + 0.5r^2 + 0.5r^3 + 0.5r^4 + 0.5r^5$

46. $1^n + 2^n + 3^n + 4^n + 5^n + 6^n$

47. $a^4 + a^3b + a^2b^2 + ab^3 + b^4$

48. $x^3 - x^2y + xy^2 - y^3$

49. $a^2 + 2ab + b^2$

50. $a^3 + 3a^2b + 3ab^2 + b^3$

51. $\frac{6+23+45}{3} = \frac{74}{3}$

52. $\frac{33+42+78+19}{4} = 43$

53. $\frac{-6+0+3+4+3+92}{6} = 16$

54. $\frac{12+20+12+30+28+28+10}{7} = 20$

55. $\frac{\sqrt{2}+\pi+33.6-19.4+52}{5} \approx 14.151$

56. $\frac{\sqrt{5}-3\sqrt{3}+\pi/2+e+98.6}{5} \approx 19.986$

57. Since $a_1 = -3$, $a_{12} = 6(12) - 9 = 63$, and $n = 12$, the sum is $S_{12} = \frac{12}{2}(-3+63) = 360$.

58. Since $a_1 = 4.5$, $a_{11} = 0.5(11) + 4 = 9.5$, and $n = 11$, the sum is $S_{11} = \frac{11}{2}(4.5+9.5) = 77$.

59. Since $a_3 = 0.7$, $a_{15} = -0.5$, and there are $n = 13$ terms in the series, the sum is $S_{13} = \frac{13}{2}(0.7-0.5) = 1.3$.

60. Since $a_4 = 0.8$, $a_{20} = -4$, and there are $n = 17$ terms in the series, so the sum is $S_{17} = \frac{17}{2}(0.8-4) = -27.2$.

61. Note $a_1 = 1$, $a_n = 47$, $n = 47$. Then the sum is $S_{47} = \frac{47}{2}(1+47) = 1128$.

62. Since $88 = 2 + (n-1)2$, $n = 44$. Then the sum is $S_{44} = \frac{44}{2}(2+88) = 1980$.

63. Since $95 = 5 + (n-1)5$, we get $n = 19$. Then the sum is $S_{19} = \frac{19}{2}(5+95) = 950$.

64. Since $450 = 10 + (n-1)10$, we get $n = 45$. The sum is $S_{45} = \frac{45}{2}(10+450) = 10,350$.

65. Since $-238 = 10 + (n-1)(-1)$, we get $n = 249$. The sum is $S_{249} = \frac{249}{2}(10-238) = -28,386$.

66. Since $888 = 1000 + (n-1)(-2)$, we get $n = 57$. The sum is $S_{57} = \frac{57}{2}(1000+888) = 53,808$.

67. Since $-16 = 8 + (n-1)(-3)$, we get $n = 9$. Thus, the sum is $S_9 = \frac{9}{2}(8+(-16)) = -36$.

68. Since $-27 = 5 + (n-1)(-4)$, we have $n = 9$. The sum is $S_9 = \frac{9}{2}(5+(-27)) = -99$.

69. Since $55 = 3 + (n-1)4$, we obtain $n = 14$. The sum is $S_{14} = \frac{14}{2}(3+55) = 406$.

70. Since $50 = -6 + (n-1)7$, we find $n = 9$. Thus, the sum is $S_9 = \frac{9}{2}(-6+50) = 198$.

71. Since $5 = \frac{1}{2} + (n-1)\frac{1}{4}$, we get $n = 19$. Thus, the sum is $S_{19} = \frac{19}{2}(1/2+5) = 52.25$.

72. Since $\frac{22}{3} = 1 + (n-1)\frac{1}{3}$, we get $n = 20$. The sum is $S_{20} = \frac{20}{2}(1+22/3) = \frac{250}{3}$.

73. $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$

74. $2\,(1+2+3+\cdots+n)=2\left(\dfrac{n(n+1)}{2}\right)=$ $n(n+1)$

75. Note, the mean of an arithmetic sequence with n terms is $\overline{a}=\dfrac{a_1+a_n}{2}$. Thus, $\overline{a}=\dfrac{2+16}{2}=9$.

76. Note, the mean of an arithmetic sequence with n terms is $\overline{a}=\dfrac{a_1+a_n}{2}$. Thus, $\overline{a}=\dfrac{3+27}{2}=$ 15.

77. Note, the mean of an arithmetic sequence with n terms is $\overline{a}=\dfrac{a_1+a_n}{2}$. Then $\overline{a}=\dfrac{2+142}{2}=$ 72.

78. Note, the mean of an arithmetic sequence with n terms is $\overline{a}=\dfrac{a_1+a_n}{2}$. Then $\overline{a}=\dfrac{3+150}{2}=$ 76.5.

79. Since $a_1=30,000$, $d=1,000$, and $a_{30}=30,000+29(1,000)=59,000$, we find his total salary for 30 years of work is

$S_{30}=\dfrac{30}{2}(30,000+59,000)=\$1,335,000.$

The mean annual salary for 30 years is

$\dfrac{S_{30}}{30}=\$44,500.$

80. If $a_n=2n+1$, then $a_1=3$. Note, $a_{31}=63$ pages will be read on October 31. There are

$S_{31}=\dfrac{31}{2}(3+63)=1023$ pages in the novel.

The mean number of pages read is $\dfrac{S_{31}}{31}=33$.

81. In the nth level, the number of cans is $a_n=12(10-n)$. In the mountain of cans there are $\sum\limits_{i=1}^{9}(120-12i)=\sum\limits_{j=1}^{9}(120-12(10-j))=$

$\sum\limits_{j=1}^{9}12j=\dfrac{9}{2}(12+108)=540$ cans.

82. In the nth level, the number of cans is $a_n=(13-n)(10-n)$ where $n=1,2,...,9$. The total number of cans is given by

$$\sum_{n=1}^{9}(13-n)(10-n)=$$

$$\sum_{i=1}^{9}(13-[10-i])(10-[10-i])=$$

$$\sum_{i=1}^{9}(3+i)i=$$

$=4+10+18+28+40+54+70+88+108$ $=420$ cans.

83. At 5% compounded annually, the future value of \$1000 after i years is $\$1000(1.05)^i$. Then the amount in the account on January 1, 2000 is $\sum\limits_{i=1}^{10}1000(1.05)^i$.

84. At 4% compounded quarterly, the future value of \$100 after i years is $\$100(1.01)^{4i}$. Note, the last deposit is three years before January 1, 2000. Then the amount in the account on January 1, 2000 is $\sum\limits_{i=3}^{10}100(1.01)^{4i}$.

85. With a calculator, one finds $200+200(.63)$ $+200(.63)^2+200(.63)^3\approx455$ mg.

86. In one week, 21 tablets of Dilantin would have been taken. The amount of Dilantin in the body after this time will be $\sum\limits_{n=0}^{20}200(0.63)^n\approx$ 540.5 mg.

87. Since $a_9=101$, $a_{60}=356$, and there are $n=52$ terms from a_9 to a_{60}, we get that the mean of the 9th through the 60th terms

is $\dfrac{1}{52}\sum\limits_{i=9}^{60}a_i=\dfrac{1}{52}\dfrac{52}{2}(101+356)=228.5$.

88. Since $a_{15}=-53$, $a_{55}=-213$, and there are $n=41$ terms from a_{15} to a_{55}, then the mean of the 15th through the 55th terms

is $\dfrac{1}{41}\sum\limits_{i=15}^{55}a_i=\dfrac{1}{41}\dfrac{41}{2}(-53-213)=-133$.

89. One finds $a_2 = a_1^2 - 3 = (-2)^2 - 3 = 1$ and $a_3 = a_2^2 - 3 = 1^2 - 3 = -2$. By a repeating pattern, one derives $a_7 = a_9 = -2$ and $a_8 = a_{10} = 1$. The mean from the 7th term through the 10th terms is

$$\frac{-2+1-2+1}{4} = -\frac{1}{2}.$$

90. One finds $a_5 = -\frac{1}{32}$, $a_6 = \frac{1}{64}$, $a_7 = -\frac{1}{128}$, and $a_8 = \frac{1}{256}$. The mean from the 5th through the 8th terms is given by

$$\begin{aligned} &= \frac{-1/32 + 1/64 - 1/128 + 1/256}{4} \\ &= \frac{1}{4}\left(\frac{-8+4-2+1}{256}\right) \\ &= -\frac{5}{1024}. \end{aligned}$$

93. $a_n = (-2)^{n-1}$

94. $a_n = 3 + 5(n-1) = 5n - 2$

95. **a)** $f(0.01) = \log(0.01) = -2$

b) The exponential form of $\log a = 3$ is $a = 10^3$. Then $a = 1000$.

96. Rewrite each side as follows:

$$\begin{aligned} 5 - 3x + 27 + 4x &= 3x - 24 - 2x - 2 \\ x + 32 &= x - 26 \\ 32 &= -26 \end{aligned}$$

No solution, i.e., the solution set is $\emptyset$.

97. Take the ln of both sides.

$$\begin{aligned} \ln\left(8^{x^2-1}\right) &= \ln 30 \\ x^2 - 1 &= \frac{\ln 30}{\ln 8} \\ x &= \pm\sqrt{1 + \frac{\ln 30}{\ln 8}} \end{aligned}$$

The solution set is approximately $\{\pm 1.623\}$.

98. The remainder is $f(2) = 71$.

Thinking Outside the Box

XCVI $\displaystyle\sum_{i=10^7}^{10^8-1} i + \sum_{i=10^9}^{10^{10}-1} i =$

$$\left(\sum_{i=1}^{10^8-1} i - \sum_{i=1}^{10^7-1} i\right) + \left(\sum_{i=1}^{10^{10}-1} i - \sum_{i=1}^{10^9-1} i\right) =$$

$$\left(\frac{(10^8-1)10^8}{2} - \frac{(10^7-1)10^7}{2}\right) +$$

$$\left(\frac{(10^{10}-1)10^{10}}{2} - \frac{(10^9-1)10^9}{2}\right) =$$

49,504,949,995,455,000,000

XCVII The sum of $1^2 + 2^2 + 3^2 + \cdots + 100^2$ may be separated using even and odd terms:

$$\begin{aligned} \sum_{n=1}^{50}(2n-1)^2 + \sum_{n=1}^{50}(2n)^2 &= \sum_{n=1}^{100} n^2 \\ \sum_{n=1}^{50}(2n-1)^2 &= \sum_{n=1}^{100} n^2 - \sum_{n=1}^{50}(2n)^2 \\ \sum_{n=1}^{50}(2n-1)^2 &= \sum_{n=1}^{100} n^2 - 4\sum_{n=1}^{50} n^2 \\ \sum_{n=1}^{50}(2n-1)^2 &= 166,650 \end{aligned}$$

where the value of the right side is calculated using $\sum_{n=1}^{k} n^2 = \frac{k(k+1)(2k+1)}{6}$. Then

$$\begin{aligned} 100^2 - 99^2 + 98^2 + \cdots + 2^2 - 1 &= \\ \sum_{n=1}^{100} n^2 - 2\sum_{n=1}^{50}(2n-1)^2 &= \\ 338,350 - 166,650 &= \\ 5050 & \end{aligned}$$

11.2 Pop Quiz

1. $1! + 2! + 3! + 4! + 5! = 1 + 2 + 6 + 24 + 120 = 153$

2. $\sum_{i=1}^{\infty} \frac{1}{2^{i-1}}$

3. $\sum_{i=5}^{24}(3i-5) = \sum_{j=1}^{20}[3(j+4)-5] = \sum_{j=1}^{20}[3j+7] = 3\sum_{j=1}^{20} j + \sum_{j=1}^{20} 7 = 3\frac{20(21)}{2} + 7(20) = 770$

11.2 Linking Concepts

a) The median is \$95.9 billion.

b) The geometric mean is

$$\sqrt[5]{189.6(100.5)(95.9)(78)(61.7)} \approx \$97.5 \text{ billion.}$$

c) The harmonic mean is

$$= \frac{5}{189.6^{-1} + 100.5^{-1} + 95.9^{-1} + 78^{-1} + 61.7^{-1}}$$
$$\approx \$91.4 \text{ billion.}$$

d) The quadratic mean is

$$= \sqrt{\frac{189.6^2 + 100.5^2 + 95.9^2 + 78^2 + 61.7^2}{5}}$$
$$\approx \$114.1 \text{ billion.}$$

For Thought

1. False, the ratios $\frac{6}{2}$ and $\frac{24}{6}$ are not equal

2. True, $a_n = 3 \cdot 2^3 \cdot 2^{-n} = 24\left(\frac{1}{2}\right)^n$.

3. True, $a_1 = 5(0.3)^1 = 1.5$.

4. False, since $a_n = \left(\frac{1}{5}\right)^n$, the common ratio is $\frac{1}{5}$.

5. True

6. False, note that the ratio 2 does not satisfy $|r| < 1$ and we cannot use the formula $S = \frac{a}{1-r}$.

7. False, $\sum_{i=1}^{9} 3(0.6)^i = \frac{1.8(1-0.6^9)}{1-0.6}$.

8. True, $\sum_{i=0}^{4} 2(10)^i = \frac{2(1-10^5)}{1-10} = 22,222$.

9. True, $\sum_{i=1}^{\infty} 3(0.1)^i = \frac{0.3}{1-0.1} = \frac{1}{3}$.

10. True, $\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \frac{1/2}{1-1/2} = 1$.

11.3 Exercises

1. geometric

2. geometric series

3. $a_1 = 3 \cdot 2^0 = 3$, $a_2 = 3 \cdot 2^1 = 6$, $a_3 = 3 \cdot 2^2 = 12$, $a_4 = 3 \cdot 2^3 = 24$

First four terms are $3, 6, 12, 24$, and common ratio is 2.

4. $a_1 = 2 \cdot 3^0 = 2$, $a_2 = 2 \cdot 3^1 = 6$, $a_3 = 2 \cdot 3^2 = 18$, $a_4 = 2 \cdot 3^3 = 54$

First four terms are $2, 6, 18, 54$, and common ratio is 3.

5. $a_1 = 800 \cdot \left(\frac{1}{2}\right)^1 = 400$, $a_2 = 800 \cdot \left(\frac{1}{2}\right)^2 = 200$, $a_3 = 800 \cdot \left(\frac{1}{2}\right)^3 = 100$, $a_4 = 800 \cdot \left(\frac{1}{2}\right)^4 = 50$

First four terms are $400, 200, 100, 50$, and common ratio is $\frac{1}{2}$.

6. $a_1 = 27 \cdot \left(\frac{1}{3}\right)^1 = 9$, $a_2 = 27 \cdot \left(\frac{1}{3}\right)^2 = 3$, $a_3 = 27 \cdot \left(\frac{1}{3}\right)^3 = 1$, $a_4 = 27 \cdot \left(\frac{1}{3}\right)^4 = \frac{1}{3}$

First four terms are $9, 3, 1, \frac{1}{3}$, and common ratio is $\frac{1}{3}$.

7. $a_1 = \left(-\frac{2}{3}\right)^0 = 1$, $a_2 = \left(-\frac{2}{3}\right)^1 = -\frac{2}{3}$,
$a_3 = \left(-\frac{2}{3}\right)^2 = \frac{4}{9}$, $a_4 = \left(-\frac{2}{3}\right)^3 = -\frac{8}{27}$
First four terms are $1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}$, and common ratio is $-\frac{2}{3}$.

8. $a_1 = \left(-\frac{3}{2}\right)^{-1} = -\frac{2}{3}$, $a_2 = \left(-\frac{3}{2}\right)^0 = 1$,
$a_3 = \left(-\frac{3}{2}\right)^1 = -\frac{3}{2}$, $a_4 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}$
First four terms are $-\frac{2}{3}, 1, -\frac{3}{2}, \frac{9}{4}$, and common ratio is $-\frac{3}{2}$.

9. $\frac{1}{2}$ **10.** 5 **11.** 10 **12.** $\frac{1}{10}$

13. -2 **14.** $-\frac{1}{3}$ **15.** -1 **16.** -1

17. Since $r = \frac{1/3}{1/6} = 2$, we get $a_n = \frac{1}{6}2^{n-1}$.

18. Since $r = \frac{0.5}{1/6} = 3$, we obtain $a_n = \frac{1}{6}3^{n-1}$.

19. Since $r = \frac{0.09}{0.9} = 0.1$, we find $a_n = (0.9)(0.1)^{n-1}$.

20. Since $r = \frac{9}{3} = 3$, we get $a_n = 3^n$.

21. Since $r = \frac{-12}{4} = -3$, we have $a_n = 4 \cdot (-3)^{n-1}$.

22. Since $r = \frac{-1}{5}$, we obtain $a_n = 5\left(-\frac{1}{5}\right)^{n-1}$.

23. Arithmetic, since $4 - 2 = 6 - 4 = ... = 2$.

24. Geometric, since $\frac{4}{2} = \frac{8}{4} = ... = 2$.

25. Neither, since there is no common difference or ratio.

26. Neither, since there is no common difference or ratio.

27. Geometric, since $\frac{-4}{2} = \frac{8}{-4} = ... = -2$.

28. Arithmetic, since $2 - 0 = 4 - 2 = ... = 2$.

29. Arithmetic, since $\frac{1}{3} - \frac{1}{6} = \frac{1}{2} - \frac{1}{3} = ... = \frac{1}{6}$.

30. Neither, since there is no common difference or ratio.

31. Geometric, since $\frac{1/3}{1/6} = \frac{2/3}{1/3} = ... = 2$.

32. Arithmetic, since $2-3 = 1-2 = 0-1 = ...-1$

33. Neither, since there is no common difference or ratio.

34. Geometric, since $\frac{1}{5} = \frac{1/5}{1} = \frac{1/25}{1/5} = ...$

35. $a_1 = 2 \cdot 1 = 2$, $a_2 = 2 \cdot 2 = 4$, $a_3 = 2 \cdot 3 = 6$, $a_4 = 2 \cdot 4 = 8$. First four terms are $2, 4, 6, 8$. Arithmetic sequence.

36. $a_1 = 2^1 = 2$, $a_2 = 2^2 = 4$, $a_3 = 2^3 = 8$, $a_4 = 2^4 = 16$. First four terms are $2, 4, 8, 16$. Geometric sequence.

37. One finds $a_1 = 1^2 = 1$, $a_2 = 2^2 = 4$, $a_3 = 3^2 = 9$, $a_4 = 4^2 = 16$. First four terms are $1, 4, 9, 16$. Neither geometric nor arithmetic.

38. One finds $a_1 = 1! = 1$, $a_2 = 2! = 2$, $a_3 = 3! = 6$, $a_4 = 4! = 24$. First four terms are $1, 2, 6, 24$. Neither geometric nor arithmetic.

39. $a_1 = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$, $a_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$,
$a_3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$, $a_4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$. First four terms are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$. Geometric sequence.

40. $a_1 = 1 + 2 = 3$, $a_2 = 2 + 2 = 4$, $a_3 = 3 + 2 = 5$, $a_4 = 4 + 2 = 6$. First four terms are $3, 4, 5, 6$. Arithmetic sequence.

41. $b_1 = 2^3 = 8$, $b_2 = 2^5 = 32$, $b_3 = 2^7 = 128$, $b_4 = 2^9 = 512$. First four terms are $8, 32, 128, 512$. Geometric sequence.

42. $b_1 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, $b_2 = \frac{1}{3}$,
$b_3 = \frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{9}$, $b_4 = \frac{1}{3^2} = \frac{1}{9}$. First four terms are $\frac{\sqrt{3}}{3}, \frac{1}{3}, \frac{\sqrt{3}}{9}, \frac{1}{9}$. Geometric sequence.

43. $c_2 = -3c_1 = -3 \cdot 3 = -9$,
$c_3 = -3c_2 = -3 \cdot (-9) = 27$,
$c_4 = -3c_3 = -3 \cdot 27 = -81$. First four terms are $3, -9, 27, -81$. Geometric sequence.

44. $h_2 = \sqrt{3}h_1 = \sqrt{3}\sqrt{2} = \sqrt{6}$,
$h_3 = \sqrt{3}h_2 = \sqrt{3}\sqrt{6} = \sqrt{18} = 3\sqrt{2}$,
$h_4 = \sqrt{3}h_3 = \sqrt{3}(3\sqrt{2}) = 3\sqrt{6}$.
First four terms are $\sqrt{2}, \sqrt{6}, 3\sqrt{2}, 3\sqrt{6}$.
Geometric sequence.

45. Since $a_n = a_1 r^{n-1}$, we obtain

$$\begin{aligned} \frac{3}{1024} &= 3\left(\frac{1}{2}\right)^{n-1} \\ \frac{1}{1024} &= \left(\frac{1}{2}\right)^{n-1} \\ \left(\frac{1}{2}\right)^{10} &= \left(\frac{1}{2}\right)^{n-1} \end{aligned}$$

So $n - 1 = 10$ and the number of terms is $n = 11$.

46. Since $a_n = a_1 r^{n-1}$, we get

$$\begin{aligned} -512 &= \frac{1}{64}(-2)^{n-1} \\ (-2)^9 &= (-2)^{-6}(-2)^{n-1} \\ (-2)^{15} &= (-2)^{n-1} \end{aligned}$$

Then $n - 1 = 15$ and the number of terms is $n = 16$.

47. Since $a_n = a_1 r^{n-1}$, we have $\frac{1}{81} = a_1\left(\frac{1}{3}\right)^5$.
Solving, one finds the first term is $a_1 = 3$.

48. Since $a_n = a_1 r^{n-1}$, we find $\frac{1}{16} = a_1\left(\frac{1}{2}\right)^6$.
Solving, one finds the first term is $a_1 = 4$.

49. Since $a_n = a_1 r^{n-1}$, we obtain $6 = \frac{2}{3}r^2$.
So $9 = r^2$ and the common ratio is $r = \pm 3$.

50. Since $a_n = a_1 r^{n-1}$, we get $-27 = (-1)r^3$.
Then $27 = r^3$ and the common ratio is $r = 3$.

51. Since $a_6 = a_3 r^3$, we obtain $96 = -12r^3$ and $r = -2$. Since $a_3 = a_1 r^2$, we get $-12 = a_1(-2)^2$. Then $a_1 = -3$ and $a_n = (-3)(-2)^{n-1}$.

52. Since $a_5 = a_2 r^3$, we have $0.04 = -40r^3$ and $r = -0.1$. Since $a_2 = a_1 r$, $-40 = a_1(-0.1)$. Thus, $a_1 = 400$ and $a_n = 400(-0.1)^{n-1}$.

53. Since $a = 1$, $r = 2$, and $n = 5$, we find that the sum is $S = \frac{1(1-2^5)}{1-2} = 31$.

54. Since $a = 100$, $r = 0.1$, and $n = 5$, we obtain that the sum is $S = \frac{100(1-0.1^5)}{1-0.1} = 111.11$.

55. Since $a = 2$, $r = 2$, and $n = 6$, we obtain that the sum is $S = \frac{2(1-2^6)}{1-2} = 126$.

56. Since $a = 1$, $r = -1$, and $n = 7$, we find that the sum is $S = \frac{1(1-(-1)^7)}{1-(-1)} = 1$.

57. Since $a = 9$, $r = 1/3$, and $n = 6$, the sum is $S = \frac{9(1-(1/3)^6)}{1-1/3} = 364/27$.

58. Since $a = 8$, $r = 1/2$, and $n = 7$, the sum is $S = \frac{8(1-(1/2)^7)}{1-1/2} = 127/8$.

59. Since $a = 6$, $r = \frac{1}{3}$, and $n = 5$, we find that

the sum is $S = \dfrac{6\left(1-\left(\frac{1}{3}\right)^5\right)}{1-\frac{1}{3}} = \dfrac{242}{27}$.

60. Since $a = 2$, $r = 5$, and $n = 5$, we obtain that the sum is $S = \frac{2(1-5^5)}{1-5} = 1562$.

61. $\sum_{i=1}^{8} 1.5\,(-2)^{i-1} = \frac{1.5(1-(-2)^8)}{1-(-2)} = -127.5$

62. $\sum_{i=1}^{7}\left(-\frac{1}{2}\right)^{i-1}=\frac{1-(-1/2)^7}{1-(-1/2)}=$
$\frac{1+1/2^7}{3/2}\cdot\frac{2^7}{2^7}=\frac{2^7+1}{3\cdot 2^6}=\frac{129}{192}=\frac{43}{64}.$

63. $\frac{2(1-1.05^{12})}{1-1.05}\approx 31.8343$

64. $\frac{300(1-1.08^{31})}{1-1.08}\approx 37,003.7604$

65. $\frac{200(1-1.01^{8})}{1-1.01}\approx 1657.1341$

66. $\frac{421(1-1.09^{20})}{1-1.09}\approx 21,538.4104$

67. $\sum_{n=1}^{5} 3\left(-\frac{1}{3}\right)^{n-1}$

68. $\sum_{n=1}^{6} 2\left(\frac{1}{2}\right)^{n-1}$

69. $\sum_{n=1}^{\infty} 0.6(0.1)^{n-1}$

70. $\sum_{n=1}^{\infty} 4\left(-\frac{1}{4}\right)^{n-1}$

71. $\sum_{n=1}^{\infty}(-4.5)\left(-\frac{1}{3}\right)^{n-1}$

72. $\sum_{n=1}^{38} ab^{n-1}$

73. $S=\frac{a_1}{1-r}=\frac{3}{1-(-1/3)}=\frac{3}{4/3}=\frac{9}{4}$

74. $S=\frac{a_1}{1-r}=\frac{1}{1-1/2}=\frac{1}{1/2}=2$

75. $S=\frac{a_1}{1-r}=\frac{0.9}{1-0.1}=\frac{0.9}{0.9}=1$

76. $S=\frac{a_1}{1-r}=\frac{-1}{1-(-1/4)}=\frac{-1}{5/4}=-\frac{4}{5}$

77. $S=\frac{a_1}{1-r}=\frac{-9.9}{1-(-1/3)}=\frac{-9.9}{4/3}=$
$-\frac{99}{10}\cdot\frac{3}{4}=-\frac{297}{40}$

78. No sum, since $|r|=|-2|>1$.

79. $S=\frac{a_1}{1-r}=\frac{0.34}{1-0.01}=\frac{0.34}{0.99}=\frac{34}{99}$

80. $S=\frac{a_1}{1-r}=\frac{300}{1-0.99}=\frac{300}{0.01}=30,000$

81. No sum, since $|r|=|-1.06|>1$.

82. $S=\frac{a_1}{1-r}=\frac{1}{1-0.98}=\frac{1}{0.02}=50$

83. $S=\frac{a_1}{1-r}=\frac{0.6}{1-0.1}=\frac{0.6}{0.9}=\frac{2}{3}$

84. $S=\frac{a_1}{1-r}=\frac{1}{1-0.1}=\frac{1}{0.9}=\frac{10}{9}$

85. $S=\frac{a_1}{1-r}=\frac{34}{1-(-0.7)}=\frac{34}{1.7}=20$

86. $S=\frac{a_1}{1-r}=\frac{0.123}{1-0.001}=\frac{0.123}{0.999}=\frac{41}{333}$

87. $\sum_{i=2}^{\infty}4(0.1)^i=\frac{0.04}{1-0.1}=\frac{0.04}{0.9}=\frac{4}{90}=\frac{2}{45}$

88. $(0.1)(0.1212...)=0.1\sum_{i=0}^{\infty}0.12(0.01)^i=$
$(0.1)\frac{0.12}{1-0.01}=(0.1)\frac{0.12}{0.99}=\frac{12}{990}=\frac{2}{165}$

89. $8.2+0.05454...=8.2+0.1(0.5454...)=$
$8.2+0.1\sum_{i=0}^{\infty}0.54(0.01)^i=$
$8.2+(0.1)\frac{0.54}{1-0.01}=8.2+\frac{0.054}{0.99}=$
$8.2+\frac{54}{990}=\frac{8118+54}{990}=\frac{8172}{990}=\frac{454}{55}$

90. $3.65+0.00176176...=$
$3.65+0.01(0.176176...)=$
$3.65+0.01\sum_{i=0}^{\infty}0.176(0.001)^i=$
$3.65+(0.01)\frac{0.176}{1-0.001}=3.65+(0.01)\frac{0.176}{0.999}=$
$3.65+(0.01)\frac{176}{999}=3.65+\frac{176}{99900}=$
$\frac{3.65(99900)+176}{99900}=\frac{364,811}{99,900}$

91. Using a formula for S_n, the sum is

$$\sum_{n=1}^{25}100(.69)^{n-1} = \frac{100\left(1-(.69)^{25}\right)}{1-.69} \approx 322.55 \text{ mg.}$$

92. The long-range build-up is the sum

$$\sum_{n=1}^{\infty} 100(.69)^{n-1} = \frac{100}{1-.69} \approx 322.58 \text{ mg.}$$

The difference between the long-range build-up and the answer in exercise #89 is 0.03 mg.

93. The total number of subscriptions in June is $\sum_{i=1}^{30} 2^{i-1} = \frac{1-2^{30}}{1-2} = 1,073,741,823.$

94. The total number of persons is $\sum_{i=1}^{43} 2^{i-1} = \frac{1-2^{43}}{1-2} \approx 8.8 \times 10^{12}.$

95. At the end of the nth quarter the amount is $a_n = 4000(1.02)^n$. At the end of the 37th quarter, it is $a_{37} = 4000(1.02)^{37} = \$8,322.74$.

96. At the end of the nth month, the amount is $a_n = 8000(1.005)^n$. At the end of the 56th month, it is $a_{56} = 8000(1.005)^{56} = \$10,577.66$.

97. At the end of the 12th month, the amount in the account is $\sum_{i=1}^{12} 200(1.01)^i =$

$$\frac{200(1.01)(1-1.01^{12})}{1-1.01} = \$2561.87.$$

98. At the end of the 40th year, the amount in the account is $\sum_{i=1}^{40} 9000(1.08)^i =$

$$\frac{9000(1.08)(1-1.08^{40})}{1-1.08} = \$2,518,029.36.$$

99. The value of the annuity immediately after the last payment is $\sum_{i=0}^{359} 100\left(1+\frac{0.09}{12}\right)^i =$

$$\frac{100(1-1.0075^{360})}{1-1.0075} = \$183,074.35.$$

100. The value of this annuity immediately after the last deposit is $\sum_{i=0}^{35} 800(1.015)^i =$

$$\frac{800(1-1.015^{36})}{1-1.015} = \$37,820.78.$$

101. Assume the ball has a small radius. Approximately, the distance it travels before it comes to a rest is

$$9 + \frac{2}{3}\cdot 9 + \frac{2}{3}\cdot 9 + \left(\frac{2}{3}\right)^2\cdot 9 + \left(\frac{2}{3}\right)^2\cdot 9 + ... =$$

$$9 + 18\left[\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + ...\right] =$$

$$9 + 18\frac{2/3}{1-2/3} = 45 \text{ feet.}$$

102. After 15 weeks of selling, the amount of sales is

$$10^6\left[\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + ... + \left(\frac{1}{2}\right)^{15}\right] =$$

$$10^6\left[\frac{1/2\,(1-(1/2)^{15})}{1-1/2}\right] =$$

$$10^6\left[1-(1/2)^{15}\right] = \$999,969.48.$$

They will be \$30.52 short of the goal.

103. The first amount spent in Hammond is \$2 million, then 75% of \$2 million is the next amount spent, and so on. So the total economic impact is

$$\sum_{i=0}^{\infty} 2(10^6)(0.75)^i = \frac{2,000,000}{1-0.75} =$$

\$8,000,000.

104. The first amount spent in Louisiana is \$300 million, then 80% of \$300 million is the next amount spent, and so on. The total economic impact is

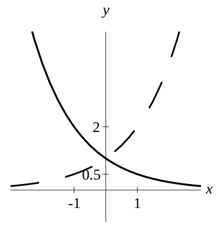
$$\sum_{i=0}^{\infty} 300(0.8)^i = \frac{300}{1-0.8} =$$

\$1,500 million = \$1.5 billion.

105. Yes, since if $a = d = 0$ and r is any positive number, then the arithmetic sequence is also a geometric sequence. This sequence is $0, 0, ...$

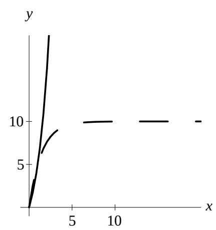
107. If $0 < r < 1$, then $r^x \to 0$ as $x \to \infty$.
If $r > 1$, then $|r^x| \to \infty$ as $x \to \infty$.

The graphs of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ are shown below.



108. If $0 < r < 1$, then
$$\frac{5(1-r^x)}{1-r} \to \frac{5}{1-r} \text{ as } x \to \infty.$$
If $r > 1$, then $\left|\frac{5(1-r^x)}{1-r}\right| \to \infty$ as $x \to \infty$.

Below are the graphs of $y = \frac{5(1-r^x)}{1-r}$ for $r = 2$ and $r = 1/2$.



109. A formula is $a_n = 3n$. The sum of the first $n = 100$ terms in an arithmetic series is
$$\frac{n}{2}(a_1 + a_{100}) = \frac{100}{2}(3 + 300) = 15,150.$$

110. $0! + 1! + 2! + 3! + 4! = 34$

111. Let $a_n = a_1 + d(n-1)$. Then $a_4 = a_1 + 3d = 5$ and $a_8 = a_1 + 7d = 11$. The solutions of the system of equations are $a_1 = \frac{1}{2}$ and $d = \frac{3}{2}$.

Then $a_{20} = \frac{1}{2} + \frac{3}{2}(20-1) = 29$.

112. $a_2 = 2(-6)+3 = -9$, $a_3 = 2(-9)+3 = -15$, $a_4 = 2(-15)+3 = -27$, and the 5th term is $a_5 = 2(-27)+3 = -51$.

113. The domain is $(-\infty, \infty)$ since the domain of the absolute value is the set of all real numbers.

Since the range of the negative absolute value function is $(-\infty, 0]$, the range of $f(x) = -30|x-50| + 200$ is $(-\infty, 200]$.

114. Since $y = -x^2 + x$ and $y = -x - 6$, we obtain
$$-x^2 + x = -x - 6.$$

Solving for x, we find $x = 1 \pm \sqrt{7}$, say by the method of completing the square. Substitute $x = 1 + \sqrt{7}$:
$$y = -x - 6 = -1 - \sqrt{7} - 6 = -7 - \sqrt{7}$$
Similarly, if $x = 1 - \sqrt{7}$, then $y = -7 + \sqrt{7}$. The solution set is
$$\{(1+\sqrt{7}, -7-\sqrt{7}), (1-\sqrt{7}, -7+\sqrt{7})\}.$$

Thinking Outside the Box

XCVIII First, the ant crawls a distance of $12r$ ft to the 13-ft hypotenuse where $r = 5/13$.

Then the ant crawls a distance of $12r^2$ ft to the hypotenuse of the smaller right triangle.

Continuing in this manner, the total distance that the ant crawls is
$$12r + 12r^2 + 12r^3 + ... = \frac{12r}{1-r} = \frac{12(5/13)}{1-5/13} = 7.5\text{ft}.$$

XCIX Since $\log a + \log b = \log(ab)$, we obtain
$$\begin{aligned}\sum_{i=1}^{9} \log\left(\frac{1+i}{i}\right) &= \\ \log\left(\frac{2}{1}\frac{3}{2}\frac{4}{3}\cdots\frac{10}{9}\right) &= \log(10) = 1\end{aligned}$$
since many terms cancel.

11.3 Pop Quiz

1. The common ratio is $r = 1/3$.

2. $a_n = 2(-2)^{n-1}$

3. Since $3 = a_6 = a(-1/2)^5$, we find $a = 3(-2)^5$. The first term is $a = -96$.

4. Since $a = -10$, $r = -2$, and $n = 8$, the sum is $S = \frac{-10(1-(-2)^8)}{1-(-2)} = 850$.

5. Since $a = 0.01$ and $r = 0.1$, the infinite sum is $S = \frac{a}{1-r} = \frac{0.01}{1-0.1} = \frac{1}{90}$.

11.3 Linking Concepts

a) $R(1+i)^{n-1}$

b) $\sum_{k=0}^{n-1} R(1+i)^k = R + R(1+i) + ... + R(1+i)^{n-1}$

c) Using a formula for the sum of a finite geometric series, one finds

$$\begin{aligned}\sum_{k=0}^{n-1} R(1+i)^k &= \frac{R\left(1-(1+i)^n\right)}{1-(1+i)} \\ &= \frac{R\left(1-(1+i)^n\right)}{-i} \\ &= \frac{R\left((1+i)^n-1\right)}{i}.\end{aligned}$$

d) If $R = 2000$, $n = 30$, and $i = 0.12$, then the amount of the annuity at the time of the last payment is

$$\begin{aligned}\frac{R\left((1+i)^n-1\right)}{i} &= \\ \frac{2000\left((1+0.12)^{30}-1\right)}{0.12} &= \\ \$482,665.37.\end{aligned}$$

e) If $R = 800$, $n = 13(12)$, and $i = \frac{0.09}{12}$, then the amount of the annuity at the time of the last payment is

$$\begin{aligned}&= \frac{R\left((1+i)^n-1\right)}{i} \\ &= \frac{800\left(\left(1+\frac{0.09}{12}\right)^{156}-1\right)}{0.09}.\end{aligned}$$

After 7 months, the above amount becomes

$$\begin{aligned}&= \left(1+\frac{.09}{12}\right)^7\left[\frac{800\left(\left(1+\frac{0.09}{12}\right)^{156}-1\right)}{0.09/12}\right] \\ &= \$248,161.69.\end{aligned}$$

For Thought

1. False, there are $90 \cdot 26$ different codes.

2. False, there are 24^3 different possible fraternity names.

3. False, there are $3 \cdot 5 \cdot 3 \cdot 1 = 45$ different outfits.

4. True, $5! = 120$.

5. True, since

$$P(10,2) = \frac{10!}{8!} = 10 \cdot 9 = 90.$$

6. False, the number of ways is 4^{20}

7. True

8. True, since $\frac{1000(999)998!}{998!} = 999,000$.

9. False, since

$$P(10,1) = \frac{10!}{9!} = 10$$

and

$$P(10,9) = \frac{10!}{1!} = 10!.$$

10. False, $P(29,1) = \frac{29!}{28!} = \frac{29 \cdot 28!}{28!} = 29$.

11.4 Exercises

1. fundamental counting principle

2. permutation

3. The number of possible outcomes is $2 \cdot 2$ or 4.

4. The number of possible different meals is $2 \cdot 3$ or 6.

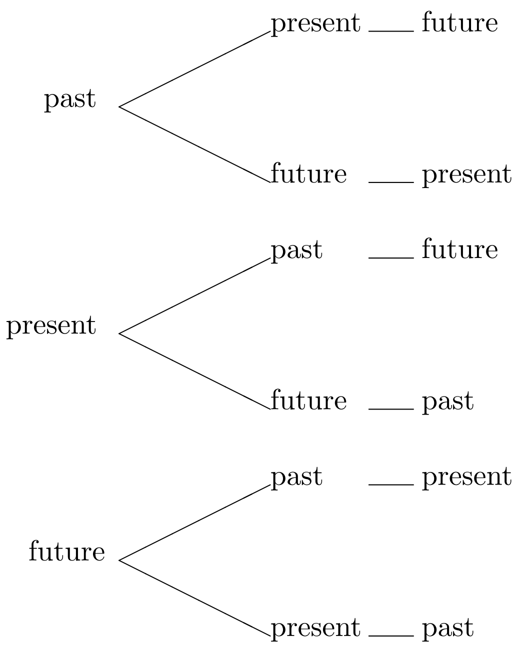
5. The number of possible outcomes is $2 \cdot 2 \cdot 2$ or 8.

6. The number of possible outfits is $3 \cdot 4$ or 12.

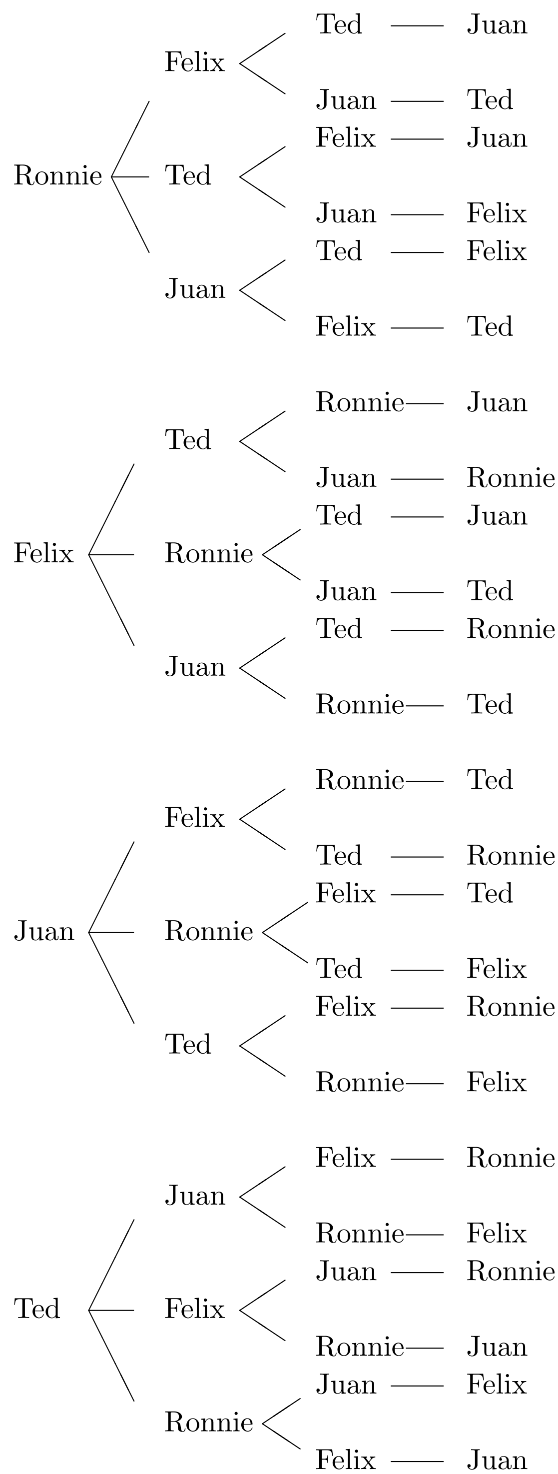
7. Number of routes is $2 \cdot 4 = 8$.

8. There are $9 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 4 = 1,296$ different cars.

9. There are $3! = 6$ different schedules.
These schedules are given by:



10. There are $4! = 24$ ways the race can end.



11. There are $4^5 = 1024$ different possible hands.

12. There are $13^4 = 28,561$ different possible hands.

13. There are 8 optional items since $2^8 = 256$.

14. There are $3 \cdot 2 \cdot 2^4 \cdot 2^5 = 3072$ different pizzas.

15. $\frac{7(6)(5)4!}{4!} = 7(6)(5) = 210$

16. $\frac{9(8)(7)6!}{6!} = 9(8)(7) = 504$

17. $\frac{5!}{0!} = \frac{120}{1} = 120$

18. $\frac{6!}{0!} = \frac{720}{1} = 720$

19. $\frac{78(77!)}{77!} = 78$

20. $\frac{56(55!)}{55!} = 56$

21. $\frac{4!}{2!} = 12$

22. $\frac{5!}{3!} = 20$

23. $\frac{5!}{2!} = 60$

24. $\frac{6!}{2!} = 360$

25. $\frac{7!}{4!} = 210$ **26.** $\frac{16!}{12!} = 43,680$

27. $\frac{9!}{4!} = 15,120$ **28.** $\frac{7!}{5!} = 42$

29. $\frac{5!}{0!} = 120$ **30.** $\frac{4!}{0!} = 24$

31. $\frac{11!}{8!} = 990$ **32.** $\frac{15!}{10!} = 360,360$

33. $\frac{99!}{99!} = 1$ **34.** $\frac{44!}{44!} = 1$

35. $\frac{105!}{103!} = \frac{105(104)(103!)}{103!} = 105(104) = 10,920$

36. $\frac{120!}{118!} = \frac{120(119)(118!)}{118!} = 120(119) = 14,280$

37. Hercules can perform 12 tasks in $12! = 479,001,600$ ways.

38. They can line up in $11! = 39,916,800$ ways.

39. The number of ways a first, second, and a third restaurant can be chosen is $P(15,3) = \frac{15!}{12!} = 2730$ ways.

40. A professor can hand out the books in $\frac{8!}{4!} = 1680$ ways.

41. Since there are 8 half-hour shows from 6:00 p.m. to 10:00 p.m., the number of possible different schedules is $P(26,8) \approx 6.3 \times 10^{10}$.

42. The number of different song arrangements is $P(20,8) = \frac{20!}{12!} = 5,079,110,400$.

43. Since each question has 4 possible answers, the number of ways to answer 6 questions is 4^6 or 4096.

44. There are $4 \cdot 3 = 12$ possible names.

45. There are $4 \cdot 3 \cdot 6 = 72$ possible prizes.

46. The number of extensions is $10^3 \cdot 9 = 9000$.

47. There are $P(26,3) = 15,600$ possible passwords.

48. The cars can be assigned in $P(10,3) = 720$ ways.

49. There are $3 \cdot 10^4 = 30,000$ possible phone numbers.

50. There are $26^4 = 456,976$ addresses.

51. Superman has $4! = 24$ ways to arrange these rescues.

52. There are $5! = 120$ ways to arrange these stops.

53. Since the true-false questions can be answered in 2^5 ways and the multiple choice questions in 4^6 ways, the test can be answered in $2^5 \cdot 4^6 = 131,072$ ways.

54. There are $2^4 = 16$ possible outcomes.

55. On her list will be $P(7,3) = 210$ possible words.

56. There are $P(5,3) = 60$ permutations. These are

ABC, ABD, ABE, ACB, ACD, ACE, ADB, ADC, ADE, AEB, AEC, AED,

BAC, BAD, BAE, BCA, BCD, BCE, BDA, BDC, BDE, BEA, BEC, BED,

CAB, CAD, CAE, CBA, CBD, CBE, CDA, CDB, CDE, CEA, CEB, CED,

DAB, DAC, DAE, DBA, DBC, DBE, DCA, DCB, DCE, DEA, DEB, DEC,

EAB, EAC, EAD, EBA, EBC, EBD, ECA, ECB, ECD, EDA, EDB, EDC

57. The number of subsets is 2^n.

59. A finite geometric series with $n = 10$ terms has the sum

$$\frac{a(1-r^n)}{1-r} = \frac{1(1-2^{11})}{1-2} = 2047.$$

60. An infinite geometric series with first term $a = 4$ and ratio $r = 0.8$ has the sum

$$\frac{a}{1-r} = \frac{4}{1-0.8} = 20.$$

61. If $a_n = ar^{n-1}$, then $ar^2 = \frac{5}{4}$ and $ar^4 = \frac{5}{16}$.

Solving for a and r, we find $a = 5$ and $r = \pm\frac{1}{2}$.

Thus, $a_{10} = 5\left(\pm\frac{1}{2}\right)^9 = \pm\frac{5}{512}$.

62. Renumber the series beginning with $k = 1$.

$$\begin{aligned}
\sum_{n=6}^{30}(3n+5) &= \sum_{k=1}^{25}(3(k+5)+5) \\
&= \sum_{k=1}^{25}(3k+20) \\
&= 3\cdot\frac{25(26)}{2} + 20(25) \\
\sum_{n=6}^{30}(3n+5) &= 1475
\end{aligned}$$

63. The two equations are multiples of each other. The solution set is $\{(x,y) : y = 5x - 12\}$.

64. If $A = Pr^t$ and $4 = 5r^{500}$, then $r = \sqrt[500]{0.8}$.

The half-life h satisfies

$$2.5 = 5(\sqrt[500]{0.8})^h$$

Then $h \approx 1553$ years.

Thinking Outside the Box C

The number of distinct routes from the corner of First Ave and First Street are

$$\frac{16!}{8!8!} = 12,870.$$

Since Ms. Peabody takes two distinct routes each day, it will take her

$$\frac{12,870}{2(365)} = 17 \text{ years}, 230 \text{ days}$$

to walk all possible routes.

11.4 Pop Quiz

1. $\dfrac{102(101)100!}{100!} = 102(101) = 10,302$

$P(10,4) = \dfrac{10!}{6!} = 5040$

2. $P(40,4) = \dfrac{40!}{36!} = 2,193,360$ ways to choose

3. $5^6 = 15,625$ ways to answer a test

4. $20(30) = 600$ ways to select an outfit

11.4 Linking Concepts

a) There are $4\cdot3\cdot2 = 24$ routes.

b) The possible routes are ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCDA, BCAD, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA.

For example, in route ABCD, starting from Cameron one visits A, first, then to B, then to C, then to D, then back to Cameron.

c) Let O represent Cameron, Louisiana. The distances between each location are $\overline{OA} \approx 11.2$, $\overline{OB} = 6$, $\overline{OC} \approx 21.5$, $\overline{OD} \approx 12.6$, $\overline{AB} \approx 10.0$, $\overline{AC} \approx 23.4$, $\overline{AD} \approx 22.0$, $\overline{BC} \approx 16.1$, $\overline{BD} \approx 12.2$, and $\overline{CD} \approx 16.5$.

d) The length of the possible routes are $\overline{ABCD} \approx 66.5$, $\overline{ABDC} \approx 71.4$, $\overline{ACBD} \approx 75.5$, $\overline{ACDB} \approx 69.3$, $\overline{ADBC} \approx 83.0$, $\overline{ADCB} \approx 71.8$, $\overline{BACD} \approx 68.5$, $\overline{BADC} \approx 76.0$, $\overline{BCAD} \approx 80.1$, $\overline{BCDA} \approx 71.8$, $\overline{BDAC} \approx 85.1$, $\overline{BDCA} \approx 69.3$, $\overline{CABD} \approx 79.7$, $\overline{CADB} \approx 85.1$, $\overline{CBAD} \approx 82.2$, $\overline{CBDA} \approx 83.0$, $\overline{CDAB} \approx 76.0$, $\overline{CDBA} \approx 71.4$, $\overline{DABC} \approx 82.2$, $\overline{DACB} \approx 80.1$, $\overline{DBAC} \approx 79.7$, $\overline{DBCA} \approx 75.5$, $\overline{DCAB} \approx 68.5$, and $\overline{DCBA} \approx 66.5$.

Yes, exactly half of them are duplications. For example, the order in which the oil rigs were visited in route ABCD was reversed in DCBA; thus, the length of these routes are the same.

e) The shortest possible route is $\overline{ABCD} \approx 66.5$ miles or $\overline{DCBA} \approx 66.5$.

f) There are 40! different routes, a number in the magnitude of

$$40! \approx 8.2 \times 10^{47}.$$

At one route per second, the number of years it would take to find the shortest route is

$$\frac{8.2 \times 10^{47}}{365(24)(60)(60)} \approx 2.6 \times 10^{40} \text{ years.}$$

For Thought

1. True, the number of ways is $C(5,3) = C(5,2) = 10$.

2. False, the number of ways is 3^5.

3. True, $P(5,2) = 20$.

4. False, it contains $n + 1$ terms. **5.** True

6. True, $C(n,r)$ is the number of subsets of size r, and a set with n elements has 2^n subsets.

7. False, $P(8,3) = 336$ and $C(8,3) = 56$.

8. True **9.** False, since $P(8,3) = 336$ and $P(8,5) = 6720$. **10.** True

11.5 Exercises

1. combination

2. n things taken r at a time

3. binomial expansion

4. Pascal's triangle

5. 10 **6.** 15 **7.** 7 **8.** 8

9. $\frac{5!}{4!1!} = 5$ **10.** $\frac{9!}{8!1!} = 9$

11. $\frac{8!}{4!4!} = 70$ **12.** $\frac{7!}{4!3!} = 35$

13. $\frac{7!}{3!4!} = 35$ **14.** $\frac{6!}{3!3!} = 20$

15. $\frac{10!}{10!0!} = 1$ **16.** $\frac{9!}{9!0!} = 1$

17. $\frac{12!}{0!12!} = 1$ **18.** $\frac{11!}{0!11!} = 1$

19. There are $C(4,2) = 6$ possible selections. They are Alice and Brenda, Alice and Carol, Alice and Dolores, Brenda and Carol, Brenda and Dolores, Carol and Dolores.

20. There are $C(3,2) = 3$ possible choices. These are Ford and Chevrolet, Ford and Toyota, Chevrolet and Toyota.

21. There are $C(5,2) = 10$ possible selections.

22. There are $C(6,2) = 15$ possible schedules.

23. $C(5,3) = 10$ ways for an in-depth interview

24. $C(20,5) = 15,504$ ways to select students

25. $C(49,6) = 13,983,816$ ways to choose six numbers

26. $C(39,5) = 575,757$ ways to choose five numbers

27. $C(52,5) = 2,598,960$ possible poker hands

28. $C(52,13) \approx 6.35 \times 10^{11}$ possible bridge hands

29. These assignments can be done in $\dfrac{10!}{5!3!2!} = 2520$ ways.

30. The officer can choose in $\dfrac{12!}{6!4!2!} = 13,860$ ways.

31. Since A occurs 4 times, the number of permutations is $\dfrac{7!}{4!} = 210$.

32. There are $7! - 1 = 5039$ incorrect spellings.

33. There are 3, 6, 10, and $C(n,2)$ distinct chords, respectively,.

34. There are $C(6,3) = 20$ triangles.

35. There are $\dfrac{10!}{3!2!5!} = 2520$ possible assignments.

36. There are $\dfrac{10!}{3!4!3!} = 4200$ possible ways.

37. There are $3^3 = 27$ ways to watch.

38. There are $5 \cdot 6 \cdot 4 = 120$ different meals.

39. $C(8,3) = 56$ ways to make the selection

40. Awards can be made in $P(6,3) = 120$ ways.

41. Since 3 men can be selected in $C(9,3)$ ways and 2 women in $C(6,2)$ ways, the team can be selected in $C(9,3) \cdot C(6,2) = 1260$ ways.

42. Since 3 hearts can be selected in $C(13,3)$ ways and 2 spades in $C(13,2)$ ways, the number of such poker hands is $C(13,3) \cdot C(13,2) = 22,308$.

43. $12! = 479,001,600$ ways to return the papers

44. $10! = 3,628,800$ ways to return the papers

45. $6 \cdot 6 = 36$ possible outcomes

46. $2^3 = 8$ possible outcomes of childrens' sexes

47. Since there are 4! ways to arrange the bands in a line and 3! ways to line up the floats, the parade can be lead in $4!3! = 144$ ways.

48. A band must lead and 2 bands cannot be next to each other, but 2 floats can be next to each other. There are 4 arrangements of bands M and floats F, namely, MFMFMFMF, MFFMFMFM , MFMFFMFM, and MFMFMFFM.

In each arrangment there are 4!4! ways to line up. So the number of ways the bands and floats can line up is $4(4!4!) = 2304$.

49. Use the numbers 1,2,1 from Pascal's triangle. Then $(x+y)^2 = 1x^2+2xy+1y^2 = x^2+2xy+y^2$.

50. Use the numbers 1,2,1 from Pascal's triangle. Then $(x+5)^2 = 1x^2 + 2x(5) + 1 \cdot 5^2 = x^2 + 10x + 25$.

51. Use the numbers 1,2,1 from Pascal's triangle. Then $(2a + (-3))^2 =$

$1(2a)^2 + 2(2a)(-3) + 1(-3)^2 = 4a^2 - 12a + 9$.

52. Use the numbers 1,2,1 from Pascal's triangle. Then $(2x + (-3y))^2 = 1(2x)^2 + 2(2x)(-3y) + 1(-3y)^2 = 4x^2 - 12xy + 9y^2$.

53. Use the numbers 1,3,3,1 from Pascal's triangle. Then $(a-2)^3 =$

$1(a^3) + 3(a^2)(-2) + 3(a)(-2)^2 + 1(-2)^3 =$

$a^3 - 6a^2 + 12a - 8$.

54. Use the numbers 1,3,3,1 from Pascal's triangle. Then $(b^2-3)^3 =$

$1(b^2)^3 + 3(b^2)^2(-3) + 3(b^2)(-3)^2 + 1(-3)^3 =$

$b^6 - 9b^4 + 27b^2 - 27$.

55. Use the numbers 1,3,3,1 from Pascal's triangle. Thus, $(2a+b^2)^3 =$

$1(2a)^3 + 3(2a)^2(b^2) + 3(2a)(b^2)^2 + 1(b^2)^3 =$

$8a^3 + 12a^2b^2 + 6ab^4 + b^6$.

56. Use the numbers 1,3,3,1 from Pascal's triangle. So $(x+3)^3 =$

$1(x)^3 + 3(x)^2(3) + 3(x)(3)^2 + 1(3)^3 =$

$x^3 + 9x^2 + 27x + 27$.

57. Use the numbers 1,4,6,4,1 from Pascal's triangle. Then we get $(x-2y)^4 =$
$1(x)^4+4(x)^3(-2y)+6(x)^2(-2y)^2+4(x)(-2y)^3$
$+1(-2y)^4 =$
$x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4.$

58. Use the numbers 1,4,6,4,1 from Pascal's triangle. Then we have $(y-2)^4 =$
$1(y)^4 + 4(y)^3(-2) + 6(y)^2(-2)^2 + 4(y)(-2)^3$
$+1(-2)^4 = y^4 - 8y^3 + 24y^2 - 32y + 16.$

59. Use the numbers 1,4,6,4,1 from Pascal's triangle. Then we obtain $(x^2+1)^4 =$
$1(x^2)^4 + 4(x^2)^3(1) + 6(x^2)^2(1)^2 + 4(x^2)(1)^3$
$+1(1)^4 = x^8 + 4x^6 + 6x^4 + 4x^2 + 1.$

60. Use the numbers 1,4,6,4,1 from Pascal's triangle. Then we obtain $(2r+3t^2)^4 =$
$1(2r)^4+4(2r)^3(3t^2)+6(2r)^2(3t^2)^2+4(2r)(3t^2)^3$
$+1(3t^2)^4 =$
$16r^4 + 96r^3t^2 + 216r^2t^4 + 216rt^6 + 81t^8.$

61. Use the numbers 1,5,10,10,5,1 from Pascal's triangle. Then we obtain $(a-3)^5 =$
$1(a)^5 - 5a^4(3) + 10a^3(3)^2 - 10a^2(3)^3+$
$5a(3)^4 - 1(3)^4 =$
$a^5 - 15a^4 + 90a^3 - 270a^2 + 405a - 243.$

62. Use the numbers 1,5,10,10,5,1 from Pascal's triangle. Then we obtain $(b+2y)^5 =$
$1(b)^5 + 5b^4(2y) + 10b^3(2y)^2 + 10b^2(2y)^3+$
$5b(2y)^4 + 1(2y)^5 =$
$b^5 + 10b^4y + 40b^3y^2 + 80b^2y^3 + 80by^4 + 32y^5.$

63. Use the numbers 1,6,15,20,15,6,1 from Pascal's triangle. Then we obtain $(x+2a)^6 =$
$1(x)^6 + 6x^5(2a) + 15x^4(2a)^2 + 20x^3(2a)^3+$
$15x^2(2a)^4 + 6x(2a)^5 + 1(2a)^6 =$
$x^6 + 12ax^5 + 60a^2x^4 + 160a^3x^3+$
$240a^4x^2 + 192a^5x + 64a^6.$

64. Use the numbers 1,6,15,20,15,6,1 from Pascal's triangle. Then we obtain $(2b-1)^6 =$
$1(2b)^6 - 6(2b)^5(1) + 15(2b)^4(1)^2 - 20(2b)^3(1)^3+$
$15(2b)^2(1)^4 - 6(2b)(1)^5 + 1(1)^6 =$
$64b^6 - 192b^5 + 240b^4 - 160b^3+$
$60b^2 - 12b + 1.$

65. Use the Binomial Theorem. So $(x+y)^9 =$
$$\binom{9}{0}x^9 + \binom{9}{1}x^8y + \binom{9}{2}x^7y^2 + ... =$$
$x^9 + 9x^8y + 36x^7y^2 + ...$

66. Use the Binomial Theorem. So $(a-b)^{10} =$
$$\binom{10}{0}a^{10} + \binom{10}{1}a^9(-b)+$$
$$\binom{10}{2}a^8(-b)^2 + ... =$$
$a^{10} - 10a^9b + 45a^8b^2 + ...$

67. Use the Binomial Theorem. Then $(2x-y)^{12} =$
$$\binom{12}{0}(2x)^{12} + \binom{12}{1}(2x)^{11}(-y)+$$
$$\binom{12}{2}(2x)^{10}(-y)^2 + ... =$$
$4096x^{12} - 24,576x^{11}y + 67,584x^{10}y^2 + ...$

68. Use the Binomial Theorem. Then $(a+2b)^{11} =$
$$\binom{11}{0}(a)^{11} + \binom{11}{1}(a)^{10}(2b)+$$
$$\binom{11}{2}(a)^9(2b)^2 + ... =$$
$a^{11} + 22a^{10}b + 220a^9b^2 + ...$

69. Use the Binomial Theorem. So $(2s-0.5t)^8 =$
$$\binom{8}{0}(2s)^8 + \binom{8}{1}(2s)^7(-0.5t)+$$
$$\binom{8}{2}(2s)^6(-0.5t)^2 + ... =$$
$256s^8 - 512s^7t + 448s^6t^2 + ...$

70. Use the Binomial Theorem. So $(3y^2+a)^{10} =$
$$\binom{10}{0}(3y^2)^{10} + \binom{10}{1}(3y^2)^9(a)+$$
$$\binom{10}{2}(3y^2)^8(a)^2 + ... =$$
$59,049y^{20} + 196,830y^{18}a+$
$295,245y^{16}a^2 + ...$

71. Use the Binomial Theorem.
Thus, $(m^2 - 2w^3)^9 =$
$$\binom{9}{0}(m^2)^9 + \binom{9}{1}(m^2)^8(-2w^3) + \binom{9}{2}(m^2)^7(-2w^3)^2 + ... =$$
$m^{18} - 18m^{16}w^3 + 144m^{14}w^6 + ...$

72. Use the Binomial Theorem.
Thus, $(ab^2 - 5c)^8 =$
$$\binom{8}{0}(ab^2)^8 + \binom{8}{1}(ab^2)^7(-5c) + \binom{8}{2}(ab^2)^6(-5c)^2 + ... =$$
$a^8b^{16} - 40a^7b^{14}c + 700a^6b^{12}c^2 + ...$

73. $\binom{8}{5} = 56$ **74.** $\binom{12}{9} = 220$

75. $\binom{13}{5}(-2)^5 = -41,184$

76. $\binom{11}{5}(0.5)^6(-1)^5 = -7.21875$

77. By looking at a particular term, one finds
$$(a + [b + c])^{12} = ... + \binom{12}{10}a^2[b + c]^{10} + ... =$$
$$... + \binom{12}{10}a^2\left[... + \binom{10}{6}b^4c^6 + ...\right] + ...$$
So the coefficient of $a^2b^4c^6$ is
$$\binom{12}{10}\binom{10}{6} = \frac{12!}{2!4!6!} = 13,860.$$

78. By looking at a particular term, one finds
$$(x+[-y-2z])^8 = ...+\binom{8}{5}x^3[-y-2z]^5+... =$$
$$...+\binom{8}{5}x^3\left[... + \binom{5}{3}(-y)^2(-2z)^3 + ...\right]+ ...$$
So the coefficient of $x^3y^2z^3$ is
$$\binom{8}{5}\binom{5}{3}(-1)^2(-2)^3 = -4480.$$

79. The coefficient of a^3b^7 in $(a + b + 2c)^{10}$ is the number of rearrangements of aaabbbbbbb, which is $\frac{10!}{3!7!} = 120$.

80. The coefficient of $w^2xy^3z^9$ in $(w+x+y+z)^{15}$ is the number of rearrangements of wwxyyyzzzzzzzzz, which is
$$\frac{15!}{2!1!3!9!} = 300,300.$$

85. $4^8 = 65,536$ ways

86. $4! = 24$ orders

87. $10(9)(8) = 720$

88. The first term of the geometric series is $a = \frac{2}{27}$, and the ratio is $r = \frac{1}{3}$. The sum is
$$S = \frac{a}{1-r} = \frac{2/27}{1-1/3} = \frac{2}{27} \cdot \frac{3}{2} = \frac{1}{9}.$$

89. $a_2 = 5 - 10 = -5$, $a_3 = 5 - (-5) = 10$, $a_4 = 5 - 10 = -5$, and the 5th term is $a_5 = 5 - (-5) = 10$.

90. $\sum_{n=1}^{5}(-1)^n(2n-3) = 1 + 1 - 3 + 5 - 7 = -3$

Thinking Outside the Box CI

Since $1+2+3+...+1413 = 998,991$, the 998,991th term is 1413. Thus, after 1009 additional terms, the next term is the one millionth term which is 1414.

11.5 Pop Quiz

1. $\frac{160!}{4!156!} = \frac{160(159)(158)(157)}{4!} = 26,294,360$

2. $C(40,3) = 9880$ ways

3. $C(30,2)C(25,2) = 130,500$ ways

4. Use the numbers 1,5,10,10,5,1 from Pascal's triangle. Then we obtain $(x + 2y)^5 =$
$1(x)^5 + 5x^4(2y) + 10x^3(2y)^2 + 10x^2(2y)^3 + 5x(2y)^4 + 1(2y)^5 =$
$x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$.

11.5 Linking Concepts

a) Note, in each suit, there are 8 possible straight flush hands. Thus, the number of straight flush hands is 32. Note that an ace is counted only as a high card here.

b) For example, there are 48 $(= 52 - 4)$ possible hands with four 10's. Then there must be 624 $(= 13 \cdot 48)$ possible four of a kind hands.

c) First, we count the number of possible full house hands with three aces. Note, the number of ways to choose 3 aces is $C(4, 3)$, there are 12 other kinds of hands besides the aces, and the number of ways to choose 2 cards from a four of a kind is $C(4, 2)$.

Thus, the number of possible full house hands with three aces is $C(4, 3) \cdot [12 \cdot C(4, 2)]$. Therefore, the number of full house hands is 3744 $(= 13 \cdot C(4, 3) \cdot [12 \cdot C(4, 2)])$.

d) The number of possible hands of the same suit (including straight flush hands) is $4 \cdot C(13, 5)$. After subtracting the number of straight flush hands and royal flush hands, one gets the number of possible flush hands, namely, 5112 hands $(= 4 \cdot C(13, 5) - 36)$.

e) The number of possible three of a kind hands is 54,912 $(= 13 \cdot C(4, 3) \cdot C(48, 2) - 3744)$. Note, 3744 is the number of full house hands and such a hand is not a three of a kind hand.

f) The number of 'two pairs' hands is 123,552 $(= C(13, 2) \cdot C(4, 2) \cdot C(4, 2) \cdot 44)$. Note, $C(13, 2)$ is the number of ways to select two distinct face values from 13 possible face values.

g) Note, in a 'one pair' hand, there are exactly four different (out of 13) kinds of cards. Thus, the number of one pair hands is given by $C(13, 4) \cdot 4 \cdot C(4, 2) \cdot 4 \cdot 4 \cdot 4$ or equivalently 1,098,240 hands.

For Thought

1. False, $P(E) = \dfrac{n(E)}{n(S)}$.

2. False, there are $2^4 = 16$ outcomes.

3. True, since possible outcomes are (H, H), (H, T), (T, H), and (T, T) then $P(\text{at least one tail}) = \dfrac{3}{4} = 0.75$.

4. True, since $\dfrac{5}{6}$ is the probability of not getting a four when a die is tossed then $P(\text{at least one four}) = 1 - P(\text{no four}) = 1 - \dfrac{5}{6} \cdot \dfrac{5}{6} = \dfrac{11}{36}$.

5. False, since $\dfrac{1}{2}$ is the probability of getting a tail when a coin is tossed, $P(\text{at least one head}) = 1 - P(\text{four tails}) = 1 - \left(\dfrac{1}{2}\right)^4 = \dfrac{15}{16}$.

6. False, the complement is getting either no head or exactly one head.

7. True, the complement of getting exactly 3 tails in a toss of three coins is getting at least one head.

8. False, the odds in favor of snow is $\dfrac{P(\text{snow})}{P(\text{no snow})} = \dfrac{7/10}{3/10} = \dfrac{7}{3}$ i.e. 7 to 3.

9. False, $P(E) = 3/7$.

10. False, it is equivalent to 1 to 4.

11.6 Exercises

1. experiment

2. sample space

3. Equally likely

4. mutually exclusive

5. complementary

6. odds in favor

7. $\{(H,H),(H,T),(T,H),(T,T)\}$

8. $\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),$
$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$
$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),$
$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$
$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),$
$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

9. $\{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),$
$(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$

10. $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),$
$(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$

11. Note, (H,T) and (T,H) are the only outcomes with exactly one head. Since the sample space has 4 elements, the probability of getting exactly one head is $\dfrac{n\{(H,T),(T,H)\}}{4} = \dfrac{1}{2}$.

12. Note, (T,T) is the only outcome with exactly two tails. Since the sample space has four elements, the probability of getting exactly two tails is $\dfrac{n\{(T,T)\}}{4} = \dfrac{1}{4}$.

13. Note, $(6,6)$ is the only outcome with exactly two sixes. Since the sample space has 36 elements, the probability of getting exactly two sixes is $\dfrac{n\{(6,6)\}}{36} = \dfrac{1}{36}$.

14. Note, there are exactly 6 outcomes with the same numbers. These are $(1,1)$,$(2,2)$, $(3,3)$, $(4,4)$,$(5,5)$ and $(6,6)$. Since the sample space has 36 elements, the probability of getting the same numbers is $\dfrac{6}{36} = \dfrac{1}{6}$.

15. Note, $(H,5)$ is the only outcome with heads and a five. Since the sample space has 12 elements, the probability of getting heads and a five is $\dfrac{n\{(H,5)\}}{12} = \dfrac{1}{12}$.

16. Note, there are exactly 3 outcomes with tails and an even number. These are $(T,2)$, $(T,4)$, $(T,6)$. Since the sample space has 12 elements, the probability of getting heads and an even number is $\dfrac{3}{12} = \dfrac{1}{4}$.

17. Note, (T,T,T) is the only outcome with all tails. Since the sample space has 8 elements, the probability of getting all tails is

$$\frac{n\{(T,T,T)\}}{8} = \frac{1}{8}.$$

18. Note, there are exactly 7 outcomes with at least one head. These are (H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,H,H), (T,H,T), (T,T,H). Since the sample space has 8 elements, the probability of getting at least one head is $\dfrac{7}{8}$.

19. (a) $\dfrac{n(\{3,4,5,6\})}{6} = \dfrac{4}{6} = \dfrac{2}{3}$,
(b) $\dfrac{n(\{1,2,3,4,5,6\})}{6} = \dfrac{6}{6} = 1$,
(c) $\dfrac{n(\{1,2,3,5,6\})}{6} = \dfrac{5}{6}$,
(d) 0, (e) $\dfrac{n(\{1\})}{6} = \dfrac{1}{6}$

20. (a) 1/2, (b) 1, (c) 0

21. (a) $\dfrac{n\{(T,T)\}}{4} = 1/4$,
(b) $1 - P(2\ tails) = 1 - 1/4 = 3/4$,
(c) $\dfrac{n\{(H,H)\}}{4} = 1/4$,
(d) $\dfrac{n\{(T,T),(H,T),(T,H)\}}{4} = 3/4$

22. (a) $\dfrac{n\{(H,T)\}}{4} = \dfrac{1}{4}$, (b) $\dfrac{n\{(T,T)\}}{4} = \dfrac{1}{4}$,
(c) $\dfrac{n\{(T,H),(H,H)\}}{4} = \dfrac{2}{4} = \dfrac{1}{2}$,
(d) $\dfrac{n\{(T,H),(H,T)\}}{4} = \dfrac{2}{4} = \dfrac{1}{2}$

23. Note there are 36 possible outcomes.
(a) $\dfrac{n\{(3,3)\}}{36} = 1/36$,

(b) Since there are 11 possible outcomes with at least one 3, $P(\textit{at least one } 3) = 11/36$,
(c) Since there are 5 possible outcomes with a sum of 6, $P(\textit{sum is } 6) = 5/36$,
(d) $1 - P(\textit{sum is } 2) = 1 - 1/36 = 35/36$,
(e) $P(\textit{sum is } 2) = 1/36$

24. Note there are 36 possible outcomes.
(a) $\dfrac{n\{(1,6)\}}{36} = 1/36$,
(b) $\dfrac{n\{(1,3),(3,1),(2,2)\}}{36} = 3/36 = 1/12$,
(c) $\dfrac{n\{(1,5),(2,5),(3,5),(4,5),(5,5),(6,5)\}}{36} = 1/6$, (d) 1, (e) $P(\textit{1st is even})P(\textit{2nd is odd}) = (1/2)(1/2) = 1/4$

25. (a) 1/3, (b) 2/3, (c) 0

26. (a) Since $\dfrac{10,000}{100,000} = \dfrac{1}{10}$, the probability of winning is $\dfrac{1}{10}$, (b) $\dfrac{9}{10}$.

27. (a) 3/13, (b) 9/13, (c) 9/13,
(d) $P(\textit{marble is yellow}) = 4/13$

28. (a) 1/14, (b) $7/14 = 1/2$,
(c) $P(\textit{Democrat or independent}) = 8/14 = 4/7$

29. There are 72 possible outcomes.
(a) $\dfrac{n\{(1,9)\}}{72} = 1/72$,
(b) $\dfrac{n\{(1,3),(3,1)\}}{72} = 2/72 = 1/36$,
(c) $\dfrac{n\{(1,4),(4,1),(2,3),(3,2)\}}{72} = 4/72 = 1/18$

30. Since a set with 12 elements has $C(12,2)$ subsets of size 2 and $\{\textit{pres.}, \textit{vice} - \textit{pres.}\}$ is the subset consisting of non-sales people, the probability that no salesperson is a winner is $1/C(12,2) = 1/66$.

31. (a) $\dfrac{1}{C(52,5)} = \dfrac{1}{2,598,960}$
(b) By using the counting principle, the probability of one 3, one 4, one 5, one 6, and one 7 is $\dfrac{4^5}{C(52,5)} = \dfrac{1024}{2,598,960}$.

32. This involves permutations.
(a) $1/4! = 1/24$ (b) $1/4! = 1/24$

33. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.6 - 0.5 = 0.9$

34. $P(C \cup D) = P(C) + P(D) - P(C \cap C) = 0.3 + 0.5 - 0.2 = 0.6$

35. Since $P(E \cup F) = P(E) + P(F) - P(E \cap F)$, we get $0.8 = 0.2 + 0.7 - P(E \cap F)$.
Then $P(E \cap F) = 0.2 + 0.7 - 0.8 = 0.1$

36. Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, we get $0.9 = 0.4 + 0.7 - P(A \cap B)$.
Then $P(A \cap B) = 0.4 + 0.7 - 0.9 = 0.2$

37. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.2 + 0.3 - 0 = 0.5$

38. $P(C \cup D) = P(C) + P(D) - P(C \cap C) = 0.3 + 0.6 - 0 = 0.9$

39. Yes, since an outcome cannot have a sum that is both 4 and 5.

40. Yes, since an outcome cannot have a sum that is both even and odd.

41. No, since the sum in the outcome $(1,1)$ is 2 (which is less than 5) and is even.

42. No, since the numbers in the outcome $(5,5)$ are the same and the sum is 10.

43. Yes, since there is no outcome with a two and with a sum greater than nine. Note, in $(2,6)$, the highest possible sum is 8.

44. Yes, since there is no outcome whose sum is twelve and the numbers are different. Note, $(6,6)$ is the only outcome with a sum of twelve.

45. $P(\textit{high}-\textit{risk or a woman}) = P(\textit{high}-\textit{risk}) + P(\textit{woman}) - P(\textit{high} - \textit{risk and a woman}) = 38\% + 64\% - 24\% = 78\%$.

46. One finds the probabilities $P(sum\ of\ 6) = 5/36$, $P(at\ least\ one\ 4) = 11/36$, $P(sum\ of\ 6\ and\ at\ least\ one\ 4) = 2/36$. By the addition rule, we obtain $P(sum\ of\ 6\ or\ at\ least\ one\ 4) = 5/36 + 11/36 - 2/36 = 7/18$.

47. One finds the probabilities $P(3\ boys) = P(3\ girls) = 1/8$. By the addition rule for mutually exclusive events, we get $P(3\ boys\ or\ 3\ girls) = 1/8 + 1/8 = 1/4$.

48. We obtain the probabilities $P(sum\ of\ 10) = 3/36$, $P(sum\ of\ 4) = 3/36$. By the addition rule for mutually exclusive events, we get $P(sum\ of\ 10\ or\ sum\ of\ 4) = 3/36 + 3/36 = 1/6$.

49. One obtains the probabilities $P(heart) = 13/52$, $P(king) = 4/52$, $P(heart\ and\ king) = 1/52$. By the addition rule, we get $P(heart\ or\ king) = 13/52 + 4/52 - 1/52 = 4/13$.

50. Since the number of cards that are either heart or diamond is 26, the probability of choosing a heart or diamond is $26/52 = 1/2$.

51. (a) 34%, (b) $22\% + 18\% = 40\%$, (c) $34\% + 22\% + 18\% = 74\%$

52. (a) $\frac{90}{900} = 0.1$, (b) $\frac{210 + 90 + 260}{900} = \frac{28}{45}$

53. $1 - P(surviving) = 1 - 0.001 = 0.999$

54. $1 - P(not\ audited) = 1 - 0.91 = 0.09$

55. Note there are 36 possible outcomes.

(a) $\frac{n\{(4,4)\}}{36} = \frac{1}{36}$,

(b) $1 - P(pair\ of\ 4's) = 1 - 1/36 = \frac{35}{36}$,

(c) $\frac{35}{36}$, since this is the same event as (b)

56. Note there are 2^3 or 8 possible outcomes.

(a) $\frac{n\{(H,H,H\}}{8} = 1/8$,

(b) $1 - P(getting\ 3\ heads) = 1 - 1/8 = 7/8$,

(c) $1 - P(getting\ 3\ tails) = 1 - 1/8 = 7/8$

57. Since $\frac{4/5}{1/5} = 4$, the odds in favor of rain is 4 to 1.

58. Since $\frac{9/10}{1/10} = 9$, the odds in favor of the Yankees winning is 9 to 1.

59. Odds in favor of the eye of the hurricane coming ashore is $\frac{0.8}{0.2} = 4$, i.e., 4 to 1

60. (a) Since $\frac{7/9}{2/9} = \frac{7}{2}$, the odds in favor of hitting is 7 to 2,

(b) Odds against hitting is 2 to 7

61. (a) since $\frac{1/4}{3/4} = \frac{1}{3}$, the odds in favor of stock market going up is 1 to 3

(b) odds against market going up is 3 to 1

62. (a) since $\frac{4/5}{1/5} = 4$, the odds in favor of new taxes is 4 to 1

(b) odds against new taxes is 1 to 4

63. If p is the probability of rain today, then

$$\begin{aligned} \frac{p}{1-p} &= \frac{4}{1} \\ p &= 4 - 4p \\ p &= \frac{4}{5}. \end{aligned}$$

The probability of rain today is $\frac{4}{5}$ or 80%.

64. If p is the probability Big Blue wins, then

$$\begin{aligned} \frac{p}{1-p} &= \frac{2}{1} \\ p &= 2 - 2p \\ p &= \frac{2}{3}. \end{aligned}$$

The probability Big Blue wins is $\frac{2}{3}$.

65. (a) 1 to 9 (b) 9/10

66. (a) 1 to 7 (b) 1/8

67. Since $P(2\ heads) = \dfrac{C(4,2)}{2^4} = 3/8$ and $\dfrac{3/8}{5/8} = 3/5$, the odds in favor of getting 2 heads in four tosses is 3 to 5

68. Since $\dfrac{P(5)}{P(not\ a\ 5)} = \dfrac{1/6}{5/6} = 1/5$, the odds in favor of getting a 5 is 1 to 5.

69. Since $\dfrac{P(sum\ of\ 7)}{P(sum\ that\ is\ not\ 7)} = \dfrac{1/6}{5/6} = 1/5$, odds in favor of getting a sum of 7 is 1 to 5.

70. Since $\dfrac{P(at\ least\ one\ 4)}{P(no\ 4)} = \dfrac{11/36}{25/36} = 11/25$, odds in favor of getting at least one 4 is 11 to 25.

71. 1 to 1,999,999

72. 1 to $(C(49,6)-1)$ or 1 to 13,983,815

73. $\dfrac{1}{1+31} = \dfrac{1}{32}$ **74.** $\dfrac{5}{5+3} = \dfrac{5}{8}$

75. A and B are mutually exclusive if $P(A \cap B) = 0$. They are complementary events if $P(A) + P(B) = 1$

76. An odds in favor or against is a ratio of probabilities.

80. If S is the sample space, then $S = A \cup A'$. Thus, $1 = P(S) = P(A \cup A') = P(A) + P(A') - P(A \cap A')$ by the addition rule.

However, $A \cap A' = \emptyset$ and $P(\emptyset) = 0$. Thus, $P(A \cap A') = 0$ and so $1 = P(A) + P(A')$.

81. $C(5,3) = 10$

82. $3 \cdot 2 \cdot 2 = 12$ possible meals

83. $2^{12} = 4096$

84. The sum of an infinite geometric series is

$$\frac{a_1}{1-r} = \frac{1}{1-0.6} = 2.5$$

85. Geometric with common ratio $\dfrac{1}{2}$

86. If $x = 0.06767\ldots$, then $10x = 0.6767\ldots$ and $1000x = 67.6767\ldots$. If we subtract the second equation from the third, we obtain $990x = 67$. Thus, the fraction is $x = \dfrac{67}{990}$.

Thinking Outside the Box CII

We consider a 60-by-60 rectangular region in the xy-plane given by

$$R = \{(x,y) : 0 \le x, y \le 60\}.$$

Let $S = \{(x,y) \in R : |x-y| \le 10\}$. Then the probability that neither friend must wait more than 10 minutes for the other is

$$\frac{\text{Area of } S}{\text{Area of } R} = \frac{1100}{3600} = \frac{11}{36}.$$

11.6 Pop Quiz

1. $\left(\dfrac{1}{2}\right)^3 = \dfrac{1}{8}$

2. Since there are four possible outcomes, namely, $(3,6)$, $(6,3)$, $(4,5)$, and $(5,4)$, the probability of getting a sum of nine is $4/36$ or $1/9$.

3. Since $0.8 = 0.6 + 0.4 - P(A \cap B)$, we obtain $P(A \cap B) = 0.2$.

4. $\dfrac{7}{7+5}$ or $\dfrac{7}{12}$

5. Since there thirteen spades and nine face cards that are not spades, the probability of either a spade or a face card when drawing a single card is $\frac{13+9}{52}$ or $\frac{11}{26}$.

11.6 Linking Concepts

a) The answers may vary and depend on the outcome of each dumping of the pennies. In one trial, it was found that (where initially

$a_1 = 100$)

n	a_n
1	100
2	48
3	22
4	7
5	6
6	5
7	4
8	2
9	1
10	1
11	0

With a graphing calculator, a linear equation approximating the data is

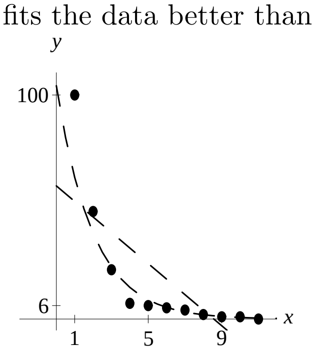
$$y = -6.936x + 59.436;$$

an exponential function approximating the data is given

$$y = 104.1587 \cdot (0.60691)^x$$

(with the point $(11, 0)$ excluded from the data since in a TI-83, ten is the maximum number of data points one can use for an exponential regression).

b) As shown in the graph, the exponential model fits the data better than the linear model.



c) One-half of the pennies, based on probability, is the expected number of pennies to be placed back in the can. One-half is close to 0.60691, a number which appears in the exponential equation $y = 104.1587 \cdot (0.60691)^x$.

d) The curve $y = 200\left(\frac{1}{2}\right)^x$, based on probability, represents the expected number of pennies on the xth dump. For example, initially when $x = 1$, there are $y = 100$ pennies on the first dump.

e) Since $200\left(\frac{1}{2}\right)^7 \approx 1.6$ and $200\left(\frac{1}{2}\right)^8 \approx 0.8$, one would expect no more pennies on the 8th dump.

For Thought

1. True, for when $n = 1$ we get $4 \cdot 1 - 2 = 2 \cdot 1^2$ which is a true staetment.

2. False, when $n = 3$ one gets $27 < 12 + 15$, which is inconsistent.

3. False, for $\frac{100-1}{100+1} \approx 0.98$.

4. False, for mathematical induction uses integers.

5. True **6.** False, rather $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$.

7. False **8.** False, since $0 > 0$ is inconsistent.

9. True

To see this, let S_n be the inequality $n^2 - n > 0$. Note S_2 is true since $2^2 - 2 > 0$ holds. Suppose S_k is true. We rewrite S_{k+1} as follows:

$$\begin{aligned}(k+1)^2 - (k+1) &= (k^2 + 2k + 1) - k - 1 \\ &= (k^2 - k) + 2k \\ &> 0 + 2k \quad \text{since } S_k \text{ is true} \\ &= 2k \\ &> 0.\end{aligned}$$

Then S_{k+1} is true if S_k is true. Thus, S_n is true for integers $n > 1$.

10. False, when $n = 1$ one gets $1^2 - 1 > 0$, which is inconsistent.

11.7 Exercises

1. If $n = 1$, then $3 - 1 = 2$ and $\frac{3 \cdot 1^2 + 1}{2} = \frac{4}{2}$.
If $n = 2$, then $(3-1) + (6-1) = 7$ and $\frac{3 \cdot 2^2 + 2}{2} = \frac{14}{2} = 7$.
If $n = 3$, then $(3-1) + (6-1) + (9-1) = 15$

and $\frac{3 \cdot 3^2 + 3}{2} = \frac{30}{2} = 15$.
It is true for $n = 1, 2$, and 3.

2. If $n = 1$, then $\frac{1}{2} = 1 - 2^{-1}$.

If $n = 2$, then $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ and $1 - 2^{-2} = 1 - \frac{1}{4} = \frac{3}{4}$.

If $n = 3$, then $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ and $1 - 2^{-3} = 1 - \frac{1}{8} = \frac{7}{8}$.

It is true for $n = 1, 2$, and 3.

3. If $n = 1$, then $\frac{1}{1(1+1)} = \frac{1}{1+1}$.

If $n = 2$, then $\frac{1}{2} + \frac{1}{6} = \frac{4}{6}$ and $\frac{2}{2+1} = \frac{2}{3}$.

If $n = 3$, then $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{9}{12}$ and $\frac{3}{3+1} = \frac{3}{4}$.

It is true for $n = 1, 2$, and 3.

4. If $n = 1$, $3^1 = 3$ and $\frac{3(3^1 - 1)}{2} = \frac{3(2)}{2} = 3$.

If $n = 2$, then $3 + 9 = 12$ and $\frac{3(3^2 - 1)}{2} = \frac{24}{2}$.

If $n = 3$, then $3 + 9 + 27 = 39$ and $\frac{3(3^3 - 1)}{2} = \frac{3(26)}{2} = 39$.

It is true for $n = 1, 2$, and 3.

5. If $n = 1$, one has $1 = 4 - 3$.

If $n = 2$, then $1 + 4 = 5$ and $4(2) - 3 = 5$.

If $n = 3$, then $1 + 4 + 9 = 14$ and $4(3) - 3 = 9$.

It is true for $n = 1, 2$ and false for $n = 3$.

6. If $n = 1$, one has $4 = 8 - 4$.

If $n = 2$, then $4 + 8 = 12$ and $8(2) - 4 = 12$.

If $n = 3$, then $4 + 8 + 12 = 24$ and $8(3) - 4 = 20$.

It is true for $n = 1, 2$ and false for $n = 3$.

7. If $n = 1$, one has $1 < 1$.

If $n = 2$, one has $4 < 8$. If $n = 3$, $9 < 27$.

It is true for $n = 2, 3$ and false for $n = 1$.

8. If $n = 1$, one has $(0.5)^0 = 1 > 0.5$

If $n = 2$, one has $0.5 > 0.5$ If $n = 3$, $0.25 > 0.5$.

It is true for $n = 1$ and false for $n = 2, 3$.

9. $S_1 : 1 = \frac{1(1+1)}{2}$ or $1 = \frac{2}{2}$ is a true statement.

$S_2 : 1 + 2 = \frac{2(2+1)}{2}$ or $3 = \frac{6}{2}$ is a true statement.

$S_3 : 1 + 2 + 3 = \frac{3(3+1)}{2}$ or $6 = \frac{12}{2}$ is a true statement.

$S_4 : 1 + 2 + 3 + 4 = \frac{4(4+1)}{2}$ or $10 = \frac{20}{2}$ is a true statement.

10. $S_1 : 1^2 = \frac{1(1+1)(2(1)+1)}{6}$ or $1 = \frac{6}{6}$ is a true statement.

$S_2 : 1^2 + 2^2 = \frac{2(2+1)(2(2)+1)}{6}$ or $5 = \frac{30}{6}$ is a true statement.

$S_3 : 1^2 + 2^2 + 3^2 = \frac{3(3+1)(2(3)+1)}{6}$ or $14 = \frac{84}{6}$ is a true statement.

$S_4 : 1^2 + 2^2 + 3^2 + 4^2 = \frac{4(4+1)(2(4)+1)}{6}$ or $30 = \frac{180}{6}$ is a true statement.

11. $S_1 : 1^3 = \frac{1^2(1+1)^2}{4}$ or $1 = \frac{4}{4}$ is a true statement.

$S_2 : 1^3 + 2^3 = \frac{2^2(2+1)^2}{4}$ or $9 = \frac{36}{4}$ is a true statement.

$S_3 : 1^3 + 2^3 + 3^3 = \frac{3^2(3+1)^2}{4}$ or $36 = \frac{144}{4}$ is a true statement.

$S_4 : 1^3 + 2^3 + 3^3 + 4^3 = \frac{4^2(4+1)^2}{4}$ or

$$100 = \frac{400}{4} \text{ is a true statement.}$$

12. $S_1 : 1^4 = \dfrac{1(1+1)(2(1)+1)(3(1)^2+3(1)-1)}{30}$

or $1 = \dfrac{30}{30}$ is a true statement.

$S_2 : 1^4 + 2^4 =$

$\dfrac{2(2+1)(2(2)+1)(3(2)^2+3(2)-1)}{30}$

or $17 = \dfrac{510}{30}$ is a true statement.

$S_3 : 1^4 + 2^4 + 3^4 =$

$\dfrac{3(3+1)(2(3)+1)(3(3)^2+3(3)-1)}{30}$

or $98 = \dfrac{2940}{30}$ is a true statement.

$S_4 : 1^4 + 2^4 + 3^4 + 4^4 =$

$\dfrac{4(4+1)(2(4)+1)(3(4)^2+3(4)-1)}{30}$

or $354 = \dfrac{10,620}{30}$ is a true statement.

13. $S_1 : 7^1 - 1$ divisible by 6 is true since the quotient $\dfrac{7^1-1}{6} = \dfrac{6}{6} = 1$ is a whole number.

$S_2 : 7^2 - 1$ divisible by 6 is true since the quotient $\dfrac{7^2-1}{6} = \dfrac{48}{6} = 8$ is a whole number.

$S_3 : 7^3 - 1$ divisible by 6 is true since the quotient $\dfrac{7^3-1}{6} = \dfrac{342}{6} = 57$ is a whole number.

$S_4 : 7^4 - 1$ divisible by 6 is true since the quotient $\dfrac{7^4-1}{6} = \dfrac{2400}{6} = 400$ is a whole number.

14. $S_1 : 3^1 > 2(1)$ or equivalently $3 > 2$ is a true statement.

$S_2 : 3^2 > 2(2)$ or equivalently $9 > 4$ is a true statement.

$S_3 : 3^3 > 2(3)$ or equivalently $27 > 6$ is a true statement.

$S_4 : 3^4 > 2(4)$ or equivalently $81 > 8$ is a true statement.

15. $S_1 : 2(1) = 1(1+1)$

$S_k : \sum_{i=1}^{k} 2i = k(k+1)$

$S_{k+1} : \sum_{i=1}^{k+1} 2i = (k+1)(k+2)$

16. $S_1 : 5(1) = \dfrac{5(2)}{2}$

$S_k : \sum_{i=1}^{k} 5i = \dfrac{5k(k+1)}{2}$

$S_{k+1} : \sum_{i=1}^{k+1} 5i = \dfrac{5(k+1)(k+2)}{2}$

17. $S_1 : 2 = 2 \cdot 1^2$
$S_k : 2 + 6 + ... + (4k-2) = 2k^2$

S_{k+1} :

$$\begin{aligned} 2 + 6 + ... + (4(k+1)-2) &= \\ 2 + 6 + ... + (4k+2) &= 2(k+1)^2 \end{aligned}$$

18. $S_1 : 3 = \dfrac{1(6)}{2}$

$S_k : 3 + 8 + ... + (5k-2) = \dfrac{k(5k+1)}{2}$

S_{k+1} :

$$\begin{aligned} 3 + ... + (5(k+1)-2) &= \frac{(k+1)(5(k+1)+1)}{2} \\ 3 + 8 + ... + (5k+3) &= \frac{(k+1)(5k+6)}{2} \end{aligned}$$

19. $S_1 : 2 = 2^2 - 2$

$S_k : \sum_{i=1}^{k} 2^i = 2^{k+1} - 2$

$S_{k+1} : \sum_{i=1}^{k+1} 2^i = 2^{k+2} - 2$

20. $S_1 : 5^2 = \dfrac{5^3 - 25}{4}$

$S_k : \sum_{i=1}^{k} 5^{i+1} = \dfrac{5^{k+2} - 25}{4}$

S_{k+1} :

$$\sum_{i=1}^{k+1} 5^{i+1} = \\ = \frac{5^{(k+1)+2} - 25}{4} \\ = \frac{5^{k+3} - 25}{4}$$

21. $S_1 : (ab)^1 = a^1b^1$
$S_k : (ab)^k = a^k b^k$
$S_{k+1} : (ab)^{k+1} = a^{k+1}b^{k+1}$

22. $S_1 : (a+b)^1 = a^1 + b^1$
$S_k : (a+b)^k = a^k + b^k$
$S_{k+1} : (a+b)^{k+1} = a^{k+1} + b^{k+1}$

23. S_1 : If $0 < a < 1$ then $0 < a^1 < 1$
S_k : If $0 < a < 1$ then $0 < a^k < 1$
S_{k+1} : If $0 < a < 1$ then $0 < a^{k+1} < 1$

24. S_1 : If $a > 1$ then $a^1 > 1$
S_k : If $a > 1$ then $a^k > 1$
S_{k+1} : If $a > 1$ then $a^{k+1} > 1$

25. Let $T_n : 1 + 2 + ... + n = \frac{n(n+1)}{2}$.

Step 1: If $n = 1$ then $T_1 : 1 = \frac{1(2)}{2}$.

So T_1 is true.

Step 2: Assume $T_k : 1 + 2 + ... + k = \frac{k(k+1)}{2}$ is true. Add $(k+1)$ to both sides. Then we obtain

$$1 + 2 + ... + k + (k+1) = \frac{k(k+1)}{2} + (k+1) \\ = \frac{k(k+1) + 2(k+1)}{2} \\ = \frac{(k+1)(k+2)}{2}.$$

Then the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

26. Let $T_n : 2 + 4 + ... + 2n = n(n+1)$.

Step 1: If $n = 1$ then $T_1 : 2 = 1(1+1)$.

So T_1 is true.

Step 2: Assume $T_k : 2 + 4 + ... + 2k = k(k+1)$ is true. Add $2(k+1)$ to both sides. Then

$$2 + 4 + ... + 2k + 2(k+1) = k(k+1) + 2(k+1) \\ = (k+1)(k+2).$$

Then the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

27. Let $T_n : 3 + 7 + ... + (4n-1) = n(2n+1)$.

Step 1: If $n = 1$ then $T_1 : 3 = 1(2+1)$.

Thus, T_1 is true.

Step 2: Assume T_k is true i.e. $3 + 7 + ... + (4k-1) = k(2k+1)$.
Note $4(k+1) - 1 = 4k + 3$. Adding $4k + 3$ to both sides, we get

$$3 + ... + (4k-1) + (4k+3) = \\ = k(2k+1) + (4k+3) \\ = 2k^2 + k + 4k + 3 \\ = 2k^2 + 5k + 3 \\ = (k+1)(2k+3) \\ = (k+1)(2(k+1)+1).$$

The truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

28. Let $T_n : 2 + 7 + 12 + ... + (5n-3) = \frac{n(5n-1)}{2}$.

Step 1: If $n = 1$ then $T_1 : 2 = \frac{1(5-1)}{2}$.

So T_1 is true.

Step 2: Assume $T_k : 2 + ... + (5k-3) = \frac{k(5k-1)}{2}$ is true. Note $5(k+1) - 3 = 5k + 2$.
Adding $5k + 2$ to both sides, we obtain

$$2 + ... + (5k-3) + (5k+2) = \\ = \frac{k(5k-1)}{2} + (5k+2) \\ = \frac{k(5k-1) + 2(5k+2)}{2} \\ = \frac{5k^2 + 9k + 4}{2} \\ = \frac{(k+1)(5k+4)}{2} \\ = \frac{(k+1)(5(k+1)-1)}{2}.$$

Then the truth of T_k implies the truth of T_{k+1}. Thus, T_n is true for every positive integer n.

29. Let $T_n : \sum_{i=1}^{n} 2^i = 2^{n+1} - 2$.

Step 1: If $n = 1$ then $T_1 : 2 = 2^2 - 2$.

So T_1 is true.

Step 2: Assume $T_k : \sum_{i=1}^{k} 2^i = 2^{k+1} - 2$ is true.

Then add 2^{k+1} to both sides.

$$\begin{aligned}\sum_{i=1}^{k} 2^i + 2^{k+1} &= 2^{k+1} - 2 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 2 \\ &= 2^{k+2} - 2 \\ &= 2^{(k+1)+1} - 2\end{aligned}$$

Thus, the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

30. Let $T_n : \sum_{i=1}^{n} 5^{i+1} = \frac{5^{n+2} - 25}{4}$.

Step 1: If $n = 1$ then $T_1 : 5^2 = \frac{5^3 - 25}{4}$.

So T_1 is true.

Step 2: Assume $T_k : \sum_{i=1}^{k} 5^{i+1} = \frac{5^{k+2} - 25}{4}$ is true. Then add 5^{k+2} to both sides.

$$\begin{aligned}\sum_{i=1}^{k} 5^{i+1} + 5^{k+2} &= \frac{5^{k+2} - 25}{4} + 5^{k+2} \\ &= \frac{5^{k+2} - 25 + 4 \cdot 5^{k+2}}{4} \\ &= \frac{5 \cdot 5^{k+2} - 25}{4} \\ &= \frac{5^{k+3} - 25}{4} \\ &= \frac{5^{(k+1)+2} - 25}{4}\end{aligned}$$

Thus, the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

31. Let $T_n : \sum_{i=1}^{n} (3i - 1) = \frac{3n^2 + n}{2}$.

Step 1: If $n = 1$ then $T_1 : 3 - 1 = \frac{3 + 1}{2}$.

So T_1 is true.

Step 2: Assume $T_k : \sum_{i=1}^{k} (3i - 1) = \frac{3k^2 + k}{2}$ is true. Note $3(k + 1) - 1 = 3k + 2$. Then add $3k + 2$ to both sides.

$$\begin{aligned}\sum_{i=1}^{k} (3i - 1) + 3k + 2 &= \frac{3k^2 + k}{2} + (3k + 2) \\ &= \frac{3k^2 + k + 2(3k + 2)}{2} \\ &= \frac{3k^2 + 7k + 4}{2} \\ &= \frac{3(k^2 + 2k + 1) + (k + 1)}{2} \\ &= \frac{3(k + 1)^2 + (k + 1)}{2}\end{aligned}$$

Thus, the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

32. Let $T_n : \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i = 1 - 2^{-n}$.

Step 1: If $n = 1$ then $T_1 : \frac{1}{2} = 1 - 2^{-1}$.

So T_1 is true.

Step 2: Assume $T_k : \sum_{i=1}^{k} \left(\frac{1}{2}\right)^i = 1 - 2^{-k}$ is true.

Add $\left(\frac{1}{2}\right)^{k+1}$ to both sides.

$$\begin{aligned}\sum_{i=1}^{k} \left(\frac{1}{2}\right)^i + \left(\frac{1}{2}\right)^{k+1} &= 1 - 2^{-k} + \left(\frac{1}{2}\right)^{k+1} \\ &= 1 - 2 \cdot 2^{-(k+1)} + 2^{-(k+1)} \\ &= 1 - 2^{-(k+1)}\end{aligned}$$

Then the truth of T_k implies the truth of T_{k+1}. Thus, T_n is true for every positive integer n.

33. Let $T_n : 1^2 + 2^2 + ... + n^2 = \frac{n(n + 1)(2n + 1)}{6}$.

Step 1: If $n = 1$ then $T_1 : 1 = \frac{1(2)(3)}{6}$.

So T_1 is true.

Step 2: Assume T_k is true i.e. we assume $1^2 + 2^2 + ... + k^2 = \frac{k(k + 1)(2k + 1)}{6}$

Add $(k+1)^2$ to both sides.

$1^2 + 2^2 + ... + k^2 + (k+1)^2 =$

$$\begin{aligned} &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\ &= \frac{(k+1)[k+2][2k+3]}{6} \\ &= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} \end{aligned}$$

Thus, the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

34. Let $T_n : \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$.

Step 1: If $n = 1$ then $T_1 : \frac{1}{1\cdot 2} = \frac{1}{1+1}$.

So T_1 is true.

Step 2: Assume T_k is true i.e. we assume

$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{k\cdot(k+1)} = \frac{k}{k+1}$.

Add $\frac{1}{(k+1)(k+2)}$ to both sides.

$\frac{1}{1\cdot 2} + ... + \frac{1}{n\cdot(n+1)} + \frac{1}{(k+1)(k+2)} =$

$$\begin{aligned} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{(k+1)+1} \end{aligned}$$

Then the truth of T_k implies the truth of T_{k+1}. Thus, T_n is true for every positive integer n.

35. Let $T_n : 1\cdot 3 + 2\cdot 4 + ... + n\cdot(n+2) = \frac{n}{6}(n+1)(2n+7)$.

Step 1: If $n = 1$ then $T_1 : 1\cdot 3 = \frac{1}{6}(2)(2+7)$.

So T_1 is true.

Step 2: Assume T_k is true i.e. we assume

$1\cdot 3 + 2\cdot 4 + ... + k\cdot(k+2) = \frac{k}{6}(k+1)(2k+7)$.

Add $(k+1)(k+3)$ to both sides. Then

$1\cdot 3 + 2\cdot 4 + ... + k\cdot(k+2) + (k+1)(k+3) =$

$$\begin{aligned} &= \frac{k}{6}(k+1)(2k+7) + (k+1)(k+3) \\ &= \frac{k(k+1)(2k+7) + 6(k+1)(k+3)}{6} \\ &= \frac{(k+1)(2k^2 + 13k + 18)}{6} \\ &= \frac{(k+1)(2k+9)(k+2)}{6} \\ &= \frac{(k+1)}{6}[k+2][2k+9] \\ &= \frac{(k+1)}{6}[(k+1)+1][2(k+1)+7] \end{aligned}$$

So the truth of T_k implies the truth of T_{k+1}. Then T_n is true for every positive integer n.

36. Let $T_n : \frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + .. + \frac{1}{(3n-2)\cdot(3n+1)} = \frac{n}{3n+1}$.

Step 1: If $n = 1$ then $T_1 : \frac{1}{1\cdot 4} = \frac{1}{3+1}$.

So T_1 is true.

Step 2: Assume T_k is true, i.e., we assume

$\frac{1}{1\cdot 4} + ... + \frac{1}{(3k-2)\cdot(3k+1)} = \frac{k}{3k+1}$.

Note $\frac{1}{[3(k+1)-2]\cdot[3(k+1)+1]} = \frac{1}{[3k+1][3k+4]}$. Then add $\frac{1}{[3k+1][3k+4]}$ to both sides. Then $\frac{1}{1\cdot 4} + ...$

$.. + \frac{1}{(3k-2)\cdot(3k+1)} + \frac{1}{[3k+1][3k+4]} =$

$$= \frac{k}{3k+1} + \frac{1}{[3k+1][3k+4]}$$

$$= \frac{k(3k+4)+1}{(3k+1)(3k+4)}$$
$$= \frac{3k^2+4k+1}{(3k+1)(3k+4)}$$
$$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$
$$= \frac{k+1}{3(k+1)+1}.$$

Thus, the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

37. Let $T_n : \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + ... + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$.

Step 1: If $n = 1$ then $T_1 : \frac{1}{1\cdot 3} = \frac{1}{2+1}$.

So T_1 is true.

Step 2: Assume T_k is true i.e. we assume $\frac{1}{1\cdot 3} + ... + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$.

Note $\frac{1}{[2(k+1)-1][2(k+1)+1]} = \frac{1}{(2k+1)(2k+3)}$. Then add $\frac{1}{(2k+1)(2k+3)}$ to both sides.

So $\frac{1}{1\cdot 3} + ...$
$.. + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} =$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$
$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$
$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$
$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$
$$= \frac{k+1}{2(k+1)+1}.$$

Thus, the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

38. Let $T_n : \frac{1}{1\cdot 2\cdot 3} + ... + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$.

Step 1: If $n = 1$ then $T_1 : \frac{1}{1\cdot 2\cdot 3} = \frac{1(4)}{4(2)(3)}$.

So T_1 is true.

Step 2: Assume T_k is true, i.e., we assume $\frac{1}{1\cdot 2\cdot 3} + ... + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$.

Add $\frac{1}{(k+1)(k+2)(k+3)}$ to both sides.

Then $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + ... + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} =$

$$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$
$$= \frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}$$
$$= \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$$
$$= \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$
$$= \frac{[k+1][(k+1)+3]}{4[(k+1)+1][(k+1)+2]}.$$

Thus, the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

39. Let T_n : If $0 < a < 1$ then $0 < a^n < 1$.

Step 1: Clearly, T_1 is true for $n = 1$.

Step 2: Assume $0 < a < 1$ implies $0 < a^k < 1$, i.e., assume T_k is true. Multiply $0 < a^k < 1$ by a to get $0 < a^{k+1} < a$. Since $a < 1$ and by transitivity, we get $0 < a^{k+1} < 1$.

Then the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

40. Let T_n : If $a > 1$ then $a^n > 1$.

Step 1: Clearly, T_1 is true for $n = 1$.

Step 2: Assume $a > 1$ implies $a^k > 1$, i.e., assume T_k is true. Multiply $a^k > 1$ by a to get $a^{k+1} > a$.

Since $a > 1$, by transitivity we get $a^{k+1} > 1$.

Then the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

41. Let $T_n : n < 2^n$.

Step 1: If $n = 1$, then $1 < 2^1$. So T_1 is true.

Step 2: Assume $k < 2^k$, i.e., assume T_k is true. Add 1 to both sides.

$$\begin{aligned} k+1 &< 2^k + 1 \\ &< 2^k + 2^k \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

Then $k + 1 < 2^{k+1}$.
So, the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

42. Let $T_n : 2^{n-1} \leq n!$.

Step 1: If $n = 1$, then $2^0 \leq 1!$. So T_1 is true.

Step 2: Assume $2^{k-1} \leq k!$, i.e., assume T_k is true. Multiply both sides by 2.

$$\begin{aligned} 2^k &\leq 2 \cdot k! \\ &\leq (k+1) \cdot k! \ \text{ since } 2 \leq k+1 \\ &= (k+1)! \end{aligned}$$

Then $2^k \leq (k+1)!$.
So, the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

43. Let $T_n : 5^n - 1$ is divisible by 4.

Step 1: If $n = 1$, then $5^1 - 1$ is divisible by 4. So T_1 is true.

Step 2: Assume $5^k - 1$ is divisible by 4, i.e., assume T_k is true.
Observe that $5^{k+1} - 1 = 5(5^k - 1) + 4$. Since sums of multiples of 4 are again multiples of 4, we get that $5^{k+1} - 1$ is a multiple of 4.

Then the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

44. Let $T_n : 7^n - 1$ is divisible by 6.

Step 1: If $n = 1$ then $7^1 - 1$ is divisible by 6. So T_1 is true.

Step 2: Assume $7^k - 1$ is divisible by 6, i.e., assume T_k is true.
Observe that $7^{k+1} - 1 = 7(7^k - 1) + 6$. Since sums of multiples of 6 are again multiples of 6, it follows that $7^{k+1} - 1$ is a multiple of 6.

Thus, the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

45. Let $T_n : (ab)^n = a^n b^n$.

Step 1: If $n = 1$ then $(ab)^1 = a^1 b^1$. So T_1 is true.

Step 2: Assume $(ab)^k = a^k b^k$, i.e., assume T_k is true. Then

$$\begin{aligned} (ab)^{k+1} &= (ab)^k (ab) \\ &= a^k b^k (ab) \text{ since } T_k \text{ is true} \\ &= a^{k+1} b^{k+1}. \end{aligned}$$

Thus, the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

46. Let $T_n : (a^m)^n = a^{mn}$.

Step 1: If $n = 1$ then $(a^m)^1 = a^m$. So T_1 is true.

Step 2: Assume $(a^m)^k = a^{mk}$, i.e., assume T_k is true. Then we get

$$\begin{aligned} (a^m)^{k+1} &= (a^m)^k a^m \\ &= a^{mk} a^m \text{ since } T_k \text{ is true} \\ &= a^{mk+m} \\ &= a^{m(k+1)}. \end{aligned}$$

Thus, the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

47. Let $T_n : \sum_{i=0}^{n} x^i = \dfrac{x^{n+1} - 1}{x - 1}$, $x \neq 1$.

Step 1: If $n = 1$, then $T_1 : 1 + x = \dfrac{x^2 - 1}{x - 1}$.

T_1 is true since $(x^2 - 1) = (x - 1)(x + 1)$.

Step 2: Assume $T_k : \sum_{i=0}^{k} x^i = \dfrac{x^{k+1}-1}{x-1}$ is true. Add x^{k+1} to both sides.

$$\begin{aligned}\sum_{i=0}^{k} x^i + x^{k+1} &= \frac{x^{k+1}-1}{x-1} + x^{k+1} \\ &= \frac{x^{k+1}-1+(x-1)x^{k+1}}{x-1} \\ &= \frac{x^{k+1}-1+x^{k+2}-x^{k+1}}{x-1} \\ &= \frac{x^{k+2}-1}{x-1}\end{aligned}$$

Then the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

48. Suppose $S = \{1, 2, 3, ..., n\}$ where n is a positive integer. Let T_n be the statement that 2^n is the number of subsets of a set with n elements.

Step 1: If $n = 1$, then $S = \{1\}$ and the subsets of S are $\{1\}$ and $\emptyset$. Thus, S has two or 2^1 subsets. So, T_1 is true.

Step 2: Assume T_k is a true statement, i.e., $S = \{1, 2, 3, ..., k\}$ has 2^k subsets. Note, if $A \subseteq S$ then $A \cup \{k+1\}$ and A are subsets of $S \cup \{k+1\}$. Conversely, each subset of $\{1, 2, 3, ..., k+1\}$ is either a subset of S or is the union of $\{k+1\}$ with a subset of S. Thus, the number of subsets of $\{1, 2, 3, ..., k+1\}$ is two times the number of subsets of S. That is, $\{1, 2, 3, ..., k+1\}$ has $2 \cdot 2^k$ or 2^{k+1} subsets. So, the truth of T_k implies the truth of T_{k+1}. Hence, T_n is true for every positive integer n.

51. The probability of getting the first answer correct is $\frac{1}{2}$. Since the ten questions are independent, the probability of getting all ten correct answers is $\left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$.

52. Since the odds in favor of rain today are 3 to 1, the probability of rain today is $\frac{3}{3+1} = \frac{3}{4}$ or 75%.

53. The probability is $1 - \dfrac{1}{5,000,000} = \dfrac{4,999,999}{5,000,000}$

54. Using Pascal's triangle, the coefficients of $(a+b)^6$ are 1, 6, 15, 20, 15, 6, and 1.

Then $(a-b)^6 =$

$a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.

55. Arithmetic, since $d = -4$ is the common difference.

56. If $x = 3.\overline{125}$, then $1000x = 3125.\overline{125}$. Subtract the first equation from the second equation. Then $999x = 3122$ or $x = \dfrac{3122}{999}$.

Thinking Outside the Box CIII

Since $(5!)^3 = 1,728,000$, the units digit and tens digits of

$$(1!)^3 + (2!)^3 + ... + (101!)^3$$

are the same as the units digit and tens digits, respectively, of

$$(1!)^3 + (2!)^3 + (3!)^3 + (4!)^3 = 14,049.$$

Thus, the units digit is 9, and the tens digit is 4.

11.7 Pop Quiz

1. Let T_n be the statement

$$T_n : \sum_{i=1}^{n} (2i-1) = n^2.$$

Step 1: If $n = 1$, then $T_1 : 2(1) - 1 = 1^2$.

Clearly, T_1 is a true statement.

Step 2: Assume $T_k : \sum_{i=1}^{k} (2i-1) = k^2$ is true. Add $2(k+1)-1$, or equivalently $2k+1$, to both sides of T_k.

$$\begin{aligned}\sum_{i=1}^{k} (2i-1) + 2k + 1 &= k^2 + 2k + 1 \\ \sum_{i=1}^{k+1} (2i-1) &= (k+1)^2.\end{aligned}$$

Then the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

Review Exercises

1. One gets $a_1 = 2^0 = 1$, $a_2 = 2^1 = 2$, $a_3 = 2^2 = 4$, $a_4 = 2^3 = 8$, $a_5 = 2^4 = 16$

First five terms are $1, 2, 4, 8, 16$

2. One gets $a_1 = 3 - 2 = 1$, $a_2 = 6 - 2 = 4$, $a_3 = 9 - 2 = 7$, $a_4 = 12 - 2 = 10$.

First four terms are $1, 4, 7, 10$.

3. $a_1 = \frac{(-1)^1}{1!} = -1$, $a_2 = \frac{(-1)^2}{2!} = \frac{1}{2}$, $a_3 = \frac{(-1)^3}{3!} = \frac{-1}{6}$, $a_4 = \frac{(-1)^4}{4!} = \frac{1}{24}$.

First four terms are $-1, \frac{1}{2}, -\frac{1}{6}, \frac{1}{24}$.

4. $a_1 = (-1)^2 = 1$, $a_2 = 0^2 = 0$, $a_3 = 1^2 = 1$, $a_4 = 2^2 = 4$, $a_5 = 3^2 = 9$, and $a_6 = 4^2 = 16$.

First six terms are $1, 0, 1, 4, 9, 16$

5. $a_1 = 3(0.5)^0 = 3$, $a_2 = 3(0.5)^1 = 1.5$, and $a_3 = 3(0.5)^2 = 0.75$.

First three terms are $3, 1.5, 0.75$.

6. $a_1 = \frac{1}{1(2)} = \frac{1}{2}$, $a_2 = \frac{1}{2(3)} = \frac{1}{6}$, and $a_3 = \frac{1}{3(4)} = \frac{1}{12}$.

First three terms are $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}$.

7. $a_1 = -3 + 6 = 3$, $a_2 = -6 + 6 = 0$, and $a_3 = -9 + 6 = -3$.

First three terms are $3, 0, -3$.

8. $a_1 = \frac{(-1)^1}{1^2} = -1$, $a_2 = \frac{(-1)^2}{2^2} = \frac{1}{4}$, and $a_3 = \frac{(-1)^3}{3^2} = \frac{-1}{9}$.

First three terms are $-1, \frac{1}{4}, -\frac{1}{9}$.

9. Since $S = \frac{a_1(1 - r^n)}{1 - r}$ is the sum of a geometric series, $S = \frac{0.5(1 - 0.5^4)}{1 - 0.5} = 0.9375$.

10. Since $S = \frac{n}{2}(a_1 + a_n)$ is the sum of an arithmetic series, $S = \frac{3}{2}(4 + 14) = 27$.

11. Since $S = \frac{n}{2}(a_1 + a_n)$ is the sum of an arithmetic series, $S = \frac{50}{2}(11 + 207) = 5450$.

12. Sum is $6 \cdot 4 = 24$.

13. Since $S = \frac{a_1}{1 - r}$ is the sum of a geometric series, $S = \frac{0.3}{1 - 0.1} = \frac{0.3}{0.9} = \frac{1}{3}$.

14. Since $S = \frac{a_1}{1 - r}$ is the sum of a geometric series, $S = \frac{-4}{1 - (-0.8)} = \frac{-4}{1.8} = -\frac{40}{18} = -\frac{20}{9}$.

15. Since $S = \frac{a_1(1 - r^n)}{1 - r}$ is the sum of a geometric series, $S = \frac{1000(1 - 1.05^{20})}{1 - 1.05} \approx 33,065.9541$

16. Since $S = \frac{a_1(1 - r^n)}{1 - r}$ is the sum of a geometric series, $S = \frac{2(1 - (1/3)^{10})}{1 - 1/3} \approx 2.999949$.

17. $a_n = \frac{(-1)^n}{n + 2}$

18. $a_n = 20 - 3(n - 1) = -3n + 23$

19. $a_n = 6\left(\frac{1}{6}\right)^{n-1}$ **20.** $a_n = (-1)^{n+1}n^2$

21. $\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i + 1}$ **22.** $\sum_{i=1}^{\infty} 5\left(\frac{1}{2}\right)^{i-1}$

23. $\sum_{i=1}^{14} 2i$ **24.** $\sum_{i=1}^{n} i^2$

25. Note, $a_n = a_1 r^{n-1}$. Since $256 = 4r^6$, we get $64 = r^6$ and the common ratio is $r = \pm 2$.

26. Note, $a_n = a_1 + d(n - 1)$. Since $29 = 5 + 6d$, $24 = 6d$ and the common difference is $d = 4$.

27. $100\left(1 + \frac{0.09}{4}\right)^{40} = \$243.52$

28. $40,000\left(1+\frac{0.06}{12}\right)^{108}=\$68,547.98$

29. $\sum_{i=1}^{10} 1000(1.06)^i=\frac{1000(1.06)(1-1.06^{10})}{1-1.06}$
$=\$13,971.64$

30. $\sum_{i=1}^{240} 50\left(1+\frac{0.06}{12}\right)^i=\sum_{i=1}^{240} 50(1.005)^i=$
$\frac{50(1.005)(1-1.005^{240})}{1-1.005}=\$23,217.55$

31. If the pattern continues, the total number of Tummy Masters that could be sold is
$\sum_{i=0}^{\infty} 100,000(0.9)^i=\frac{100,000}{1-0.9}=1$ million.

32. Suppose the ball has a small radius. The maximum distance it will travel is given by

$$\begin{aligned}6+2[6(0.97)+6(0.97)^2+6(0.97)^3+...] &=\\ 6+12[0.97+0.97^2+0.97^3+...] &=\\ 6+12\sum_{i=1}^{\infty}(0.97)^i &=\\ 6+12\frac{0.97}{1-0.97} &=\\ 394 \text{ feet.}\end{aligned}$$

33. $a^4+\binom{4}{1}a^3(2b)+$
$\binom{4}{2}a^2(2b)^2+\binom{4}{3}a(2b)^3+(2b)^4=$
$a^4+8a^3b+24a^2b^2+32ab^3+16b^4$

34. $x^3+\binom{3}{1}x^2(-5)+\binom{3}{2}x^1(-5)^2+(-5)^3$
$=x^3-15x^2+75x-125$

35. $(2a)^5+\binom{5}{1}(2a)^4(-b)+\binom{5}{2}(2a)^3(-b)^2+$
$\binom{5}{3}(2a)^2(-b)^3+\binom{5}{4}(2a)(-b)^4+(-b)^5=$
$32a^5-80a^4b+80a^3b^2-40a^2b^3+10ab^4-b^5$

36. $w^6+\binom{6}{1}w^5(2)+\binom{6}{2}w^4(2)^2+$
$\binom{6}{3}w^3(2)^3+\binom{6}{4}w^2(2)^4+\binom{6}{5}w(2)^5$
$+2^6=w^6+12w^5+60w^4+160w^3+240w^2+$
$192w+64$

37. $a^{10}+\binom{10}{1}a^9b+\binom{10}{2}a^8b^2+...=$
$a^{10}+10a^9b+45a^8b^2+...$

38. $x^9+\binom{9}{1}x^8(-2y)+\binom{9}{2}x^7(-2y)^2+...=$
$x^9-18x^8y+144x^7y^2+...$

39. $(2x)^8+\binom{8}{1}(2x)^7\left(\frac{y}{2}\right)+\binom{8}{2}(2x)^6\left(\frac{y}{2}\right)^2+$
$...$
$=256x^8+512x^7y+448x^6y^2+...$

40. $(2a)^7+\binom{7}{1}(2a)^6(-3b)+$
$\binom{7}{2}(2a)^5(-3b)^2+...=$
$128a^7-1344a^6b+6048a^5b^2+...$

41. $\binom{13}{9}=715$

42. $\binom{15}{7}2^8(-1)^7=-1,647,360$

43. $\frac{11!}{2!3!6!}2^3=36,960$

44. $\frac{12!}{5!7!}=792$ **45.** 24 terms

46.

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

47. $5^9=1,953,125$ ways to mark the answers

48. $6!=720$ ways to line up for departure

49. $P(7,3) = 210$ possible three-letter 'words'

50. $5 \cdot 6 = 30$ possible teams

51. $3 \cdot 5 \cdot 4 = 60$ ways to place advertisements

52. $P(9,3) = 504$ different signals and note that the order of the flags is important

53. $C(8,5) = 56$ possible vacations where order is not taken into account

54. There are $C(7,4) = 35$ different subsets

55. (a) $C(8,4) = 70$ possible councils,
(b) $C(5,4) = 5$ possible councils from Democrats,
(c) $C(5,2) \cdot C(3,2) = 30$ possible councils with two Democrats and two Republicans

56. $C(4,3) \cdot C(4,2) = 4 \cdot 6 = 24$ such full houses

57. For KANSAS, $\frac{6!}{2!2!} = 180$ arrangements; for TEXAS, $5! = 120$ arrangements.

58. $\frac{8!}{2!2!4!} = 420$ ways to mark the pickups

59. $2^7 = 128$ different families

60. $3! = 6$ arrangements for a triple feature

61. Since there are 2^{10} ways to answer the test, the probabilities are $P(\textit{all } 10 \textit{ correct}) = \frac{1}{1024}$ and $P(\textit{all } 10 \textit{ wrong}) = \frac{1}{1024}$.

62. There are 2^6 or 64 possible families with six children according to sex and order of birth. Then $P(\textit{all } 6 \textit{ girls}) = 1/64$

63. (a) 5/13, (b) 8/13, (c) 0, (d) 1

64. (a) Note there are 9 outcomes that show two odd numbers. So $P(\textit{at least one even})$ $= 1 - P(\textit{both odd}) = 1 - \frac{9}{36} = \frac{3}{4}$.
(b) Note there are 18 outcomes whose sum is even. So $P(\textit{sum is even}) = \frac{18}{36} = \frac{1}{2}$.
(c) Note there are 3 outcomes whose sum is four. So $P(\textit{sum is four}) = \frac{3}{36} = \frac{1}{12}$.
(d) The only possible outcome is $(2,2)$. Then $P(\textit{sum is } 4 \textit{ and at least one even}) = \frac{1}{36}$.
(e) Use (c), (a), and (d).
$P(\textit{sum is } 4 \textit{ or at least one even}) =$
$\frac{1}{12} + \frac{3}{4} - \frac{1}{36} = \frac{29}{36}$

65. Assuming the probabilities of a boy or girl being born are equal, $P(3 \textit{ boys}) = \frac{1}{8}$. Then the odds in favor of 3 boys is $\frac{1/8}{7/8}$, i.e., 1 to 7.

66. Note, $P(\textit{six}) = \frac{5}{36}$. Then the odds in favor of six is $\frac{5/36}{31/36}$, i.e., 5 to 31.

67. Odds in favor of catching a perch is $\frac{0.9}{0.1}$, i.e., 9 to 1.

68. Odds against attending college is $\frac{0.2}{0.8}$, i.e., 1 to 4.

69. By the Addition Rule, $P(\textit{Math orEnglish}) =$ $0.7 + 0.6 - 0.4 = 0.9$, i.e., 90%.

70. Note, $P(5) = \frac{4}{36}$ and $P(6) = \frac{5}{36}$.
By the Addition Rule, $P(5 \textit{ or } 6) =$
$\frac{4}{36} + \frac{5}{36} = \frac{1}{4}$.

71. 40,320 **72.** 1

73. 20 **74.** $\frac{7!}{4!} = 210$

75. 1680 **76.** 2520

77. $\frac{8!}{2!6!} = 28$ **78.** $\frac{12!}{9!3!} = 220$

79. $\frac{8!}{4!} = 1680$ **80.** $\frac{4!}{0!} = 24$

81. $\frac{8!}{7!1!} = 8$ **82.** $\frac{12!}{0!12!} = 1$

83. Let $T_n : 3 + 6 + 9 + ... + 3n = \frac{3}{2}(n^2 + n)$.

Step 1: If $n = 1$, then $T_1 : 3 = \frac{3}{2}(1 + 1)$.

So T_1 is true.

Step 2: Assume $T_k : 3+6+...+3k = \frac{3}{2}(k^2+k)$ is true. Add $3(k + 1)$ to both sides.

$3 + 6 + ... + 3k + 3(k + 1) =$

$$\begin{aligned} &= \frac{3}{2}(k^2 + k) + 3(k + 1) \\ &= \frac{3(k^2 + k) + 6(k + 1)}{2} \\ &= \frac{3k^2 + 9k + 6}{2} \\ &= \frac{3}{2}(k^2 + 3k + 2) \\ &= \frac{3}{2}((k + 1)^2 + (k + 1)) \end{aligned}$$

Then the truth of T_k implies the truth of T_{k+1}. T_n is true for every positive integer n.

84. Let $T_n : \sum_{i=1}^{n} 2(3)^{i-1} = 3^n - 1$.

Step 1: If $n = 1$, then $T_1 : 2 = 3^1 - 1$.

So T_1 is true.

Step 2: Assume $T_k : \sum_{i=1}^{k} 2(3)^{i-1} = 3^k - 1$ is true. Add $2(3)^k$ to both sides.

$$\begin{aligned} \sum_{i=1}^{k} 2(3)^{i-1} + 2(3)^k &= 3^k - 1 + 2(3)^k \\ &= 3(3)^k - 1 \\ &= 3^{k+1} - 1 \end{aligned}$$

Then the truth of T_k implies the truth of T_{k+1}. Thus, T_n is true for every positive integer n.

Thinking Outside the Box CIV

The 1st generation ancestor of a male bee is described by the ordered pair (0 male, 1 female).

The 2nd generation ancestors of a male bee is described by the ordered pair (1 male, 1 female).

The 3rd generation ancestors of a male bee is described by the ordered pair (1 male, 2 females).

The 4th generation ancestors of a male bee is described by the ordered pair (2 males, 3 females).

A formula from the nth generation to the $(n + 1)$st generation ancestors is given by the rule

$$(x \text{ males}, y \text{ females}) \text{ to } (y \text{ males}, x + y \text{ females}).$$

Using the above rule, we find that the 10th generation ancestors of a male bee is described by the ordered pair (34 males, 55 females).

Thus, the number of ancestors going back 10 generations is

$$1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 + 89 = 231.$$

Chapter 11 Test

1. $a_1 = 2.3$, $a_2 = 2.3 + 0.5$, $a_3 = 2.3 + 1$, and $a_4 = 2.3 + 1.5$. First four terms are 2.3, 2.8, 3.3, 3.8.

2. One finds $c_2 = \frac{1}{2} \cdot 20 = 10$,

$c_3 = \frac{1}{2} \cdot 10 = 5$, and $c_4 = \frac{1}{2} \cdot 5 = 2.5$.

First four terms are 20, 10, 5, 2.5.

3. $a_n = (-1)^{n-1}(n - 1)^2$

4. $a_n = 7 + 3(n - 1) = 3n + 4$

5. $a_n = \frac{1}{3}\left(-\frac{1}{2}\right)^{n-1}$

6. Since $S = \frac{n}{2}(a_1 + a_n)$, the sum is $\frac{54}{2}(-5 + 154) = 4023$.

7. Since $S = \frac{a_1(1 - r^n)}{1 - r}$, the sum is

$$\frac{300(1.05)(1 - 1.05^{23})}{1 - 1.05} \approx 13,050.5997.$$

8. Since $S = \frac{a_1}{1 - r}$, the sum is $\frac{0.98}{1 - 0.98} = 49$.

9. Since $a_n = a_1 + d(n-1)$, $9 = -3 + 8d$.
Solving for d, one gets $d = 1.5$.
So $a_n = -3 + 1.5(n-1)$ or

$$a_n = 1.5n - 4.5.$$

10. $\frac{9!}{4!2!2!} = 3780$ possible nine-letter 'words'

11. Since $S = \frac{n}{2}(a_1 + a_n)$ is the sum of an arithmetic series, the mean daily sales is

$$\frac{1}{30}\sum_{i=0}^{29} 300 + 10i = \frac{1}{30}\frac{30}{2}(300 + 590) = \$445$$

12. There are $10^4 = 10,000$ possible secret numbers. The probability of guessing the correct number in 3 tries is the probability of getting it in the 1st try plus the probability of getting it in the 2nd try plus the probability of getting it in the 3rd try. Assuming the person remembers what he tries and tries a different number after a failure, this probability is
$\frac{1}{10,000} + \frac{9,999}{10,000} \cdot \frac{1}{9,999} + \frac{9,999}{10,000} \cdot \frac{9,998}{9,999} \cdot$
$\frac{1}{9,998} = \frac{3}{10,000}$.

13. At the end of the 300th month the value of the annuity is $\sum_{i=1}^{300} 700\left(1 + \frac{0.06}{12}\right)^i =$
$\sum_{i=1}^{300} 700(1.005)^i = 700(1.005)\left(\frac{1 - 1.005^{300}}{1 - 1.005}\right)$
$= \$487,521.25$.
The cost of the house at the end of 25 years is

$$120,000(1.06)^{25} = \$515,024.49.$$

No, they will not have enough money to buy the house.

14. $a^5 + \binom{5}{1}a^4(-2x) + \binom{5}{2}a^3(-2x)^2 +$
$\binom{5}{3}a^2(-2x)^3 + \binom{5}{4}a(-2x)^4 + (-2x)^5 =$
$a^5 - 10a^4x + 40a^3x^2 - 80a^2x^3 + 80ax^4 - 32x^5$

15. $x^{24} + \binom{24}{1}x^{23}(y^2) + \binom{24}{2}x^{22}(y^2)^2 + ... =$
$x^{24} + 24x^{23}y^2 + 276x^{22}y^4 + ...$

16. $\sum_{i=0}^{30}\binom{30}{i}m^{30-i}y^i$

17. Note, $P(sum\ is\ 7) = \frac{6}{36} = \frac{1}{6}$. The odds in favor of 7 is $\frac{6/36}{30/36} = \frac{1}{5}$, i.e., 1 to 5.

18. $C(12,3) = 220$ selections

19. $C(12,2) \cdot C(10,2) = 66 \cdot 45 = 2970$ outcomes

20. $\frac{1}{P(8,3)} = \frac{1}{336}$ is the probability of getting the horses and the order of finish correctly.

21. Let $T_n : \sum_{i=1}^{n}\left(\frac{1}{2}\right)^i = 1 - 2^{-n}$.

Step 1: If $n = 1$, then $T_1 : \frac{1}{2} = 1 - 2^{-1}$.
So T_1 is true.

Step 2: Assume $T_k : \sum_{i=1}^{k}\left(\frac{1}{2}\right)^i = 1 - 2^{-k}$ is true.
Add $\left(\frac{1}{2}\right)^{k+1}$ to both sides.

$$\begin{aligned}\sum_{i=1}^{k}\left(\frac{1}{2}\right)^i + \left(\frac{1}{2}\right)^{k+1} &= 1 - 2^{-k} + \left(\frac{1}{2}\right)^{k+1} \\ &= 1 - 2 \cdot 2^{-(k+1)} + 2^{-(k+1)} \\ &= 1 - 2^{-(k+1)}\end{aligned}$$

Then the truth of T_k implies the truth of T_{k+1}. Thus, T_n is true for every positive integer n.
Since $1 - 2^{-n} < 1$ for all positive integers n, then by transitivity we obtain $\sum_{i=1}^{n}\left(\frac{1}{2}\right)^i < 1$ for all positive integers n.

Concepts of Calculus

1.

a.

n	1	2	3	4	5
$1/n^2$	1	1/4	1/9	1/16	1/25

b. Since $\frac{1}{n^2}$ gets closer to zero as n gets larger, we conclude $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

2.

a.

n	1	2	3	4	5
$1/n^3$	1	1/8	1/27	1/64	1/125

Since $\frac{1}{n^3}$ gets closer to zero as n gets larger, we conclude $\lim_{n\to\infty} \frac{1}{n^3} = 0$.

b.

n	1	2	3	4	5
$5/(3n)$	5/3	5/6	5/9	5/12	5/15

Since $\frac{5}{3n}$ gets closer to zero as n gets larger, we obtain $\lim_{n\to\infty} \frac{5}{3n} = 0$.

c.

n	1	2	3	4	5
$2n+1$	3	5	7	9	11

Since $2n+1$ increases without bound as n gets larger, the limit $\lim_{n\to\infty}(2n+1)$ does not exist.

d.

n	1	2	3	4	5
$(2n+1)/n$	3	5/2	7/3	9/4	11/5

Since $\frac{2n+1}{n}$ gets closer to two as n gets larger, we obtain $\lim_{n\to\infty} \frac{2n+1}{n} = 2$.

e.

n	1	2	3	4	5
$\cos(n\pi/2)$	0	-1	0	1	0

Since $\cos\left(\frac{n\pi}{2}\right)$ does not approach exactly one fixed number as n gets larger, the limit $\lim_{n\to\infty} \cos\left(\frac{n\pi}{2}\right)$ does not exist.

f.

n	1	2	3	4	5
$n\sin(1/n)$	0.841	0.959	0.982	0.990	0.993

Since $n\sin\left(\frac{1}{n}\right)$ approaches one as n gets larger, we obtain $\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = 1$.

g.

n	1	2	3	4	5	20
$\frac{n+1}{n-1}$	undefined	3	2	5/3	3/2	21/19

Since $\frac{n+1}{n-1}$ approaches one as n gets larger, we obtain $\lim_{n\to\infty} \frac{n+1}{n-1} = 1$.

h.

n	1	2	3	4	5	20
$\frac{n^2+5}{n-5}$	$-3/2$	-3	-7	-21	undefined	27

Since $\frac{n^2+5}{n-5}$ increases without bound as n gets larger, the limit $\lim_{n\to\infty} \frac{n^2+5}{n-5}$ does not exist.

3.

a. Since $\frac{1}{2n}$ gets closer to zero as n gets larger, we conclude $\lim_{n\to\infty} \frac{1}{2n} = 0$.

b. Since $\ln(n)$ increases without bound as n gets larger, it follows that $\lim_{n\to\infty} \ln(n)$ does not exist.

c. Since $\tan(n\pi) = 0$ for each integer n, $\lim_{n\to\infty} \tan(n\pi) = 0$.

d. Since $-1 < \sin(n) < 1$, it follows that $\frac{\sin(n)}{n}$ approaches zero as n gets larger. Thus, $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$.

e. Note, $\frac{4n^2+5}{3n^2-1} = \frac{4+\frac{5}{n^2}}{3-\frac{1}{n^2}}$ for $n > 0$.

Since $\frac{5}{n^2}$ and $\frac{1}{n^2}$ both approach zero as n gets larger, it follows that

$$\lim_{n\to\infty} \frac{4n^2+5}{3n^2-1} = \lim_{n\to\infty} \frac{4+\frac{5}{n^2}}{3-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{4+0}{3-0} = \frac{4}{3}.$$

f. Note, $\frac{n-3}{6n+5} = \frac{1-\frac{3}{n}}{6+\frac{5}{n}}$ for $n > 0$.

Since $\frac{3}{n}$ and $\frac{5}{n}$ both approach zero as n gets larger, it follows that

$$\lim_{n\to\infty} \frac{n-3}{6n+5} = \lim_{n\to\infty} \frac{1-\frac{3}{n}}{6+\frac{5}{n}} = \lim_{n\to\infty} \frac{1-0}{6+0} = \frac{1}{6}.$$

g. Note, $\frac{n^3+99}{n-1} = \frac{n^2+\frac{99}{n}}{1-\frac{1}{n}}$ for $n > 0$.

Since $\frac{99}{n}$ and $\frac{1}{n}$ both approach zero as n gets larger, it follows that $\frac{n^2+\frac{99}{n}}{1-\frac{1}{n}}$, which is approximately n^2, must increase without bound as n gets larger. Thus, $\lim_{n\to\infty} \frac{n^3+99}{n-1}$. does not exist.

h. Note, $\frac{n+5000}{n^2} = \frac{1+5000/n}{n}$ for $n > 0$.

Since $\frac{5000}{n}$ approaches zero as n gets larger, it follows that $\frac{1+5000/n}{n}$, which is approximately $\frac{1}{n}$, approaches zero as n gets larger. Thus, we obtain $\lim_{n\to\infty} \frac{n+5000}{n^2} = 0$.