For Thought

1. False, vertex is $(0, -1/2)$. **2.** True

3. True, since $p = 3/2$ and the focus is $(4, 5)$, vertex is $(4 - 3/2, 5) = (5/2, 5)$.

4. False, focus is at $(0, 1/4)$ since parabola opens upward. **5.** True, $p = 1/4$.

6. False. Since $p = 4$ and the vertex is $(2, -1)$, equation of parabola is $y = \frac{1}{16}(x-2)^2 - 1$ and x-intercepts are $(6, 0)$, $(-2, 0)$.

7. False, if $x = 0$ then $y = 0$ and y-intercept is $(0, 0)$. **8.** True

9. False; since $p = 1/4$ and the vertex is $(5, 4)$, the focus is $(5, 4 + 1/4) = (-5, 17/4)$.

10. False, it opens to the left.

10.1 Exercises

1. conic section

2. parabola

3. axis of symmetry

4. vertex

5. Note $p = 1$.
Vertex $(0, 0)$, focus $(0, 1)$, and directrix $y = -1$

6. Note $p = -1$.
Vertex $(0, 0)$, focus $(0, -1)$, and directrix $y = 1$

7. Note $p = -1/2$. Vertex $(1, 2)$, focus $(1, 3/2)$, and directrix $y = 5/2$

8. Note $p = 1/2$. Vertex $(-2, 1)$, focus $(-2, 3/2)$, and directrix $y = 1/2$

9. Note $p = 3/4$. Vertex $(3, 1)$, focus $(15/4, 1)$, and directrix $x = 9/4$

10. Note $p = -1/4$. Vertex $(1, 2)$, focus $(3/4, 2)$, and directrix $x = 5/4$

11. Since the vertex is equidistant from $y = -2$ and $(0, 2)$, the vertex is $(0, 0)$ and $p = 2$. Then $a = \frac{1}{4p} = \frac{1}{8}$ and an equation is $y = \frac{1}{8}x^2$.

12. Since the vertex is equidistant from $y = -1$ and $(0, 1)$, the vertex is $(0, 0)$ and $p = 1$. Then $a = \frac{1}{4p} = \frac{1}{4}$ and an equation is $y = \frac{1}{4}x^2$.

13. Since the vertex is equidistant from $y = 3$ and $(0, -3)$, the vertex is $(0, 0)$ and $p = -3$. Then $a = \frac{1}{4p} = -\frac{1}{12}$ and an equation is $y = -\frac{1}{12}x^2$.

14. Since the vertex is equidistant from $(0, -1)$ and $y = 1$, the vertex is $(0, 0)$ and $p = -1$. Then $a = \frac{1}{4p} = -\frac{1}{4}$ and an equation is $y = -\frac{1}{4}x^2$.

15. One finds $p = \frac{3}{2}$. So $a = \frac{1}{4p} = \frac{1}{6}$ and vertex is $\left(3, 5 - \frac{3}{2}\right) = \left(3, \frac{7}{2}\right)$. Parabola is given by $y = \frac{1}{6}(x-3)^2 + \frac{7}{2}$.

16. One finds $p = 1$. So $a = \frac{1}{4p} = \frac{1}{4}$ and vertex is $(-1, 5 - 1) = (-1, 4)$. Parabola is given by $y = \frac{1}{4}(x+1)^2 + 4$.

17. One finds $p = -\frac{5}{2}$. So $a = \frac{1}{4p} = -\frac{1}{10}$ and vertex is $\left(1, -3 + \frac{5}{2}\right) = \left(1, -\frac{1}{2}\right)$. Parabola is given by $y = -\frac{1}{10}(x-1)^2 - \frac{1}{2}$.

18. One finds $p = -2$. So $a = \frac{1}{4p} = -\frac{1}{8}$ and vertex is $(1, -4 + 2) = (1, -2)$. Parabola is given by $y = -\frac{1}{8}(x-1)^2 - 2$

19. One finds $p = 0.2$. So $a = \frac{1}{4p} = 1.25$ and vertex is $(-2, 1.2-0.2) = (-2,1)$. Parabola is given by $y = 1.25(x+2)^2 + 1$.

20. One finds $p = \dfrac{1}{8}$. So $a = \dfrac{1}{4p} = 2$ and vertex is $\left(3, \dfrac{9}{8} - \dfrac{1}{8}\right) = (3,1)$. Parabola is given by $y = 2(x-3)^2 + 1$.

21. Since $p = 1$, we get $a = \dfrac{1}{4p} = \dfrac{1}{4}$.

An equation is $y = \dfrac{1}{4}x^2$.

22. Since $p = -2$, we get $a = \dfrac{1}{4p} = -\dfrac{1}{8}$.

An equation is $y = -\dfrac{1}{8}x^2$.

23. Since $p = -\dfrac{1}{4}$, we get $a = \dfrac{1}{4p} = -1$.

An equation is $y = -x^2$

24. Since $p = \dfrac{1}{16}$, we get $a = \dfrac{1}{4p} = 4$.

An equation is $y = 4x^2$.

25. Note, vertex is $(1,0)$ and since $a = \dfrac{1}{4p} = 1$, we find $p = \dfrac{1}{4}$. The focus is $(1, 0+p) = \left(1, \dfrac{1}{4}\right)$ and the directrix is $y = 0 - p$ or $y = -\dfrac{1}{4}$.

26. Note, vertex is $(-2,0)$ and since $a = \dfrac{1}{4p} = 1$, we find $p = \dfrac{1}{4}$. The focus is $(-2, 0+p) = \left(-2, \dfrac{1}{4}\right)$ and the directrix is $y = 0 - p$ or $y = -\dfrac{1}{4}$.

27. Note, vertex is $(3,0)$ and since $a = \dfrac{1}{4p} = \dfrac{1}{4}$, we find $p = 1$. The focus is $(3, 0+p) = (3,1)$ and the directrix is $y = 0 - p$ or $y = -1$.

28. Note, vertex is $(-5,0)$ and since $a = \dfrac{1}{4p} = \dfrac{1}{2}$, we obtain $p = \dfrac{1}{2}$. The focus is $(-5, 0+p) = \left(-5, \dfrac{1}{2}\right)$ and the directrix is $y = 0 - p$ or $y = -\dfrac{1}{2}$.

29. Note, vertex is $(3,4)$ and since $a = \dfrac{1}{4p} = -2$, we find $p = -\dfrac{1}{8}$. The focus is $(3, 4+p) = \left(3, \dfrac{31}{8}\right)$ and the directrix is $y = 4 - p$ or $y = \dfrac{33}{8}$.

30. Note, vertex is $(1,3)$ and since $a = \dfrac{1}{4p} = -4$, we find $p = -\dfrac{1}{16}$. The focus is $(1, 3+p) = \left(1, \dfrac{47}{16}\right)$ and the directrix is $y = 3 - p$ or $y = \dfrac{49}{16}$.

31. Completing the square, we obtain

$$\begin{aligned} y &= (x^2 - 8x + 16) - 16 + 3 \\ y &= (x-4)^2 - 13. \end{aligned}$$

Since $\dfrac{1}{4p} = 1$, $p = 0.25$.

Since vertex is $(4, -13)$, focus is $(4, -13 + 0.25) = (4, -51/4)$, and directrix is $y = -13 - p = -53/4$.

32. Completing the square, we find

$$\begin{aligned} y &= (x^2 + 2x + 1) - 1 - 5 \\ y &= (x+1)^2 - 6. \end{aligned}$$

Since $\dfrac{1}{4p} = 1$, $p = 1/4$.

Since vertex is $(-1, -6)$, focus is $(-1, -6 + 1/4) = (-1, -23/4)$, and directrix is $y = -6 - 1/4 = -25/4$.

33. Completing the square, we obtain

$$\begin{aligned} y &= 2(x^2+6x+9)+5-18 \\ y &= 2(x+3)^2-13. \end{aligned}$$

Since $\dfrac{1}{4p}=2$, $p=1/8$.

Since vertex is $(-3,-13)$, the focus is $(-3,-13+1/8)=(-3,-103/8)$, and directrix is $y=-13-1/8=-105/8$.

34. Completing the square, we get

$$\begin{aligned} y &= 3(x^2+4x+4)+1-12 \\ y &= 3(x+2)^2-11. \end{aligned}$$

Since $\dfrac{1}{4p}=3$, $p=1/12$.

Since vertex is $(-2,-11)$, the focus is $(-2,-11+1/12)=(-2,-131/12)$, and directrix is $y=-11-1/12=-133/12$.

35. Completing the square, we get

$$\begin{aligned} y &= -2(x^2-3x+9/4)+1+9/2 \\ y &= -2(x-3/2)^2+11/2. \end{aligned}$$

Since $\dfrac{1}{4p}=-2$, $p=-1/8$.

Since vertex is $(3/2,11/2)$, the focus is $(3/2,11/2-1/8)=(3/2,43/8)$, and directrix is $y=11/2+1/8=45/8$.

36. Completing the square,

$$\begin{aligned} y &= -3(x^2+2x+1)+5+3 \\ y &= -3(x+1)^2+8. \end{aligned}$$

Since $\dfrac{1}{4p}=-3$, $p=-1/12$.

Since vertex is $(-1,8)$, the focus is $(-1,8-1/12)=(-1,95/12)$, and directrix is $y=8+1/12=97/12$.

37. Completing the square,

$$\begin{aligned} y &= 5(x^2+6x+9)-45 \\ y &= 5(x+3)^2-45. \end{aligned}$$

Since $\dfrac{1}{4p}=5$, we have $p=0.05$.

Since vertex is $(-3,-45)$, the focus is $(-3,-45+0.05)=(-3,-44.95)$, and directrix is $y=-45-.05=-45.05$.

38. Completing the square,

$$\begin{aligned} y &= -2(x^2-6x+9)+18 \\ y &= -2(x-3)^2+18. \end{aligned}$$

Since $\dfrac{1}{4p}=-2$, we get $p=-1/8$.

Since vertex is $(3,18)$, the focus is $(3,18-1/8)=(3,143/8)$, and directrix is $y=18+1/8=145/8$.

39. Completing the square, we get

$$\begin{aligned} y &= \frac{1}{8}(x^2-4x+4)+\frac{9}{2}-\frac{1}{2} \\ y &= \frac{1}{8}(x-2)^2+4. \end{aligned}$$

Since $\dfrac{1}{4p}=1/8$, we find $p=2$.

Since vertex is $(2,4)$, the focus is $(2,4+2)=(2,6)$, and directrix is $y=4-2$ or $y=2$.

40. Completing the square, we get

$$\begin{aligned} y &= \frac{1}{4}(x^2+2x+1)-\frac{7}{4}-\frac{1}{4} \\ y &= \frac{1}{4}(x+1)^2-2. \end{aligned}$$

Since $\dfrac{1}{4p}=1/4$, we find $p=1$.

Since vertex is $(-1,-2)$, the focus is $(-1,-2+1)=(-1,-1)$, and directrix is $y=-2-1$ or $y=-3$.

41. Since $a = 1$ and $b = -4$, we find $x = \dfrac{-b}{2a} = \dfrac{4}{2} = 2$. Since $\dfrac{1}{4p} = a = 1$, we obtain $p = 1/4$. Substituting $x = 2$, we get $y = 2^2 - 4(2) + 3 = -1$. Thus, the vertex is $(2, -1)$, the focus is $(2, -1 + p) = (2, -3/4)$, the directrix is $y = -1 - p = -5/4$, and the parabola opens upward since $a > 0$.

42. Note, $a = 1$, $b = -6$. So $x = \dfrac{-b}{2a} = \dfrac{6}{2} = 3$ and since $\dfrac{1}{4p} = a = 1$, $p = 1/4$.
Substituting $x = 3$, $y = 3^2 - 6(3) - 7 = -16$.
The vertex is $(3, -16)$ and
focus is $(3, -16 + p) = (3, -63/4)$,
directrix is $y = -16 - p = -65/4$, and parabola opens up since $a > 0$.

43. Note, $a = -1$, $b = 2$. So $x = \dfrac{-b}{2a} = \dfrac{-2}{-2} = 1$ and since $\dfrac{1}{4p} = a = -1$, $p = -1/4$.
Substituting $x = 1$, $y = -(1)^2 + 2(1) - 5 = -4$.
The vertex is $(1, -4)$ and
focus is $(1, -4 + p) = (1, -17/4)$,
directrix is $y = -4 - p = -15/4$, and parabola opens down since $a < 0$.

44. Note, $a = -1$, $b = 4$. So $x = \dfrac{-b}{2a} = \dfrac{-4}{-2} = 2$ and since $\dfrac{1}{4p} = a = -1$, $p = -1/4$.
Substituting $x = 2$, $y = -(2)^2 + 4(2) + 3 = 7$.
The vertex is $(2, 7)$ and
focus is $(2, 7 + p) = (2, 27/4)$,
directrix is $y = 7 - p = 29/4$, and parabola opens down since $a < 0$.

45. Note, $a = 3$, $b = -6$. So $x = \dfrac{-b}{2a} = \dfrac{6}{6} = 1$ and since $\dfrac{1}{4p} = a = 3$, $p = 1/12$. Substituting $x = 1$, $y = 3(1)^2 - 6(1) + 1 = -2$.
The vertex is $(1, -2)$ and
focus is $(1, -2 + p) = (1, -23/12)$,
directrix is $y = -2 - p = -25/12$, and parabola opens up since $a > 0$.

46. Note, $a = 2$, $b = 4$. So $x = \dfrac{-b}{2a} = \dfrac{-4}{4} = -1$ and since $\dfrac{1}{4p} = a = 2$, $p = 1/8$. Substituting $x = -1$, $y = 2(-1)^2 + 4(-1) - 1 = -3$.
The vertex is $(-1, -3)$ and
focus is $(-1, -3 + p) = (-1, -23/8)$,
directrix is $y = -3 - p = -25/8$, and parabola opens up since $a > 0$.

47. Note, $a = -1/2$, $b = -3$. So $x = \dfrac{-b}{2a} = \dfrac{3}{-1} = -3$ and since $\dfrac{1}{4p} = a = -1/2$, $p = -1/2$. Substituting $x = -3$, we have $y = -\dfrac{1}{2}(-3)^2 - 3(-3) + 2 = \dfrac{13}{2}$. The vertex is $\left(-3, \dfrac{13}{2}\right)$ and focus is $\left(-3, \dfrac{13}{2} + p\right) = (-3, 6)$, directrix is $y = \dfrac{13}{2} - p = 7$, and parabola opens down since $a < 0$.

48. Note, $a = -1/2$, $b = 3$.
So $x = \dfrac{-b}{2a} = \dfrac{-3}{-1} = 3$ and since $\dfrac{1}{4p} = a = -1/2$ then $p = -1/2$.
Substituting $x = 3$, we get $y = -\dfrac{1}{2}(3)^2 + 3(3) - 1 = \dfrac{7}{2}$.
The vertex is $\left(3, \dfrac{7}{2}\right)$ and
focus is $\left(3, \dfrac{7}{2} + p\right) = (3, 3)$,
directrix is $y = \dfrac{7}{2} - p = 4$, and
parabola opens down since $a < 0$.

49. Note, $y = \frac{1}{4}x^2 + 5$ is of the form $y = a(x-h)^2 + k$. So $h = 0$, $k = 5$, and $\frac{1}{4p} = a = \frac{1}{4}$ from which we have $p = 1$. The vertex is $(h, k) = (0, 5)$, focus is $(0, 5+p) = (0, 6)$, directrix is $y = 5 - p = 4$ and parabola opens up since $a > 0$.

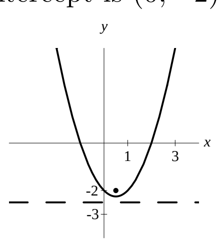
50. Note, $y = -\frac{1}{8}x^2 - 6$ is of the form $y = a(x-h)^2 + k$. So $h = 0$, $k = -6$, and $\frac{1}{4p} = a = -\frac{1}{8}$ from which we have $p = -2$. The vertex is $(h, k) = (0, -6)$, focus is $(0, -6+p) = (0, -8)$, directrix is $y = -6 - p = -4$ and parabola opens down since $a < 0$.

51. From the given focus and directrix, one finds $p = 1/4$. So $a = \frac{1}{4p} = 1$, vertex is $(h, k) = (1/2, -2-p) = (1/2, -9/4)$, axis of symmetry is $x = 1/2$, and parabola is given by $y = a(x-h)^2 + k = \left(x - \frac{1}{2}\right)^2 - \frac{9}{4}$.

If $y = 0$ then $x - \frac{1}{2} = \pm\frac{3}{2}$ or $x = 2, -1$.

The x-intercepts are $(2, 0), (-1, 0)$.

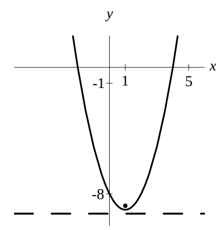
If $x = 0$ then $y = \left(0 - \frac{1}{2}\right)^2 - \frac{9}{4} = -2$ and y-intercept is $(0, -2)$.



52. From the given focus and directrix one finds $p = 1/4$. So $a = \frac{1}{4p} = 1$, vertex is $(h, k) = (1, -35/4 - p) = (1, -9)$, axis of symmetry is $x = 1$, and parabola is given by $y = a(x-h)^2 + k = (x-1)^2 - 9$.

If $y = 0$ then $x - 1 = \pm 3$ or $x = 4, -2$. x-intercepts are $(4, 0), (-2, 0)$.

If $x = 0$ then $y = (0-1)^2 - 9 = -8$ and y-intercept is $(0, -8)$.



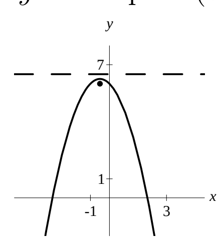
53. From the given focus and directrix one finds $p = -1/4$. So $a = \frac{1}{4p} = -1$, vertex is $(h, k) = (-1/2, 6-p) = (-1/2, 25/4)$, axis of symmetry is $x = -1/2$, and parabola is given by

$$y = a(x-h)^2 + k = -\left(x + \frac{1}{2}\right)^2 + \frac{25}{4}.$$

If $y = 0$ then $x + \frac{1}{2} = \pm\frac{5}{2}$ or $x = -3, 2$.

The x-intercepts are $(-3, 0), (2, 0)$.

If $x = 0$ then $y = -\left(0 + \frac{1}{2}\right)^2 + \frac{25}{4} = 6$ and y-intercept is $(0, 6)$.

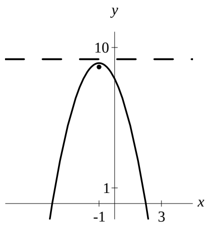


54. From the given focus and directrix one finds $p = -1/4$. So $a = \frac{1}{4p} = -1$, vertex is $(h, k) = (-1, 35/4 - p) = (-1, 9)$, axis of symmetry is $x = -1$, and parabola is given by $y = a(x-h)^2 + k = -(x+1)^2 + 9$.

If $y = 0$ then $x + 1 = \pm 3$ or $x = 2, -4$.

The x-intercepts are $(2, 0), (-4, 0)$.

If $x = 0$ then $y = -(0+1)^2 + 9 = 8$ and y-intercept is $(0, 8)$.

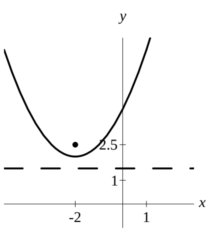


55. Since $\frac{1}{2}(x+2)^2+2$ is of the form $a(x-h)^2+k$, vertex is $(h,k)=(-2,2)$ and axis of symmetry is $x=-2$. If $y=0$ then $0=\frac{1}{2}(x+2)^2+2$; this has no solution since left-hand side is always positive. No x-intercept. If $x=0$ then

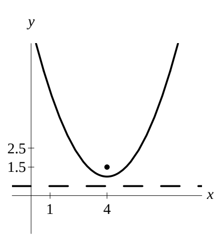
$y=\frac{1}{2}(0+2)^2+2=4$. y-intercept is $(0,4)$.

Since $\frac{1}{4p}=a=\frac{1}{2}$, $p=\frac{1}{2}$, focus is $(h,k+p)=$

$(-2,5/2)$, and directrix is $y=k-p=3/2$.



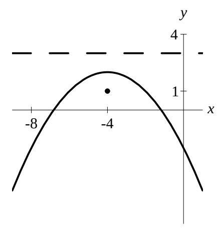
56. Since $\frac{1}{2}(x-4)^2+1$ is of the form $a(x-h)^2+k$, vertex is $(h,k)=(4,1)$ and axis of symmetry is $x=4$. If $y=0$ then $0=\frac{1}{2}(x-4)^2+1$; this has no solution since right-hand side is always positive. No x-intercept. If $x=0$ then $y=\frac{1}{2}(0-4)^2+1=9$. y-intercept is $(0,9)$. Since $\frac{1}{4p}=a=\frac{1}{2}$, $p=\frac{1}{2}$, focus is $(h,k+p)=$ $(4,3/2)$, and directrix is $y=k-p=1/2$.



57. Since $-\frac{1}{4}(x+4)^2+2$ is of the form $a(x-h)^2+k$, vertex is $(h,k)=(-4,2)$ and axis of symmetry is $x=-4$. If $y=0$ then

$$\begin{aligned}\frac{1}{4}(x+4)^2 &= 2\\ x+4 &= \pm\sqrt{8}.\end{aligned}$$

x-intercepts are $(-4\pm 2\sqrt{2},0)$. If $x=0$, then $y=-\frac{1}{4}(0+4)^2+2=-2$. The y-intercept is $(0,-2)$. Since $\frac{1}{4p}=a=-\frac{1}{4}$, $p=-1$, focus is $(h,k+p)=(-4,1)$, and directrix is $y=k-p=3$.



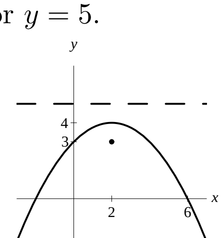
58. Since $-\frac{1}{4}(x-2)^2+4$ is of the form $a(x-h)^2+k$, vertex is $(h,k)=(2,4)$ and axis of symmetry is $x=2$. If $y=0$, then

$$\begin{aligned}\frac{1}{4}(x-2)^2 &= 4\\ x-2 &= \pm\sqrt{16}.\end{aligned}$$

x-intercepts are $(6,0),(-2,0)$. If $x=0$, then $y=-\frac{1}{4}(0-2)^2+4=3$. y-intercept is $(0,3)$.

Since $\frac{1}{4p}=a=-\frac{1}{4}$, $p=-1$, focus is

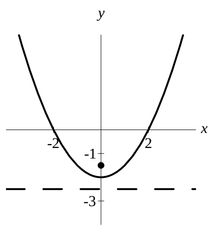
$(h,k+p)=(2,3)$, and directrix is $y=k-p=$ 5 or $y=5$.



59. Since $\frac{1}{2}x^2-2$ is of the form $a(x-h)^2+k$, vertex is $(h,k)=(0,-2)$ and axis of symmetry is $x=0$. If $y=0$, then

$$\begin{aligned}\frac{1}{2}x^2 &= 2\\ x^2 &= 4.\end{aligned}$$

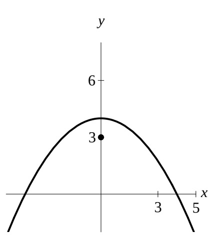
x-intercepts are $(\pm 2, 0)$. If $x = 0$ then $y = \frac{1}{2}(0)^2 - 2 = -2$. The y-intercept is $(0, -2)$. Since $\frac{1}{4p} = a = \frac{1}{2}$, $p = 1/2$, focus is $(h, k+p) = (0, -3/2)$, and directrix is $y = k - p = -5/2$.



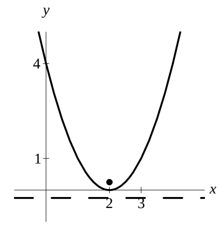
60. Since $-\frac{1}{4}x^2 + 4$ is of the form $a(x-h)^2 + k$, vertex is $(h, k) = (0, 4)$ and axis of symmetry is $x = 0$. If $y = 0$ then

$$\begin{aligned} \frac{1}{4}x^2 &= 4 \\ x^2 &= 16. \end{aligned}$$

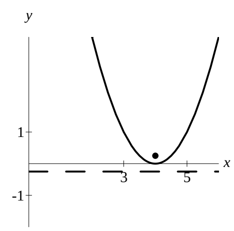
x-intercepts are $(\pm 4, 0)$. If $x = 0$ then $y = -\frac{1}{4}(0)^2 + 4 = 4$. The y-intercept is $(0, 4)$. Since $\frac{1}{4p} = a = -\frac{1}{4}$, $p = -1$, focus is $(h, k+p) = (0, 3)$, and directrix is $y = k - p = 5$ or $y = 5$.



61. Since $y = (x-2)^2$ is of the form $a(x-h)^2 + k$, vertex is $(h, k) = (2, 0)$, and axis of symmetry is $x = 2$. If $y = 0$ then $(x-2)^2 = 0$ and x-intercept is $(2, 0)$. If $x = 0$ then $y = (0-2)^2 = 4$ and y-intercept is $(0, 4)$. Since $\frac{1}{4p} = a = 1$, $p = 1/4$, focus is $(h, k+p) = (2, 1/4)$, and directrix is $y = k - p = -1/4$.



62. Since $y = (x-4)^2$ is of the form $a(x-h)^2 + k$, vertex is $(h, k) = (4, 0)$ and axis of symmetry is $x = 4$. If $y = 0$ then $(x-4)^2 = 0$ and x-intercept is $(4, 0)$. If $x = 0$ then $y = (0-4)^2 = 16$ and y-intercept is $(0, 16)$. Since $\frac{1}{4p} = a = 1$, $p = 1/4$, focus is $(h, k+p) = (4, 1/4)$, and directrix is $y = k - p = -1/4$.



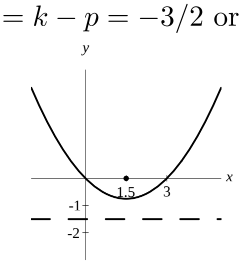
63. By completing the square, we obtain

$$y = \frac{1}{3}(x - 3/2)^2 - 3/4.$$

Vertex is $(h, k) = (3/2, -3/4)$ and axis of symmetry is $x = 3/2$. If $y = 0$, then

$$\begin{aligned} \frac{1}{3}\left(x - \frac{3}{2}\right)^2 &= \frac{3}{4} \\ \left(x - \frac{3}{2}\right)^2 &= \frac{9}{4} \\ x &= \frac{3}{2} \pm \frac{3}{2}. \end{aligned}$$

x-intercepts are $(3, 0), (0, 0)$. If $x = 0$, then $y = \frac{1}{3}\left(0 - \frac{3}{2}\right)^2 - \frac{3}{4} = 0$ and y-intercept is $(0, 0)$. Since $\frac{1}{4p} = a = 1/3$, $p = 3/4$, focus is $(h, k+p) = (3/2, 0)$, and directrix is $y = k - p = -3/2$ or $y = -3/2$.



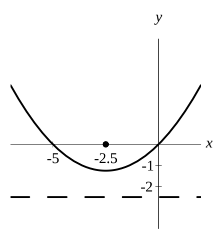
64. By completing the square, we obtain

$$y = \frac{1}{5}(x + 5/2)^2 - 5/4.$$

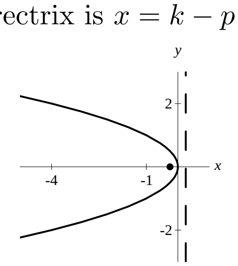
Vertex is $(h, k) = (-5/2, -5/4)$ and axis of symmetry is $x = -5/2$. If $y = 0$, then

$$\begin{aligned} \frac{1}{5}\left(x + \frac{5}{2}\right)^2 &= \frac{5}{4} \\ \left(x + \frac{5}{2}\right)^2 &= \frac{25}{4} \\ x &= -\frac{5}{2} \pm \frac{5}{2}. \end{aligned}$$

x-intercepts are $(-5, 0), (0, 0)$. If $x = 0$ then $y = \frac{1}{5}\left(0 + \frac{5}{2}\right)^2 - \frac{5}{4} = 0$ and y-intercept is $(0, 0)$. Since $\frac{1}{4p} = a = 1/5$, $p = 5/4$, focus is $(h, k + p) = (-5/2, 0)$, and directrix is $y = k - p$ or $y = -5/2$.



65. Since $x = -y^2$ is of the form $x = a(y-h)^2 + k$, vertex is $(k, h) = (0, 0)$ and axis of symmetry is $y = 0$. If $y = 0$ then $x = -0^2 = 0$ and x-intercept is $(0, 0)$. If $x = 0$ then $0 = -y^2$ and y-intercept is $(0, 0)$. Since $\frac{1}{4p} = a = -1$, $p = -1/4$, focus is $(k + p, h) = (-1/4, 0)$, and directrix is $x = k - p = 1/4$.



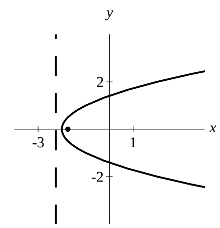
66. Since $x = y^2 - 2$ is of the form

$$x = a(y - h)^2 + k,$$

the vertex is $(k, h) = (-2, 0)$ and axis of symmetry is $y = 0$. If $y = 0$ then $x = 0^2 - 2 = -2$ and x-intercept is $(-2, 0)$. If $x = 0$ then $y^2 = 2$ and y-intercepts are $(0, \pm\sqrt{2})$.

Since $\frac{1}{4p} = a = 1$, we get $p = 1/4$,

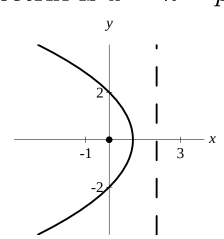
focus is $(k + p, h) = (-7/4, 0)$, and directrix is $x = k - p = -9/4$.



67. Since $x = -\frac{1}{4}y^2 + 1$ is of the form

$$x = a(y - h)^2 + k,$$

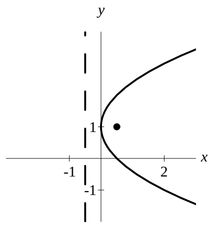
vertex is $(k, h) = (1, 0)$ and axis of symmetry is $y = 0$. If $y = 0$ then $x = 1$ and x-intercept is $(1, 0)$. If $x = 0$ then $\frac{1}{4}y^2 = 1$, $y^2 = 4$, and y-intercepts are $(0, \pm 2)$. Since $\frac{1}{4p} = a = -\frac{1}{4}$, $p = -1$, focus is $(k + p, h) = (0, 0)$, and directrix is $x = k - p = 2$ or $x = 2$.



68. Since $x = \frac{1}{2}(y - 1)^2$ is of the form

$$x = a(y - h)^2 + k,$$

vertex is $(k, h) = (0, 1)$ and axis of symmetry is $y = 1$. If $y = 0$ then $x = \frac{1}{2}(-1)^2 = \frac{1}{2}$ and x-intercept is $(1/2, 0)$. If $x = 0$ then $\frac{1}{2}(y - 1)^2 = 0$, $y = 1$, and y-intercept is $(0, 1)$. Since $\frac{1}{4p} = a = \frac{1}{2}$, $p = 1/2$, focus is $(k + p, h) = (1/2, 1)$, and directrix is $x = k - p = -1/2$.



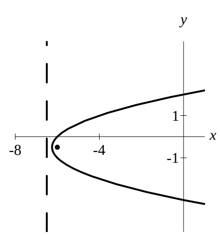
69. By completing the square, we obtain

$$x = (y + 1/2)^2 - 25/4.$$

Vertex is $(k, h) = (-25/4, -1/2)$ and axis of symmetry is $y = -1/2$. If $y = 0$, then $x = 1/4 - 25/4 = -6$. The x-intercept is $(-6, 0)$. If $x = 0$, then

$$\begin{aligned} \left(y + \frac{1}{2}\right)^2 &= \frac{25}{4} \\ y &= -\frac{1}{2} \pm \frac{5}{2} \\ y &= 2, -3. \end{aligned}$$

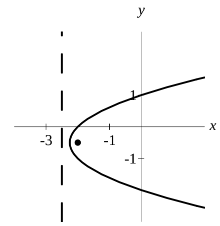
y-intercepts are $(0, 2), (0, -3)$. Since $\dfrac{1}{4p} = a = 1$, $p = 1/4$, focus is $(k + p, h) = (-6, -1/2)$, and directrix is $x = k - p = -13/2$.



70. By completing the square, $x = (y+1/2)^2 - 9/4$. Vertex is $(k, h) = (-9/4, -1/2)$ and axis of symmetry is $y = -1/2$. If $y = 0$ then $x = 1/4 - 9/4 = -2$. The x-intercept is $(-2, 0)$. If $x = 0$, then

$$\begin{aligned} \left(y + \frac{1}{2}\right)^2 &= \frac{9}{4} \\ y &= -\frac{1}{2} \pm \frac{3}{2} \\ y &= 1, -2. \end{aligned}$$

y-intercepts are $(0, 1), (0, -2)$. Since $\dfrac{1}{4p} = a = 1$, $p = 1/4$, focus is $(k + p, h) = (-2, -1/2)$, and directrix is $x = k - p = -5/2$.



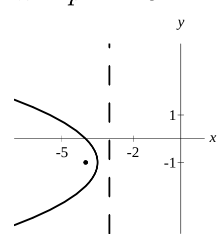
71. By completing the square, we get

$$x = -\frac{1}{2}(y + 1)^2 - 7/2.$$

Vertex is $(k, h) = (-7/2, -1)$ and axis of symmetry is $y = -1$. If $y = 0$, then $x = -1/2 - 7/2 = -4$. The x-intercept is $(-4, 0)$. If $x = 0$, then

$$0 = -\frac{1}{2}(y + 1)^2 - 7/2 < 0$$

which is inconsistent and so there is no y-intercept. Since $\dfrac{1}{4p} = a = -1/2$, $p = -1/2$, focus is $(k + p, h) = (-4, -1)$, and directrix is $x = k - p = -3$.



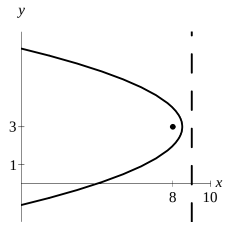
72. By completing the square, we find

$$x = -\frac{1}{2}(y - 3)^2 + 17/2.$$

Vertex is $(k, h) = (17/2, 3)$ and axis of symmetry is $y = 3$. If $y = 0$, then $x = -\dfrac{9}{2} + \dfrac{17}{2} = 4$. The x-intercept is $(4, 0)$. If $x = 0$, then

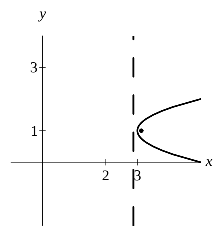
$$\begin{aligned} \frac{1}{2}(y - 3)^2 &= \frac{17}{2} \\ (y - 3)^2 &= 17 \\ y &= 3 \pm \sqrt{17}. \end{aligned}$$

y-intercepts are $(0, 3 \pm \sqrt{17})$. Since $\dfrac{1}{4p} = a = -\dfrac{1}{2}$, $p = -\dfrac{1}{2}$, focus is $(k + p, h) = (8, 3)$, and directrix is $x = k - p = 9$.



73. Since $x = 2(y-1)^2 + 3$ is of the form $a(y-h)^2+k$, we find that the vertex is $(k,h) = (3,1)$ and axis of symmetry is $y = 1$. If $y = 0$ then $x = 2(-1)^2 + 3 = 5$ and x-intercept is $(5,0)$. If $x = 0$, we obtain $2(y-1)^2 + 3 = 0$ which is inconsistent since the left-hand side is always positive. No y-intercept.

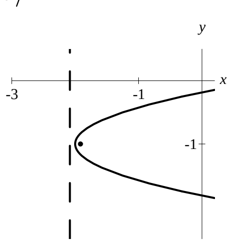
Since $\dfrac{1}{4p} = a = 2$, $p = 1/8$, focus is $(k+p, h) = (25/8, 1)$, and directrix is $x = k - p = 23/8$.



74. Since $x = 3(y+1)^2 - 2$ is of the form $a(y-h)^2 + k$, vertex is $(k,h) = (-2,-1)$ and axis of symmetry is $y = -1$. If $y = 0$, then $x = 3(1)^2 - 2 = 1$ and x-intercept is $(1,0)$. If $x = 0$, then

$$\begin{aligned} (y+1)^2 &= \frac{2}{3} \\ y &= -1 \pm \frac{\sqrt{6}}{3}. \end{aligned}$$

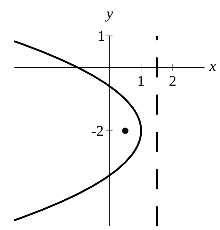
y-intercepts are $\left(0, -1 \pm \dfrac{\sqrt{6}}{3}\right)$. Since $\dfrac{1}{4p} = a = 3$, $p = 1/12$, focus is $(k+p, h) = (-23/12, -1)$, and directrix is $x = k - p = -25/12$.



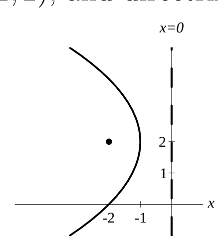
75. Since $x = -\dfrac{1}{2}(y+2)^2 + 1$ is of the form $a(y-h)^2 + k$, vertex is $(k,h) = (1,-2)$ and axis of symmetry is $y = -2$. If $y = 0$, then $x = -\dfrac{1}{2}(2)^2 + 1 = -1$ and x-intercept is $(-1,0)$. If $x = 0$, then

$$\begin{aligned} \frac{1}{2}(y+2)^2 &= 1 \\ (y+2)^2 &= 2 \\ y &= -2 \pm \sqrt{2}. \end{aligned}$$

The y-intercepts are $(0, -2 \pm \sqrt{2})$. Since $\dfrac{1}{4p} = a = -\dfrac{1}{2}$, we find $p = -\dfrac{1}{2}$, focus is $(k+p, h) = (1/2, -2)$, and directrix is $x = k - p = 3/2$.



76. Since $x = -\dfrac{1}{4}(y-2)^2 - 1$ is of the form $a(y-h)^2 + k$, vertex is $(k,h) = (-1,2)$ and axis of symmetry is $y = 2$. If $y = 0$, then $x = -\dfrac{1}{4}(-2)^2 - 1 = -2$ and x-intercept is $(-2,0)$. If $x = 0$, then $0 = -\dfrac{1}{4}(y-2)^2 - 1$ which is inconsistent, since the right-hand side is always negative. No y-intercept. Since $\dfrac{1}{4p} = a = -\dfrac{1}{4}$, we get $p = -1$, focus is $(k+p, h) = (-2, 2)$, and directrix is $x = k - p = 0$.



77. Since focus is 1 unit above the vertex $(1,4)$, we obtain $p = 1$. Then $a = \dfrac{1}{4p} = \dfrac{1}{4}$ and parabola is given by $y = \dfrac{1}{4}(x-1)^2 + 4$.

78. Since vertex $(2,3)$ is 2 units below the directrix, $p=-2$ and parabola opens down. Then $a=\dfrac{1}{4p}=-\dfrac{1}{8}$ and parabola is given by $y=-\dfrac{1}{8}(x-2)^2+3$.

79. Since vertex $(0,0)$ is 2 units to the right of the directrix, we find $p=2$ and parabola opens to the right. Then $a=\dfrac{1}{4p}=\dfrac{1}{8}$ and the parabola is given by $x=\dfrac{1}{8}y^2$.

80. Since the focus is $1/4$ unit to the right of the vertex $(-9/4,3)$, we obtain $p=1/4$. Thus, $a=\dfrac{1}{4p}=1$ and the parabola is given by $x=(y-3)^2-\dfrac{9}{4}$.

81. Since the parabola opens up, $p=55(12)$ inches, and the vertex is $(0,0)$. Note, $\dfrac{1}{4p}=\dfrac{1}{2640}$. Thus, the parabola is given by $y=\dfrac{1}{2640}x^2$. Thickness at the outside edge is $23+\dfrac{1}{2640}(100)^2\approx 26.8$ in.

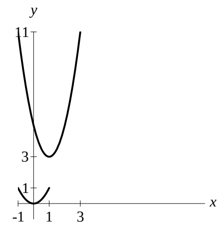
82. Since the parabola opens up, we find $p=6$ in. and the vertex is $(0,0)$. Note, $\dfrac{1}{4p}=\dfrac{1}{24}$. The parabola is given by $y=\dfrac{1}{24}x^2$ and the depth of the shield is

$$y=\frac{1}{24}\left(\frac{18.75}{2}\right)^2\approx 3.66 \text{ in.}$$

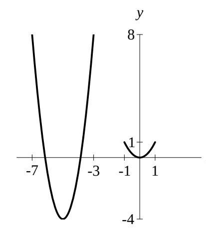
83. $y=x^2$ has vertex $(0,0)$ and opens up. The second parabola can be written as

$$y=2(x-1)^2+3$$

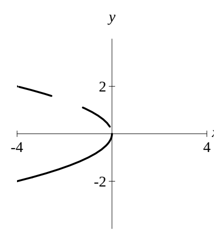
and its vertex is $(1,3)$. In the given viewing window these graphs look alike.



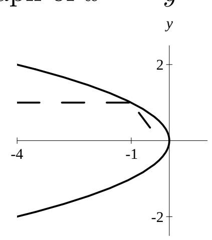
84. Note, $y=3x^2+30x+71=3(x+5)^2-4$. The vertex is $(-5,-4)$. Two viewing windows are
(a) $-7\le x\le -3$, $-4\le y\le 8$ and
(b) $-6\le x\le -4$, $-4\le y\le -1$



85. Two functions are $f_1(x)=\sqrt{-x}$ and $f_2(x)=-\sqrt{-x}$ where $x\le 0$.



86. Graph of $x=-y^2$



89. $\dfrac{\sqrt{3}}{\sqrt{6}-\sqrt{3}}\cdot\dfrac{\sqrt{6}+\sqrt{3}}{\sqrt{6}+\sqrt{3}}=\dfrac{\sqrt{18}+3}{3}=$
$\dfrac{3\sqrt{2}+3}{3}=\sqrt{2}+1$

90. $\dfrac{2}{x+5}+\dfrac{x}{x-5}-\dfrac{3}{x^2}=$
$\dfrac{2\cdot x^2(x-5)+x\cdot x^2(x+5)-3(x+5)(x-5)}{x^2(x-5)(x+5)}=$
$\dfrac{x^4+7x^3-13x^2+75}{x^2(x-5)(x+5)}=$

91.

$$\begin{aligned}\frac{9x-15}{(x-3)(x+3)} &= \frac{A}{x-3}+\frac{B}{x+3}\\ 9x-15 &= A(x+3)+B(x-3)\end{aligned}$$

If $x = 3$ in the above equation, we find

$$\begin{aligned}12 &= 6A\\ 2 &= A\end{aligned}$$

Likewise, if $x = -3$ then

$$\begin{aligned}-42 &= -6B\\ 7 &= B\end{aligned}$$

The partial fraction decomposition is

$$\frac{2}{x-3}+\frac{7}{x+3}.$$

92. Solve for y:

$$\begin{aligned}x &= \log_5(y-1)+6\\ x-6 &= \log_5(y-1)\\ 5^{x-6} &= y-1\\ 5^{x-6}+1 &= y.\end{aligned}$$

The inverse function is

$$f^{-1}(x) = 5^{x-6}+1.$$

93. Use the method of completing the square.

$$\begin{aligned}f(x) &= 3\left(x^2-\frac{4}{3}x\right)+7\\ f(x) &= 3\left(x^2-\frac{4}{3}x+\frac{4}{9}\right)+7-\frac{4}{3}\\ f(x) &= 3\left(x-\frac{2}{3}\right)^2+\frac{17}{3}\end{aligned}$$

94. If we multiply the second equation by -5 and add to the third equation, we obtain

$$\begin{array}{rcl}-5x+15y-5z &=& 50\\ 2y+5z &=& 50\\ \hline -5x+17y &=& 100\end{array}$$

If we add first equation $5x+7y = 20$ to the above equation, we find

$$\begin{aligned}24y &= 120\\ y &= 5\end{aligned}$$

Then substitute back as follows:

$$\begin{aligned}5x+7(5) &= 20\\ 5x &= -15\\ x &= -3\end{aligned}$$

Likewise,

$$\begin{aligned}2(5)+5z &= 50\\ 5z &= 40\\ z &= 8\end{aligned}$$

Hence, the solution set is $\{-3, 5, 8\}$.

Thinking Outside the Box LXXXVII

Let r be the radius of the pipe that is placed on top of two pipes with radii 2 ft and 3 ft. Draw a triangle with vertices at the center of the three circles (cross section of the pipes). Let θ be the angle of the triangle at the center of the circle with radius 2 ft. Using the Law of Cosines, we obtain

$$(r+3)^2 = 5^2+(r+2)^2-2(5)(r+2)\cos\theta.$$

Solving for r, we find

$$r = \frac{10(1-\cos\theta)}{5\cos\theta+1}.$$

Note, the angle between the side of the triangle joining the centers of the circle with radii 2 ft and 3 ft and the horizontal is $\cos^{-1}(\sqrt{24}/5)$. Since the top circle is tangent to the circle with radius 3 feet, θ must lie in the interval

$$\left[0, \frac{\pi}{2}-\cos^{-1}(\sqrt{24}/5)\right].$$

The maximum of r occurs at the right endpoint of the above closed interval. That is, the maximum radius is

$$r = \frac{10(1-\cos\theta)}{5\cos\theta+1} = \frac{10(1-1/5)}{5(1/5)+1} = 4 \text{ ft}.$$

10.1 Pop Quiz

1. Since the vertex is equidistant from $(0, 3)$ and $y = 1$, the vertex is $(0, 2)$ and $p = 1$. Using $y = \frac{1}{4p}(x-h)^2 + k$, the equation of the parabola is $y = \frac{1}{4}x^2 + 2$.

2. Comparing $y = -\frac{1}{16}(x-2)^2 + 3$ with

$y = \frac{1}{4p}(x-h)^2 + k$, we obtain that $p = -4$

and the vertex is $(2, 3)$. Moreover, the focus is $(h, k+p) = (2, -1)$ and the directrix is $y = k - p = 7$.

3. Since $x = (y-2)^2 - 1$, the vertex is $(-1, 2)$ and the axis of symmetry is $y = 2$. If $y = 0$, then $x = 4 - 1 = 3$ and the x-intercept is $(3, 0)$. If $x = 0$, then $(y-2)^2 = 1$ or $y = 2 \pm 1$ and the y-intercepts are $(0, 1)$ and $(0, 3)$.

Note, $p = 1/4$. Then the focus is $(h+p, k) = (-3/4, 2)$ and the directrix is $x = h - p = -5/4$.

10.1 Linking Concepts

a) Substitute $y = x^2$ into $y = m(x-3) + 9$ and solve for m. Then

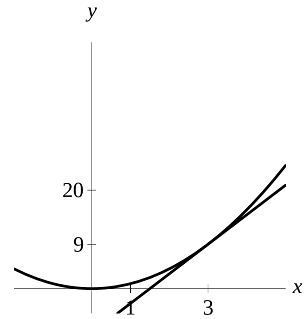
$$\begin{aligned} x^2 &= mx + 9 - 3m \\ x^2 - mx + (3m - 9) &= 0. \end{aligned}$$

If there is exactly one solution for x then the discriminant $b^2 - 4ac$ must be zero. Thus,

$$\begin{aligned} m^2 - 4(3m - 9) &= 0 \\ m^2 - 12m + 36 &= 0 \\ (m - 6)^2 &= 0. \end{aligned}$$

The value is $m = 6$.

b) Sketched are the graphs of $y = x^2$ and $y = 6x - 9$. They intersect at the point $(3, 9)$.



c) The other line is $x = 3$ and this line has no slope. It did not appear in part a) since $x = 3$ cannot be expressed in the form $y = mx + b$.

d) Substitute $y = x^2$ into $y - y_1 = m(x - x_1)$ and solve for m. Then

$$\begin{aligned} x^2 - y_1 &= m(x - x_1) \\ x^2 - mx + (mx_1 - y_1) &= 0 \\ x^2 - mx + (mx_1 - x_1^2) &= 0. \end{aligned}$$

If there is exactly one solution for x, then the discriminant $b^2 - 4ac$ must be zero. Thus,

$$\begin{aligned} m^2 - 4(mx_1 - x_1^2) &= 0 \\ 4x_1^2 - 4mx_1 + m^2 &= 0 \\ (2x_1 - m)^2 &= 0. \end{aligned}$$

The value of the slope at (x_1, y_1) is $m = 2x_1$.

e) Since the slope at the point $(-2.5, 6.25)$ is $m = 2x_1 = -5$, by using part d), an equation of the tangent line at the same point is $y = -5x - 6.25$.

For Thought

1. False, y-intercepts are $(\pm 3, 0)$.

2. True, since it can be written as $\frac{x^2}{1/2} + y^2 = 1$.

3. True, length of the major axis is $2a = 2(5) = 10$.

4. True, if $y = 0$ then $x^2 = \frac{1}{0.5} = 2$ and $x = \pm\sqrt{2}$.

5. True, if $x = 0$ then $y^2 = 3$ and $y = \pm\sqrt{3}$.

6. False, the center is not a point on the circle.

7. True **8.** False, $(3, -1)$ satisfies equation.

9. False. No point satisfies the equation since the left-hand side is always positive.

10. False, since the circle can be written as $(x-2)^2 + (y+1/2)^2 = 53/4$ and so the radius is $\sqrt{53}/2$.

10.2 Exercises

1. ellipse

2. major axis

3. circle

4. eccentricity

5. Foci $(\pm\sqrt{5}, 0)$, vertices $(\pm 3, 0)$, center $(0, 0)$

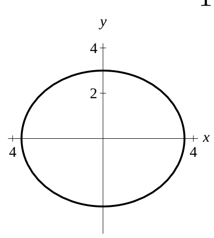
6. Foci $(0, \pm\sqrt{21})$, vertices $(0, \pm 5)$, center $(0, 0)$

7. Foci $(2, 1 \pm \sqrt{5})$, vertices $(2, 4)$ and $(2, -2)$, center $(2, 1)$

8. Foci $(-1 \pm \sqrt{7}, 2)$, vertices $(-5, 2)$ and $(3, 2)$, center $(-1, 2)$

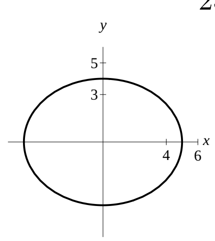
9. Since $c = 2$ and $b = 3$, we get $a^2 = b^2 + c^2 = 9 + 4 = 13$ and $a = \sqrt{13}$.

Ellipse is given by $\dfrac{x^2}{13} + \dfrac{y^2}{9} = 1$.



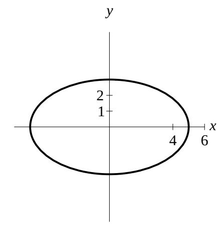
10. Since $c = 3$ and $b = 4$, $a^2 = b^2 + c^2 = 16 + 9 = 25$ and $a = 5$.

Ellipse is given by $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.



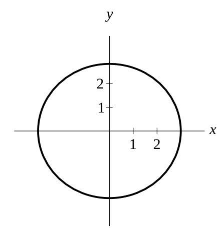
11. Since $c = 4$ and $a = 5$, we find $b^2 = a^2 - c^2 = 25 - 16 = 9$ and $b = 3$.

Ellipse is given by $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$.



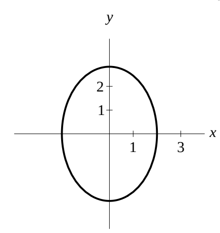
12. Since $c = 1$ and $a = 3$, $b^2 = a^2 - c^2 = 9 - 1 = 8$ and $b = \sqrt{8}$.

Ellipse is given by $\dfrac{x^2}{9} + \dfrac{y^2}{8} = 1$.



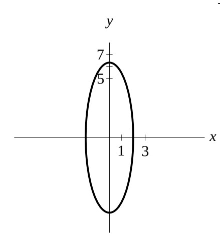
13. Since $c = 2$ and $b = 2$, we obtain $a^2 = b^2 + c^2 = 4 + 4 = 8$ and $a = \sqrt{8}$.

Ellipse is given by $\dfrac{x^2}{4} + \dfrac{y^2}{8} = 1$.



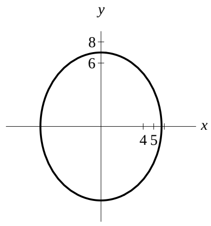
14. Since $c = 6$ and $b = 2$, $a^2 = b^2 + c^2 = 4 + 36 = 40$ and $b = \sqrt{40}$.

Ellipse is given by $\dfrac{x^2}{4} + \dfrac{y^2}{40} = 1$.



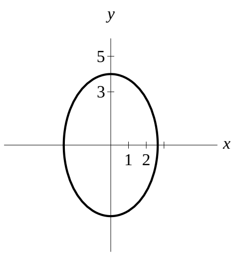
15. Since $c = 4$ and $a = 7$, we obtain $b^2 = a^2 - c^2 = 49 - 16 = 33$ and $b = \sqrt{33}$.

Ellipse is given by $\dfrac{x^2}{33} + \dfrac{y^2}{49} = 1$.

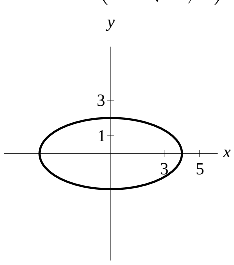


16. Since $c = 3$ and $a = 4$, we find $b^2 = a^2 - c^2 = 16 - 9 = 7$ and $b = \sqrt{7}$.

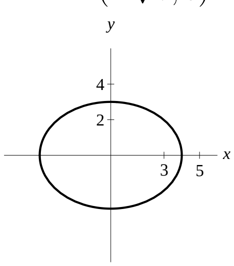
Ellipse is given by $\dfrac{x^2}{7} + \dfrac{y^2}{16} = 1$.



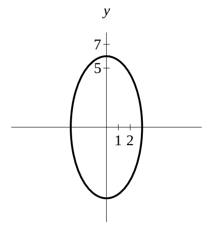
17. Since $c = \sqrt{a^2 - b^2} = \sqrt{16 - 4} = 2\sqrt{3}$, the foci are $(\pm 2\sqrt{3}, 0)$



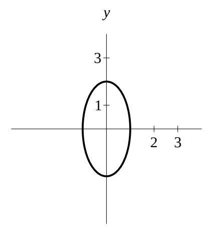
18. Since $c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$, the foci are $(\pm\sqrt{7}, 0)$



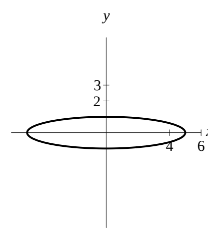
19. Since $c = \sqrt{a^2 - b^2} = \sqrt{36 - 9} = \sqrt{27}$, the foci are $(0, \pm 3\sqrt{3})$



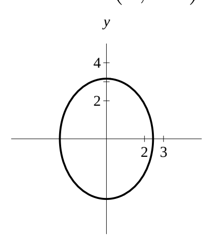
20. Since $c = \sqrt{a^2 - b^2} = \sqrt{4 - 1} = \sqrt{3}$, the foci are $(0, \pm\sqrt{3})$



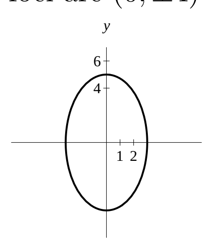
21. Since $c = \sqrt{a^2 - b^2} = \sqrt{25 - 1} = \sqrt{24}$, the foci are $(\pm 2\sqrt{6}, 0)$



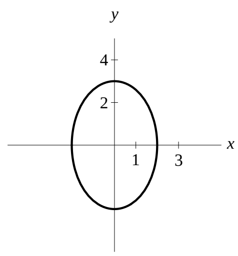
22. Since $c = \sqrt{a^2 - b^2} = \sqrt{10 - 6} = \sqrt{4}$, the foci are $(0, \pm 2)$



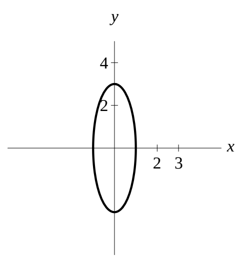
23. Since $c = \sqrt{a^2 - b^2} = \sqrt{25 - 9} = \sqrt{16}$, the foci are $(0, \pm 4)$



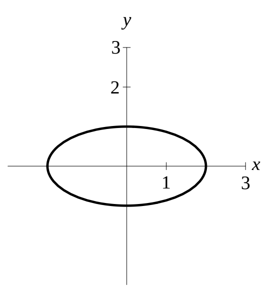
24. Since $c = \sqrt{a^2 - b^2} = \sqrt{9-4} = \sqrt{5}$, the foci are $(0, \pm\sqrt{5})$



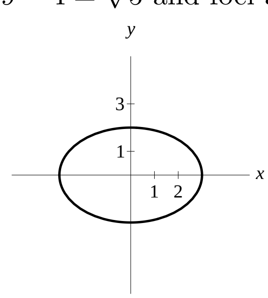
25. From $x^2 + \dfrac{y^2}{9} = 1$, one finds $c = \sqrt{a^2 - b^2}$ $= \sqrt{9-1} = \sqrt{8}$ and foci are $(0, \pm 2\sqrt{2})$.



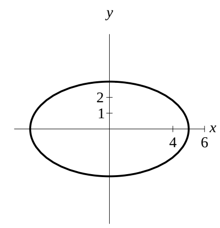
26. From $\dfrac{x^2}{4} + y^2 = 1$, one finds $c = \sqrt{a^2 - b^2}$ $= \sqrt{4-1} = \sqrt{3}$ and foci are $(\pm\sqrt{3}, 0)$.



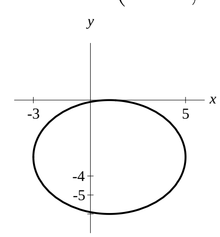
27. From $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$, one finds $c = \sqrt{a^2 - b^2}$ $= \sqrt{9-4} = \sqrt{5}$ and foci are $(\pm\sqrt{5}, 0)$.



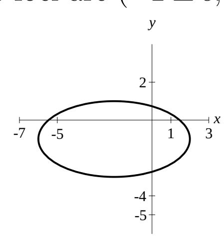
28. From $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$, one finds $c = \sqrt{a^2 - b^2}$ $= \sqrt{25-9} = \sqrt{16}$ and foci are $(\pm 4, 0)$.



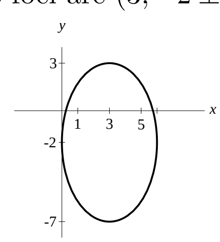
29. Since $c = \sqrt{a^2 - b^2} = \sqrt{16-9} = \sqrt{7}$, the foci are $(1 \pm c, -3) = (1 \pm \sqrt{7}, -3)$.



30. Since $c = \sqrt{a^2 - b^2} = \sqrt{16-4} = \sqrt{12}$, the foci are $(-2 \pm c, -1) = (-2 \pm 2\sqrt{3}, -1)$.

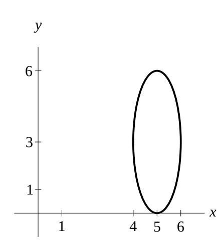


31. Since $c = \sqrt{a^2 - b^2} = \sqrt{25-9} = \sqrt{16} = 4$, the foci are $(3, -2 \pm c)$, or $(3, 2)$ and $(3, -6)$.



32. Since $c = \sqrt{a^2 - b^2} = \sqrt{9-1} = \sqrt{8}$, the foci are

$$(5, 3 \pm c) = (5, 3 \pm 2\sqrt{2}).$$

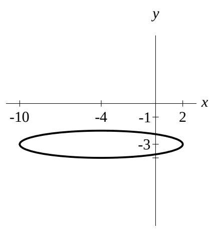


33. Since $\dfrac{(x+4)^2}{36} + (y+3)^2 = 1$, we get

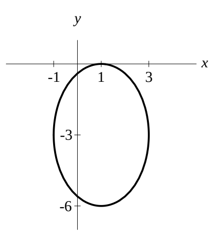
$$c = \sqrt{a^2 - b^2} = \sqrt{36 - 1} = \sqrt{35}$$

and the foci are

$$(-4 \pm \sqrt{35}, -3).$$



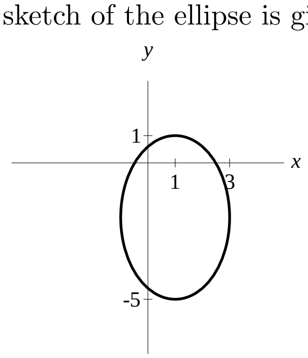
34. Since $\dfrac{(x-1)^2}{4} + \dfrac{(y+3)^2}{9} = 1$, we get $c = \sqrt{a^2 - b^2} = \sqrt{9 - 4} = \sqrt{5}$. The foci are $(1, -3 \pm \sqrt{5})$.



35. If one applies the method of completing the square, one obtains

$$\begin{aligned} 9(x^2 - 2x + 1) + 4(y^2 + 4y + 4) &= 11 + 9 + 16 \\ 9(x-1)^2 + 4(y+2)^2 &= 36 \\ \frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} &= 1. \end{aligned}$$

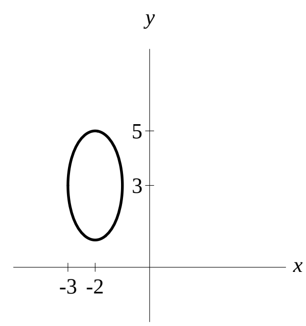
From $a^2 = b^2 + c^2$ with $a = 3$ and $b = 2$, one finds $c = \sqrt{5}$. The foci are $(1, -2 \pm \sqrt{5})$ and a sketch of the ellipse is given.



36. If one applies the method of completing the square, one obtains

$$\begin{aligned} 4(x^2 + 4x + 4) + (y^2 - 6y + 9) &= -21 + 16 + 9 \\ 4(x+2)^2 + (y-3)^2 &= 4 \\ (x+2)^2 + \frac{(y-3)^2}{4} &= 1. \end{aligned}$$

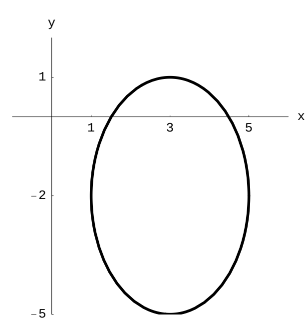
From $a^2 = b^2 + c^2$ with $a = 2$ and $b = 1$, one finds $c = \sqrt{3}$. The foci are $(-2, 3 \pm \sqrt{3})$ and a sketch of the ellipse is given.



37. Applying the method of completing the square, we find

$$\begin{aligned} 9(x^2 - 6x + 9) + 4(y^2 + 4y + 4) &= -61 + 81 + 16 \\ 9(x-3)^2 + 4(y+2)^2 &= 36 \\ \frac{(x-3)^2}{4} + \frac{(y+2)^2}{9} &= 1. \end{aligned}$$

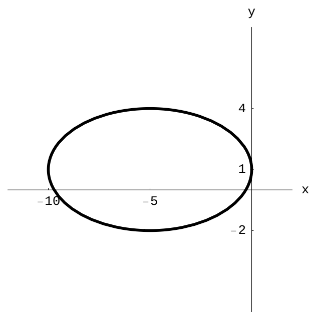
From $a^2 = b^2 + c^2$ with $a = 3$ and $b = 2$, we find $c = \sqrt{5}$. The foci are $(3, -2 \pm \sqrt{5})$ and a sketch of the ellipse is given.



38. Applying the method of completing the square, we find

$$\begin{aligned} 9(x^2 + 10x + 25) + 25(y^2 - 2y + 1) &= -25 + 225 + 25 \\ 9(x+5)^2 + 25(y-1)^2 &= 225 \\ \frac{(x+5)^2}{25} + \frac{(y-1)^2}{9} &= 1. \end{aligned}$$

From $a^2 = b^2 + c^2$ with $a = 5$ and $b = 3$, we find $c = 4$. The foci are $(-5 \pm c, 1)$, i.e., $(-1, 1)$ and $(-9, 1)$ are the foci. A sketch of the ellipse is given.



39. Since $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$, we get $c = \sqrt{a^2 - b^2}$ $= \sqrt{16-4} = \sqrt{12}$, and the foci are $(\pm 2\sqrt{3}, 0)$.

40. Since $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$, we find $c = \sqrt{a^2 - b^2}$ $= \sqrt{25-9} = \sqrt{16}$, and the foci are $(0, \pm 4)$.

41. Since $\dfrac{(x+1)^2}{4} + \dfrac{(y+2)^2}{16} = 1$, we get $c = \sqrt{a^2 - b^2} = \sqrt{16-4} = \sqrt{12}$, and the foci are $(-1, -2 \pm 2\sqrt{3})$.

42. Since $\dfrac{(x-2)^2}{16} + \dfrac{(y+1)^2}{9} = 1$, $c = \sqrt{a^2 - b^2}$ $= \sqrt{16-9} = \sqrt{7}$. The foci are $(2 \pm \sqrt{7}, -1)$.

43. $x^2 + y^2 = 4$ **44.** $x^2 + y^2 = 5$

45. Since $r = \sqrt{(4-0)^2 + (5-0)^2} = \sqrt{41}$, the circle is given by $x^2 + y^2 = 41$.

46. Since $r = \sqrt{(-3-0)^2 + (-4-0)^2} = \sqrt{25}$, the circle is given by $x^2 + y^2 = 25$.

47. Since $r = \sqrt{(4-2)^2 + (1+3)^2} = \sqrt{20}$, the circle is given by $(x-2)^2 + (y+3)^2 = 20$.

48. Since $r = \sqrt{(1+2)^2 + (-1+4)^2} = \sqrt{18}$, the circle is given by $(x+2)^2 + (y+4)^2 = 18$.

49. Since center is $\left(\dfrac{3-1}{2}, \dfrac{4+2}{2}\right) = (1, 3)$ and $r = \sqrt{(3-1)^2 + (4-3)^2} = \sqrt{5}$, the circle is given by $(x-1)^2 + (y-3)^2 = 5$.

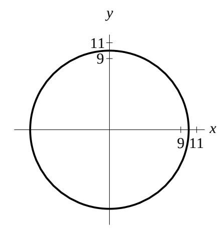
50. Since the center is

$$\left(\frac{3-4}{2}, \frac{-1+2}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

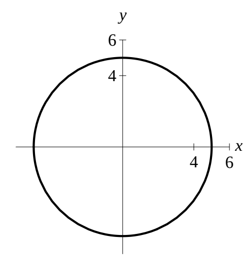
and $r = \sqrt{\left(3 + \dfrac{1}{2}\right)^2 + \left(-1 - \dfrac{1}{2}\right)^2} = \sqrt{\dfrac{29}{2}}$, the circle is given by

$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{29}{2}.$$

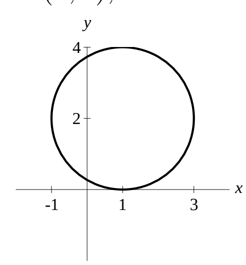
51. center $(0, 0)$, radius 10



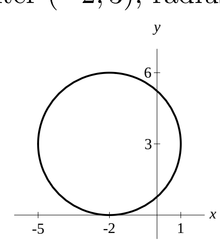
52. center $(0, 0)$, radius 5



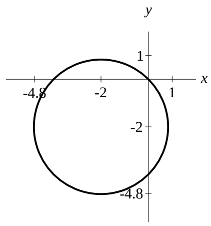
53. center $(1, 2)$, radius 2



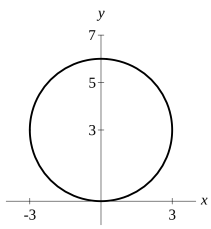
54. center $(-2, 3)$, radius 3



55. center $(-2,-2)$, radius $\sqrt{8}$ or $2\sqrt{2}$



56. center $(0,3)$, radius 3



57. Completing the square, we obtain

$$\begin{aligned} x^2+(y^2+2y+1) &= 8+1 \\ x^2+(y+1)^2 &= 9. \end{aligned}$$

The center is $(0,-1)$ with radius 3.

58. Completing the square, we get

$$\begin{aligned} (x^2-6x+9)+y^2 &= 1+9 \\ (x-3)^2+y^2 &= 10. \end{aligned}$$

The center is $(3,0)$ with radius $\sqrt{10}$.

59. Completing the square, we find

$$\begin{aligned} x^2+8x+16+y^2-10y+25 &= 16+25 \\ (x+4)^2+(y-5)^2 &= 41. \end{aligned}$$

The center is $(-4,5)$ with radius $\sqrt{41}$.

60. Completing the square, we get

$$\begin{aligned} x^2-12x+36+y^2+12y+36 &= 36+36 \\ (x-6)^2+(y+6)^2 &= 72. \end{aligned}$$

The center is $(6,-6)$ with radius $\sqrt{72}$ or $6\sqrt{2}$.

61. Completing the square, we find

$$\begin{aligned} (x^2+4x+4)+y^2 &= 5+4 \\ (x+2)^2+y^2 &= 9. \end{aligned}$$

The center is $(-2,0)$ with radius 3.

62. Completing the square, we obtain

$$\begin{aligned} x^2+(y^2-6y+9) &= 0+9 \\ x^2+(y-3)^2 &= 9. \end{aligned}$$

The center is $(0,3)$ with radius 3.

63. Completing the square, we have

$$\begin{aligned} x^2-x+\frac{1}{4}+y^2+y+\frac{1}{4} &= \frac{1}{2}+\frac{1}{4}+\frac{1}{4} \\ (x-1/2)^2+(y+1/2)^2 &= 1. \end{aligned}$$

The center is $(0.5,-0.5)$ with radius 1.

64. Completing the square, we find

$$\begin{aligned} x^2+5x+\frac{25}{4}+y^2+3y+\frac{9}{4} &= \frac{1}{2}+\frac{25}{4}+\frac{9}{4} \\ (x+5/2)^2+(y+3/2)^2 &= 9. \end{aligned}$$

The center is $(-5/2,-3/2)$ with radius 3.

65. Completing the square, we get

$$\begin{aligned} x^2+\frac{2}{3}x+\frac{1}{9}+y^2+\frac{1}{3}y+\frac{1}{36} &= \frac{1}{9}+\frac{1}{9}+\frac{1}{36} \\ (x+1/3)^2+(y+1/6)^2 &= 1/4. \end{aligned}$$

The center is $(-1/3,-1/6)$ with radius $1/2$.

66. Completing the square, we obtain

$$\begin{aligned} x^2+\frac{1}{2}x+\frac{1}{16}+y^2+\frac{1}{2}y+\frac{1}{16} &= \frac{1}{8}+\frac{1}{16}+\frac{1}{16} \\ (x+1/4)^2+(y+1/4)^2 &= 1/4. \end{aligned}$$

The center is $(-1/4,-1/4)$ with radius $1/2$.

67. Divide equation by 2 and complete the square.

$$\begin{aligned} x^2+2x+y^2 &= 1/2 \\ (x^2+2x+1)+y^2 &= 1/2+1 \\ (x+1)^2+y^2 &= 3/2 \end{aligned}$$

The center is $(-1,0)$ with radius $\sqrt{\dfrac{3}{2}}$ or $\dfrac{\sqrt{6}}{2}$.

68. Completing the square, we find

$$\begin{aligned} x^2+y^2-\frac{3}{2}y+\frac{9}{16} &= \frac{9}{16} \\ x^2+(y-3/4)^2 &= \frac{9}{16}. \end{aligned}$$

The center is $(0,3/4)$ with radius $3/4$.

69. Completing the square, we get

$$\begin{aligned} y^2 - y + x^2 &= 0 \\ y^2 - y + \frac{1}{4} + x^2 &= \frac{1}{4} \\ (y - 1/2)^2 + x^2 &= \frac{1}{4}, \end{aligned}$$

which is a circle.

70. Completing the square, we obtain

$$\begin{aligned} x^2 - 4x + y^2 &= 0 \\ x^2 - 4x + 4 + y^2 &= 4 \\ (x-2)^2 + y^2 &= 4 \end{aligned}$$

and which is a circle.

71. Divide equation by 4.

$$\begin{aligned} x^2 + 3y^2 &= 1 \\ x^2 + \frac{y^2}{1/3} &= 1 \end{aligned}$$

This is an ellipse.

72. Divide equation by 2 and complete the square.

$$\begin{aligned} x^2 + y^2 + \frac{1}{2}y &= 2 \\ x^2 + y^2 + \frac{1}{2}y + \frac{1}{16} &= 2 + \frac{1}{16} \\ x^2 + (y + 1/4)^2 &= 33/16 \end{aligned}$$

This is a circle.

73. Solve for y and complete the square.

$$\begin{aligned} y &= -2x^2 - 4x + 4 \\ y &= -2(x^2 + 2x) + 4 \\ y &= -2(x^2 + 2x + 1) + 4 + 2 \\ y &= -2(x+1)^2 + 6 \end{aligned}$$

We find a parabola.

74. Completing the square, we obtain

$$\begin{aligned} 2x^2 + 4y^2 + y &= 4 \\ 2x^2 + 4\left(y^2 + \frac{1}{4}y + \frac{1}{64}\right) &= \frac{65}{16} \\ \frac{x^2}{65/32} + \frac{(y + 1/8)^2}{65/64} &= 1. \end{aligned}$$

This is an ellipse.

75. Note, $(y-2)^2 = (2-y)^2$. Then we solve for x.

$$\begin{aligned} 2 - x &= (y-2)^2 \\ x &= -(y-2)^2 + 2 \end{aligned}$$

This is a parabola.

76. Note, $(x-3)^2 = (3-x)^2$. Solving for y, we find

$$\begin{aligned} 3(x-3)^2 &= 9 - y \\ y &= -3(x-3)^2 + 9 \end{aligned}$$

which is a parabola.

77. Simplify and note $(x-4)^2 = (4-x)^2$.

$$\begin{aligned} 2(x-4)^2 &= 4 - y^2 \\ 2(x-4)^2 + y^2 &= 4 \\ \frac{(x-4)^2}{2} + \frac{y^2}{4} &= 1 \end{aligned}$$

We find an ellipse.

78. Multiply given equation by 4 to get the circle given by $x^2 + y^2 = 4$.

79. Divide given equation by 9 to get the circle given by $x^2 + y^2 = \dfrac{1}{9}$.

80. Solve for y.

$$\begin{aligned} 9x^2 - 1 &= -9y \\ y &= -x^2 + \frac{1}{9} \end{aligned}$$

We have a parabola.

81. From the foci, we obtain $c = 2$ and $a^2 = b^2 + c^2 = b^2 + 4$. Equation of the ellipse is of the form $\dfrac{x^2}{b^2+4} + \dfrac{y^2}{b^2} = 1$. Substitute $x = 2$ and $y = 3$.

$$\begin{aligned} \frac{4}{b^2+4} + \frac{9}{b^2} &= 1 \\ 4b^2 + 9(b^2 + 4) &= b^2(b^2 + 4) \\ 0 &= b^4 - 9b^2 - 36 \\ 0 &= (b^2 - 12)(b^2 + 3) \end{aligned}$$

So $b^2 = 12$ and $a^2 = 12 + 4 = 16$.

The ellipse is given by $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$.

82. Since $c = 5$ and $e = \frac{1}{2} = \frac{c}{a} = \frac{5}{a}$, we get $\frac{1}{2} = \frac{5}{a}$ and so $a = 10$.
Since $b^2 = a^2 - c^2 = 100 - 25 = 75$, the ellipse is given by $\frac{x^2}{100} + \frac{y^2}{75} = 1$.

83. If c is the distance between the center and focus $(0,0)$ then the other focus is $(2c, 0)$. Since the distance between the x-intercepts is $6+2c$, which is also the length of the major axis, then $6 + 2c = 2a$. Since $2a$ is the sum of the distances of $(0,5)$ from the foci,

$$\begin{aligned} 5 + \sqrt{25 + 4c^2} &= 2a \\ 5 + \sqrt{25 + 4c^2} &= 6 + 2c \\ \sqrt{25 + 4c^2} &= 1 + 2c \\ 25 + 4c^2 &= 1 + 4c + 4c^2 \\ 24 &= 4c \\ 6 &= c. \end{aligned}$$

The other focus is $(2c, 0) = (12, 0)$.

84. The amount of string needed is 12 ft. From the lengths of the major and minor axes one finds $2a = 12$ and $b = 4$. So $c = \sqrt{a^2 - b^2} = \sqrt{6^2 - 4^2} = 2\sqrt{5}$. The foci are located at $2\sqrt{5}$ units from the center of the board.

85. Since the sun is a focus of the elliptical orbit, the length of the major axis is $2a = 521$ (the sum of the shortest distance, $P = 1$ AU, and longest distance, $A = 520$ AU, between the orbit and the sun, respectively). In addition, $c = 259.5$ AU (which is the distance from the center to a focus). The eccentricity is given by $e = \frac{c}{a} = \frac{259.5}{260.5} \approx .996$. Orbit's equation is

$$\frac{x^2}{260.5^2} + \frac{y^2}{520} = 1.$$

86. Assume the center of the elliptical orbit is $(0,0)$. So $c = \frac{405,500 - 363,300}{2} = 21,100$ and $a = 21,100 + 363,300 = 384,400$.
The eccentricity is $e = \frac{c}{a} = \frac{21,100}{384,400} \approx 0.055$

87. If $2a$ is the sum of the distances from Haley's comet to the two foci and c is the distance from the sun to the center of the ellipse then, $c = a - 8 \times 10^7$.
Since $0.97 = c/a$, we get

$$0.97 = \frac{a - 8 \times 10^7}{a}$$

and the solution of this equation is $a \approx 2.667 \times 10^9$. So $c = a(0.97) \approx 2.587 \times 10^9$. The maximum distance from the sun is $c + a \approx 5.25 \times 10^9$ km.

88. Since $6.5^2 = 42.25$, an equation for the boundary of the 13-inch mag wheel is

$$(x - 6.5)^2 + (y - 6.5)^2 = 42.25.$$

One can let $(h, 8)$ be the center of the 16-inch mag wheel. Since distance between the centers of the two wheels is 14.5 inches, we obtain

$$\begin{aligned} (h - 6.5)^2 + 1.5^2 &= 14.5^2 \\ (h - 13/2)^2 &= 208 \\ h &= \frac{13}{2} \pm \sqrt{208} \\ h &= \frac{13}{2} \pm 4\sqrt{13} \\ h &= \frac{13 \pm 8\sqrt{13}}{2}. \end{aligned}$$

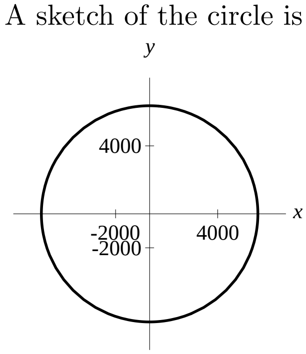
The other wheel is given by

$$\left(x - \frac{13 + 8\sqrt{13}}{2}\right)^2 + (y - 8)^2 = 64.$$

89. Solving for y, one finds

$$\begin{aligned} y^2 &= 6360^2 - x^2 \\ y &= \pm\sqrt{6360^2 - x^2}. \end{aligned}$$

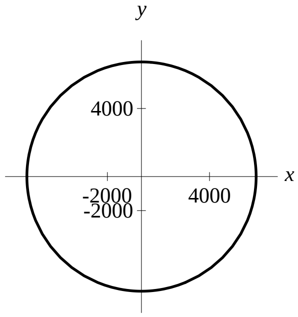
A sketch of the circle is given.



90. Solving for y, one finds

$$\begin{aligned} \frac{y^2}{6734.998^2} &= 1-\frac{(x-5)^2}{6735^2} \\ y^2 &= 6734.998^2\left(1-\frac{(x-5)^2}{6735^2}\right) \\ y &= \pm\sqrt{6734.998^2\left(1-\frac{(x-5)^2}{6735^2}\right)}. \end{aligned}$$

A sketch of the elliptical orbit is shown. The eccentricity of the orbit is $\frac{5.2}{6735} \approx 0.0008$.

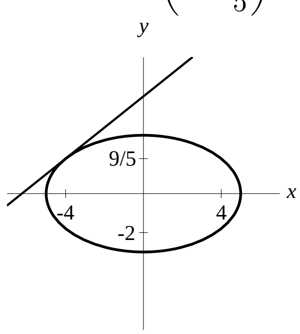


91.

a) As derived below, an equation of the tangent line is given by

$$\begin{aligned} \frac{x_1x}{a^2}+\frac{y_1y}{b^2} &= 1 \\ \frac{-4x}{25}+\frac{\frac{9}{5}y}{9} &= 1 \\ \frac{1}{5}y &= 1+\frac{4x}{25} \\ y &= \frac{4x}{5}+5. \end{aligned}$$

b) The tangent line and the ellipse intersect at the point $\left(-4, \frac{9}{5}\right)$ as shown.



92. From $\frac{x^2}{25}+\frac{y^2}{9}=1$, one finds that the foci are at $A(4,0)$ and $B(-4,0)$. Moreover, $y^2 = 9\left(1-\frac{x^2}{25}\right)$. Suppose (x, y) is a point on the ellipse whose distance from B is twice the distance from A. Then

$$\begin{aligned} 2\sqrt{(x-4)^2+y^2} &= \sqrt{(x+4)^2+y^2} \\ 4((x-4)^2+y^2) &= (x+4)^2+y^2 \end{aligned}$$

$$\begin{aligned} 4(x-4)^2-(x+4)^2+3y^2 &= 0 \\ 3x^2-40x+48+3y^2 &= 0 \\ 3x^2-40x+48+27\left(1-\frac{x^2}{25}\right) &= 0 \\ 75x^2-1000x+1200+675-27x^2 &= 0 \\ 48x^2-1000x+1875 &= 0. \end{aligned}$$

Solving for x, one finds $x = \frac{25}{12}$ and $x = \frac{75}{4}$; the second value must be excluded since it is out of the domain. Substituting $x = \frac{25}{12}$ into $y^2 = 9\left(1-\frac{x^2}{25}\right)$, one obtains $y = \pm\frac{\sqrt{119}}{4}$.

By the symmetry of the ellipse, there are four points that are twice as far from one focus as they are from the other focus.

Namely, these points are $\left(\pm\frac{25}{12}, \pm\frac{\sqrt{119}}{4}\right)$.

93. A parabolic reflector is preferable since otherwise one would have to place the moving quarterback on one focus of the ellipse.

94. If (x, y) is any point on the ellipse and the sum of the distances from the two foci $(\pm c, 0)$ is $2a$, then

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2} = 2a.$$

Simplify the inside of the radical and transpose a term to the right-hand side. So

$$\sqrt{x^2-2xc+c^2+y^2} = 2a - \sqrt{x^2+2cx+c^2+y^2}.$$

Squaring both sides, we get
$x^2 - 2xc + c^2 + y^2 =$
$4a^2 - 4a\sqrt{x^2 + 2cx + c^2 + y^2} + x^2 + 2cx + c^2 + y^2$.
Cancel like terms and simplify.

$$\begin{aligned} -4xc - 4a^2 &= -4a\sqrt{x^2 + 2cx + c^2 + y^2} \\ xc + a^2 &= a\sqrt{x^2 + 2cx + c^2 + y^2} \\ x^2c^2 + 2xca^2 + a^4 &= a^2(x^2 + 2cx + c^2 + y^2) \\ x^2c^2 + a^4 &= a^2(x^2 + y^2 + c^2) \\ a^4 - a^2c^2 &= x^2(a^2 - c^2) + a^2y^2 \end{aligned}$$

Let $b^2 = a^2 - c^2$. So $a^2b^2 = x^2b^2 + a^2y^2$ and $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is an ellipse with foci $(\pm c, 0)$.

95. If (x, y) is any point on the ellipse and the sum of the distances from the two foci $(0, \pm c)$ is $2a$, then

$$\sqrt{x^2 + (y - c)^2} + \sqrt{x^2 + (y + c)^2} = 2a.$$

Simplify the inside of the radical and transpose a term to the right-hand side.

$$\sqrt{x^2 + y^2 - 2yc + c^2} = 2a - \sqrt{x^2 + y^2 + 2cy + c^2}$$

Squaring both sides, we find $x^2 + y^2 - 2yc + c^2 =$ $4a^2 - 4a\sqrt{x^2 + y^2 + 2cy + c^2} + x^2 + y^2 + 2cy + c^2$.
Cancel like terms and simplify.

$$\begin{aligned} -4yc - 4a^2 &= -4a\sqrt{x^2 + y^2 + 2cy + c^2} \\ yc + a^2 &= a\sqrt{x^2 + y^2 + 2cy + c^2} \\ y^2c^2 + 2yca^2 + a^4 &= a^2(x^2 + y^2 + 2cy + c^2) \\ y^2c^2 + a^4 &= a^2(x^2 + y^2 + c^2) \\ a^4 - a^2c^2 &= a^2x^2 + y^2(a^2 - c^2) \end{aligned}$$

Let $b^2 = a^2 - c^2$. So $a^2b^2 = a^2x^2 + b^2y^2$ and $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ is an ellipse with foci $(0, \pm c)$ and one finds that the x-intercepts are $(\pm b, 0)$.

96. If (s, t) is any point on the semicircle $y = \sqrt{r^2 - x^2}$, whose radius is r, then $(2.5s, t)$ is a point on the boundary of the drapes to be cut. Since this point on the boundary satisfies $\frac{x^2}{2.5^2} + y^2 = 1$, one must cut the drapes in the shape of an ellipse.

97. Factoring, we find

$$\begin{aligned} y &= (2x - 1)^2 \\ y &= 4\left(x - \frac{1}{2}\right)^2. \end{aligned}$$

If $\frac{1}{4p} = 4$, then $p = \frac{1}{16}$.
Since the vertex is $(\frac{1}{2}, 0)$, the focus is $(\frac{1}{2}, 0 + p) = (\frac{1}{2}, \frac{1}{16})$, and directrix is $y = 0 - p = -\frac{1}{16}$.

98. Completing the square, we find

$$\begin{aligned} x &= 2(y^2 + y) \\ x &= 2\left(y + \frac{1}{2}\right)^2 - \frac{1}{2}. \end{aligned}$$

If $\frac{1}{4p} = 2$, then $p = \frac{1}{8}$.
Since the vertex is $(-\frac{1}{2}, -\frac{1}{2})$, the focus is $(-\frac{1}{2} + p, -\frac{1}{2}) = (-\frac{3}{8}, -\frac{1}{2})$, and directrix is $x = -\frac{1}{2} - p = -\frac{5}{8}$.

99. The distance is

$$\begin{aligned} \sqrt{\left(\frac{1}{2} - \frac{1}{4}\right)^2 + \left(-\frac{1}{2} - 1\right)^2} &= \\ \sqrt{\left(\frac{1}{4}\right)^2 + \left(-\frac{3}{2}\right)^2} &= \\ \sqrt{\frac{1}{16} + \frac{9}{4}} &= \\ \sqrt{\frac{37}{16}} &= \\ \frac{\sqrt{37}}{4}. \end{aligned}$$

The midpoint is

$$\left(\frac{\frac{1}{2} + \frac{1}{4}}{2}, \frac{-\frac{1}{2} + 1}{2}\right) = \left(\frac{3}{8}, \frac{1}{4}\right).$$

100. The distance is

$$\begin{aligned} \frac{\frac{2}{3(x+h)} - \frac{2}{3x}}{h} &= \\ \frac{\frac{2x}{3x(x+h)} - \frac{2(x+h)}{3x(x+h)}}{h} &= \\ \frac{\frac{-2h}{3x(x+h)}}{h} &= \frac{-2}{3x(x+h)} \end{aligned}$$

101. Since $f(-x) = -f(x)$, the graph is symmetric about the origin.

102. Since $A = \pi \left(\frac{d}{2}\right)^2$, we find

$$\begin{aligned} A &= \pi\left(\frac{d^2}{4}\right) \\ \frac{4A}{\pi} &= d^2. \end{aligned}$$

Then $d = \sqrt{\frac{4A}{\pi}}$.

Thinking Outside the Box LXXXVIII

Let (h, k) be the center of the third circle. The radius of the third circle is h. Since the third circle is tangent to the circle centered at $(0.5, 0)$ with radius 0.5, we get

$$(0.5 - h)^2 + k^2 = (h + 0.5)^2.$$

The above equation reduces to

$$k^2 = 2h.$$

The common point between the third circle and the circle centered at the origin with radius 1 is

$$T\left(\frac{h}{\sqrt{h^2+k^2}}, \frac{k}{\sqrt{h^2+k^2}}\right).$$

SInce h is the distance from the center (h, k) to T, we obtain

$$\left(\frac{h}{\sqrt{h^2+k^2}} - h\right)^2 + \left(\frac{k}{\sqrt{h^2+k^2}} - k\right)^2 = h^2.$$

The above equation simplifies to

$$1 + k^2 = 2\sqrt{h^2+k^2}.$$

Since $k^2 = 2h$, we obtain

$$h = \frac{1}{4}, k = \frac{\sqrt{2}}{2}.$$

The center is

$$(h, k) = \left(\frac{1}{4}, \frac{\sqrt{2}}{2}\right)$$

and the radius is $h = 1/4$.

10.2 Pop Quiz

1. The center of the ellipse is $(0, 0)$, $c = 3$, and $b = 5$. Since $a^2 = b^2 + c^2 = 25 + 9$, we find $a = \sqrt{34}$. Thus, the ellipse is given by

$$\frac{x^2}{34} + \frac{y^2}{25} = 1.$$

2. The center of the ellipse is $(1, 3)$, $a = 5$, and $b = 3$. Since $25 = a^2 = b^2 + c^2 = 9 + c^2$, we obtain $16 = c^2$ or $c = 4$. The foci are $(1, 3 \pm c)$, i.e., the foci are $(1, 7)$ and $(1, -1)$.

3. Completing the square, we get

$$\begin{aligned} (x^2 + 4x + 4) + (y^2 - 10y + 25) &= 4 + 25 \\ (x+2)^2 + (y-5)^2 &= 29. \end{aligned}$$

The center of the circle is $(-2, 5)$ and the radius is $\sqrt{29}$.

10.2 Linking Concepts

a) Let P and A be the perigee and apogee, respectively. Then the length of the major axis is $2a = A + P$ and the distance from the center of the ellipse to its focus is $c = \dfrac{A+P}{2} - P$. One obtains and simplifies that the eccentricity is

$$e = \frac{c}{a} = \frac{\frac{A+P}{2}}{\frac{A-P}{2}} = \frac{A-P}{A+P}.$$

b) For Sputnik I, its perigree is $P = 3950 + 132 = 4082$ miles and its apogee is $A = 3950 + 583 = 4533$ miles.

c) Using the answers from parts a) and b), the eccentricity of the orbit of Sputnik I is
$$e = \frac{A-P}{A+P} \approx \frac{4533 - 4082}{4533 + 4082} \approx 0.052.$$

d) Solving for the apogee A in

$$\begin{aligned} e &= \frac{A-P}{A+P} \\ 0.2484 &= \frac{A - 29.64}{A + 29.64} \end{aligned}$$

one finds $A \approx 49.23$ AU.

e) Solving for the perigee P in

$$\begin{aligned} e &= \frac{A-P}{A+P} \\ 0.0934 &= \frac{2.492 \times 10^8 - P}{2.492 \times 10^8 + P} \end{aligned}$$

one finds $P \approx 2.066 \times 10^8$ km.

For Thought

1. False, it is a parabola.

2. False, there is no y-intercept.

3. True **4.** True

5. False, $y = \dfrac{b}{a}x$ is an asymptote.

6. True **7.** True, for $c = \sqrt{16+9} = 5$.

8. True, since $c = \sqrt{3+5} = \sqrt{8}$.

9. False, for $y = \dfrac{2}{3}x$ is an asymptote.

10. False, it is a circle centered at $(0,0)$.

10.3 Exercises

1. hyperbola

2. transverse axis

3. center

4. fundamental triangle

5. Vertices $(\pm 1, 0)$, foci $(\pm\sqrt{2}, 0)$, asymptotes $y = \pm x$

6. Vertices $(0, \pm 2)$, foci $(0, \pm\sqrt{5})$, asymptotes $y = \pm 2x$

7. Vertices $(1, \pm 3)$, foci $(1, \pm\sqrt{10})$

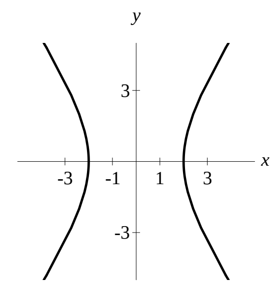
Since the slope of the asymptotes are ± 3, the equations of the asymptotes are of the form $y = \pm 3x + b$. If we substitute $(1, 0)$ into $y = \pm 3x + b$, then $0 = \pm 3(1) + b$ or $b = \mp 3$. Thus, the asymptotes are $y = 3x - 3$ and $y = -3x + 3$.

8. Vertices $(-2, 1)$ and $(0, 1)$, foci $(-1 \pm \sqrt{2}, 1)$

Since the slopes of the asymptotes are ± 1, the equations of the asymptotes are of the form $y = \pm x + b$. If we substitute $(-1, 1)$ into $y = \pm x + b$, then $1 = \pm(-1) + b$. Then $1 = \mp 1 + b$ or $1 \pm 1 = b$. Thus, the asymptotes are $y = x + 2$ and $y = -x$.

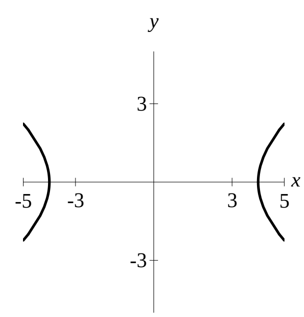
9. Note, $c = \sqrt{a^2 + b^2} = \sqrt{2^2 + 3^2} = \sqrt{13}$.

Foci $(\pm\sqrt{13}, 0)$, asymptotes $y = \pm\dfrac{b}{a}x = \pm\dfrac{3}{2}x$



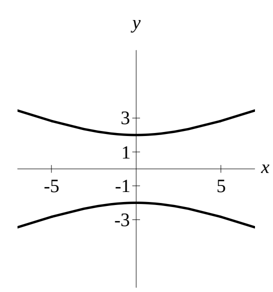
10. Note, $c = \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = 5$.

Foci $(\pm 5, 0)$, asymptotes $y = \pm\dfrac{b}{a}x = \pm\dfrac{3}{4}x$



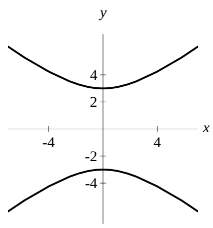
11. Note, $c = \sqrt{a^2 + b^2} = \sqrt{2^2 + 5^2} = \sqrt{29}$.

Foci $(0, \pm\sqrt{29})$, asymptotes $y = \pm\dfrac{a}{b}x = \pm\dfrac{2}{5}x$



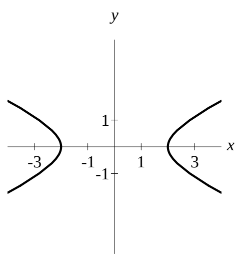
12. Note, $c = \sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = 5$.

Foci $(0, \pm 5)$, asymptotes $y = \pm \dfrac{a}{b} x = \pm \dfrac{3}{4} x$



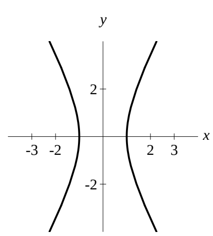
13. Note, $c = \sqrt{a^2 + b^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$.

Foci $(\pm\sqrt{5}, 0)$, asymptotes $y = \pm \dfrac{b}{a} x = \pm \dfrac{1}{2} x$



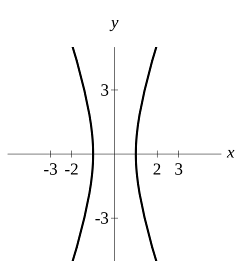
14. Note, $c = \sqrt{a^2 + b^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$.

Foci $(\pm\sqrt{5}, 0)$, asymptotes $y = \pm \dfrac{b}{a} x = \pm 2x$



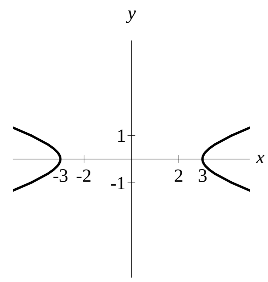
15. Note, $c = \sqrt{a^2 + b^2} = \sqrt{1^2 + 3^2} = \sqrt{10}$.

Foci $(\pm\sqrt{10}, 0)$, asymptotes $y = \pm \dfrac{b}{a} x = \pm 3x$



16. Note, $c = \sqrt{a^2 + b^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$.

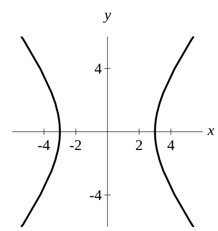
Foci $(\pm\sqrt{10}, 0)$, asymptotes $y = \pm \dfrac{b}{a} x = \pm \dfrac{1}{3} x$



17. Dividing by 144, we get $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$.

Note, $c = \sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = 5$.

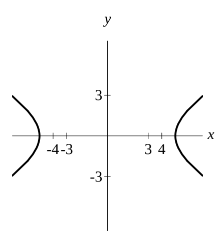
Foci $(\pm 5, 0)$, asymptotes $y = \pm \dfrac{b}{a} x = \pm \dfrac{4}{3} x$



18. Dividing by 225, we obtain $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$.

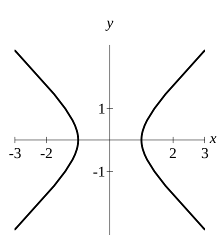
Note, $c = \sqrt{a^2 + b^2} = \sqrt{5^2 + 3^2} = \sqrt{34}$.

Foci $(\pm\sqrt{34}, 0)$, asymptotes $y = \pm \dfrac{b}{a} x = \pm \dfrac{3}{5} x$



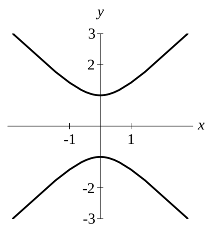
19. Note, $c = \sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$.

Foci $(\pm\sqrt{2}, 0)$, asymptotes $y = \pm \dfrac{b}{a} x = \pm x$

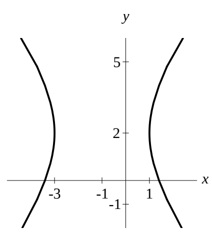


20. Note, $c = \sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$.

Foci $(0, \pm\sqrt{2})$, asymptotes $y = \pm \frac{a}{b} x = \pm x$



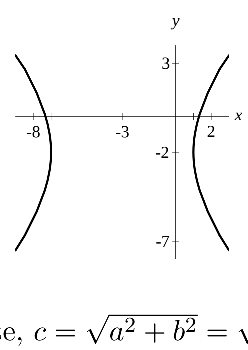
21. Note, $c = \sqrt{a^2 + b^2} = \sqrt{2^2 + 3^2} = \sqrt{13}$. Since the center is $(-1, 2)$, we find that the foci are $(-1 \pm \sqrt{13}, 2)$. Solving for y in $y - 2 = \pm \frac{3}{2}(x+1)$, we obtain that the asymptotes are $y = \frac{3}{2}x + \frac{7}{2}$ and $y = -\frac{3}{2}x + \frac{1}{2}$.



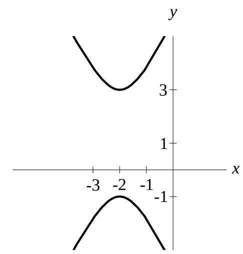
22. Note, $c = \sqrt{a^2 + b^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$. Since the center is $(-3, -2)$, we get that the the foci are $(-3 \pm \sqrt{41}, -2)$. Solving for y in

$y + 2 = \pm \frac{5}{4}(x+3)$, we obtain that the

asymptotes are $y = \frac{5}{4}x + \frac{7}{4}$ and $y = -\frac{5}{4}x - \frac{23}{4}$.

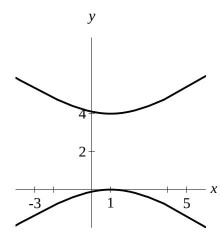


23. Note, $c = \sqrt{a^2 + b^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$. Since the center is $(-2, 1)$, the foci are $(-2, 1 \pm \sqrt{5})$. Solving for y in $y - 1 = \pm 2(x+2)$, we obtain that the asymptotes are $y = 2x + 5$ and $y = -2x - 3$.



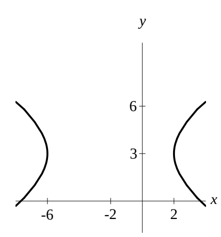
24. Note, $c = \sqrt{a^2 + b^2} = \sqrt{2^2 + 3^2} = \sqrt{13}$. Since the center is $(1, 2)$, the foci are $(1, 2 \pm \sqrt{13})$. Solving for y in

$y - 2 = \pm \frac{2}{3}(x-1)$, we find that the asymptotes are $y = \frac{2}{3}x + \frac{4}{3}$ and $y = -\frac{2}{3}x + \frac{8}{3}$.



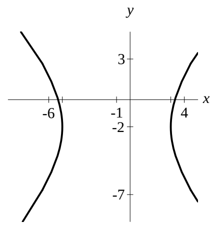
25. Note, $c = \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = 5$. Since the center is $(-2, 3)$, the foci are $(3, 3)$ and $(-7, 3)$. Solving for y in

$y - 3 = \pm \frac{3}{4}(x+2)$, we find that the asymptotes are $y = \frac{3}{4}x + \frac{9}{2}$ and $y = -\frac{3}{4}x + \frac{3}{2}$.

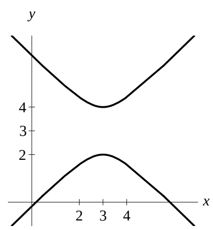


26. Note, $c = \sqrt{a^2 + b^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$. Since the center is $(-1, -2)$, the foci are $(-1 \pm \sqrt{41}, -2)$. Solving for y in

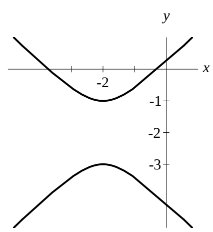
$y + 2 = \pm \frac{5}{4}(x+1)$, we get that the asymptotes are $y = \frac{5}{4}x - \frac{3}{4}$ and $y = -\frac{5}{4}x - \frac{13}{4}$.



27. Note, $c = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$. Since the center is $(3,3)$, the foci are $(3, 3 \pm \sqrt{2})$. Solving for y in $y-3 = \pm(x-3)$, we obtain that the asymptotes are $y = x$ and $y = -x+6$.



28. Note, $c = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$. Since the center is $(-2,-2)$, the foci are $(-2, -2 \pm \sqrt{2})$. Solving for y in $y+2 = \pm(x+2)$, we find that the asymptotes are $y = x$ and $y = -x-4$.



29. Since the x-intercepts are $(\pm 6, 0)$, the hyperbola is given by $\dfrac{x^2}{6^2} - \dfrac{y^2}{b^2} = 1$. From the asymptotes one gets $\dfrac{1}{2} = \dfrac{b}{6}$ and $b = 3$. An equation is $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$.

30. Since y-intercepts are $(0, \pm 2)$, the hyperbola is given by $\dfrac{y^2}{2^2} - \dfrac{x^2}{b^2} = 1$. From the asymptotes one gets $1 = \dfrac{2}{b}$ and $b = 2$. An equation is $\dfrac{y^2}{4} - \dfrac{x^2}{4} = 1$.

31. Since the x-intercepts are $(\pm 3, 0)$, the hyperbola is given by $\dfrac{x^2}{3^2} - \dfrac{y^2}{b^2} = 1$. From the foci, $c = 5$ and $b^2 = c^2 - a^2$ $= 5^2 - 3^2 = 16$. An equation is $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$.

32. Since the y-intercepts are $(0, \pm 4)$, the hyperbola is given by $\dfrac{y^2}{4^2} - \dfrac{x^2}{b^2} = 1$. From the foci, $c = 5$ and $b^2 = c^2 - a^2$ $= 5^2 - 4^2 = 9$. Equation is $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$.

33. By using the vertices of the fundamental rectangle and since it opens sideways, one gets $a = 3$, $b = 5$, and the center is at the origin. An equation is $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$.

34. By using the vertices of the fundamental rectangle and because it opens up and down, one gets $a = 7$, $b = 1$, and the center is at the origin.

An equation is $\dfrac{y^2}{49} - x^2 = 1$.

35. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$ **36.** $\dfrac{y^2}{16} - \dfrac{x^2}{4} = 1$

37. $\dfrac{y^2}{9} - \dfrac{(x-1)^2}{9} = 1$ **38.** $\dfrac{(x+2)^2}{16} - \dfrac{y^2}{4} = 1$

39. Completing the square,

$$\begin{aligned} y^2 - (x^2 - 2x) &= 2 \\ y^2 - (x^2 - 2x + 1) &= 2 - 1 \\ y^2 - (x-1)^2 &= 1 \end{aligned}$$

we obtain a hyperbola.

40. Completing the square,

$$\begin{aligned} (x^2 - 2x + 1) + 4y^2 &= 15 + 1 \\ (x-1)^2 + 4y^2 &= 16 \\ \frac{(x-1)^2}{16} + \frac{y^2}{4} &= 1 \end{aligned}$$

we get an ellipse.

41. $y = x^2 + 2x$ is a parabola.

42. Completing the square,

$$\begin{aligned} x^2 - 2x + 1 + y^2 &= 1 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

we find a circle.

43. Simplifying,

$$\begin{aligned} 25x^2 + 25y^2 &= 2500 \\ x^2 + y^2 &= 100 \end{aligned}$$

we obtain a circle.

44. Simplifying,

$$\begin{aligned} 100x^2 - 25y^2 &= 2500 \\ \frac{x^2}{25} - \frac{y^2}{100} &= 1 \end{aligned}$$

we obtain a hyperbola.

45. Simplifying,

$$\begin{aligned} 25x &= -100y^2 + 2500 \\ x &= -4y^2 + 100 \end{aligned}$$

we find a parabola.

46. Simplifying,

$$\begin{aligned} 25x^2 + 100y^2 &= 2500 \\ \frac{x^2}{100} + \frac{y^2}{25} &= 1 \end{aligned}$$

we find an ellipse.

47. Completing the square,

$$\begin{aligned} 2(x^2 - 2x) + 2(y^2 - 4y) &= -9 \\ 2(x^2 - 2x + 1) + 2(y^2 - 4y + 4) &= -9 + 2 + 8 \\ 2(x-1)^2 + 2(y-2)^2 &= 1 \\ (x-1)^2 + (y-2)^2 &= \frac{1}{2} \end{aligned}$$

we find a circle.

48. Completing the square,

$$\begin{aligned} y &= -2x^2 - 4x - 7 \\ y &= -2(x^2 + 2x) - 7 \\ y &= -2(x^2 + 2x + 1) - 7 + 2 \\ y &= -2(x+1)^2 - 5 \end{aligned}$$

we obtain a parabola.

49. Completing the square,

$$\begin{aligned} 2(x^2 + 2x) + y^2 + 6y &= -7 \\ 2(x^2 + 2x + 1) + y^2 + 6y + 9 &= -7 + 2 + 9 \\ 2(x+1)^2 + (y+3)^2 &= 4 \\ \frac{(x+1)^2}{2} + \frac{(y+3)^2}{4} &= 1 \end{aligned}$$

we get an ellipse.

50. Completing the square,

$$\begin{aligned} 9(x^2 - 2x) + 4(y^2 + 4y) &= 11 \\ 9(x^2 - 2x + 1) + 4(y^2 + 4y + 4) &= 11 + 9 + 16 \\ 9(x-1)^2 + 4(y+2)^2 &= 36 \\ \frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} &= 1 \end{aligned}$$

we find an ellipse.

51. Completing the square, we find

$$\begin{aligned} 25(x^2 - 6x + 9) - 4(y^2 + 2y + 1) &= -121 + 225 - 4 \\ 25(x-3)^2 - 4(y+1)^2 &= 100 \\ \frac{(x-3)^2}{4} - \frac{(y+1)^2}{25} &= 1. \end{aligned}$$

We have a hyperbola.

52. Completing the square, we obtain

$$\begin{aligned} 100y^2 - (x^2 - 4x + 4) &= 104 - 4 \\ 100y^2 - (x-2)^2 &= 100 \\ y^2 - \frac{(x-2)^2}{100} &= 1. \end{aligned}$$

We have a hyperbola.

53. From the center $(0,0)$ and vertex $(0,8)$ one gets $a = 8$. From the foci $(0, \pm 10)$, $c = 10$. So $b^2 = c^2 - a^2 = 10^2 - 8^2 = 36$.

Hyperbola is given by $\dfrac{y^2}{64} - \dfrac{x^2}{36} = 1$.

54. Since $p = 10$ is the distance from the vertex of the parabola to its focus and $a = \frac{1}{4p} = \frac{1}{40}$, the parabola is given by $y = \frac{1}{40}x^2$.

55. Multiply $16y^2 - x^2 = 16$ by 9 and add to $9x^2 - 4y^2 = 36$.

$$\begin{array}{rcr} -9x^2 + 144y^2 & = & 144 \\ 9x^2 - 4y^2 & = & 36 \\ \hline 140y^2 & = & 180 \\ y^2 & = & \frac{9}{7} \\ y & = & \frac{3\sqrt{7}}{7} \end{array}$$

Using $y^2 = \frac{9}{7}$ in $x^2 = 16(y^2 - 1)$, we get $x^2 = 16\left(\frac{2}{7}\right)$ or $x = \frac{4\sqrt{14}}{7}$.

The exact location is $\left(\frac{4\sqrt{14}}{7}, \frac{3\sqrt{7}}{7}\right)$.

56. Let (s, t) be the exact location where $s, t > 0$. Then we obtain

$$\begin{aligned} 4 + \sqrt{(s-4)^2 + t^2} &= \sqrt{(s+4)^2 + t^2} \\ 2 + \sqrt{s^2 + (t-4)^2} &= \sqrt{s^2 + (t+4)^2}. \end{aligned}$$

Square the first equation, then simplify to obtain

$$\sqrt{(s-4)^2 + t^2} = 2(s-1).$$

Square once more and simplify to get $12 + t^2 = 3s^2$. Similarly, from the second equation one gets $s^2 + 15 = 15t^2$. By substitution, we find

$$\begin{aligned} s^2 + 15 &= 15(3s^2 - 12) \\ 195 &= 44s^2 \\ \frac{195}{44} &= s^2 \\ \frac{195(11)}{4(11)(11)} &= s^2 \\ \frac{\sqrt{2145}}{22} &= s. \end{aligned}$$

So $t^2 = 3s^2 - 12 = 3\frac{195}{44} - 12 = \frac{57}{44}$ and $t = \frac{\sqrt{627}}{22}$. The exact location is

$$\left(\frac{\sqrt{2145}}{22}, \frac{\sqrt{627}}{22}\right).$$

Since $c^2 = 16$, the hyperbola with foci $(\pm 4, 0)$ is given by $\frac{x^2}{a^2} - \frac{y^2}{16 - a^2} = 1$. Since this hyperbola passes through the exact location, we have

$$\frac{195/44}{a^2} - \frac{57/44}{16 - a^2} = 1$$

from which one solves and finds that $a^2 = 4$.

This hyperbola is given by $\frac{x^2}{4} - \frac{y^2}{12} = 1$.

Similarly, the other hyperbola with foci $(0, \pm 4)$ is given by

$$y^2 - \frac{x^2}{15} = 1.$$

57. Since $c^2 = a^2 + b^2 = 1^2 + 1^2 = 2$, the foci of $x^2 - y^2 = 1$ are $A(\sqrt{2}, 0)$ and $B(-\sqrt{2}, 0)$. Note, $y^2 = x^2 - 1$. Suppose (x, y) is a point on the hyperbola whose distance from B is twice the distance between (x, y) and A. Then

$$\begin{aligned} 2\sqrt{(x-\sqrt{2})^2 + y^2} &= \sqrt{(x+\sqrt{2})^2 + y^2} \\ 4((x-\sqrt{2})^2 + y^2) &= (x+\sqrt{2})^2 + y^2 \end{aligned}$$

$$\begin{aligned} 4(x-\sqrt{2})^2 - (x+\sqrt{2})^2 + 3y^2 &= 0 \\ 3x^2 - 10\sqrt{2}x + 6 + 3y^2 &= 0 \\ 3x^2 - 10\sqrt{2}x + 6 + 3(x^2 - 1) &= 0 \\ 6x^2 - 10\sqrt{2}x + 3 &= 0. \end{aligned}$$

Solving for x, one finds $x = \frac{3\sqrt{2}}{2}$ and $x = \frac{\sqrt{2}}{6}$; the second value must be excluded since it is out of the domain. Substituting $x = \frac{3\sqrt{2}}{2}$ into $y^2 = x^2 - 1$, one obtains $y = \pm\frac{\sqrt{14}}{2}$.

By the symmetry of the hyperbola, there are four points that are twice as far from one focus as they are from the other focus. Namely, these points are

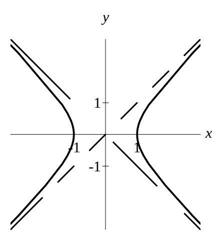
$$\left(\pm\frac{3\sqrt{2}}{2}, \pm\frac{\sqrt{14}}{2}\right).$$

58. The asymptotes of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ are $y = \pm\dfrac{b}{a}x$.

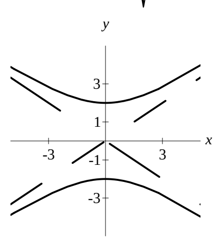
The slopes are perpendicular precisely when $\dfrac{b}{a}\left(-\dfrac{b}{a}\right) = -1$. This happens exactly when $\dfrac{b^2}{a^2} = 1$ or equivalently when $a = b$ (since $a, b > 0$).

59. Note, the asymptotes are $y = \pm x$. The difference is

$$50 - \sqrt{50^2 - 1} \approx 0.01$$



60. Note, the asymptotes are $y = \pm\dfrac{2}{3}x$. The difference is $2\sqrt{1 + \dfrac{50^2}{9}} - \dfrac{2}{3}(50) \approx 0.06$.



62. Let $\mathcal{D} = B^2 - 4AC$.

If $\mathcal{D} = 0$, the graph is a parabola, or for degenerate cases, the graph is a line, parallel lines, or no graph at all.

If $\mathcal{D} < 0$, the graph is an ellipse, or for degenerate cases, the graph is a circle, point, or no graph at all.

If $\mathcal{D} > 0$, the graph is a hyperbola, or a pair of intersecting lines.

63. $\begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix} =$

$\begin{bmatrix} 1(-1) + 2(2) & 1(3) + 2(4) \\ -3(-1) + 5(2) & -3(3) + 5(4) \end{bmatrix} =$

$\begin{bmatrix} 3 & 11 \\ 13 & 11 \end{bmatrix}$

64. Completing the square, we obtain

$$y = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}$$

Since $\dfrac{1}{4p} = 1$, we find $p = \dfrac{1}{4}$.

Since vertex is $\left(\dfrac{1}{2}, -\dfrac{1}{4}\right)$, the focus is $\left(\dfrac{1}{2}, -\dfrac{1}{4} + p\right) = \left(\dfrac{1}{2}, 0\right)$, and directrix is $y = -\dfrac{1}{4} - p = -\dfrac{1}{2}$ or $y = -\dfrac{1}{2}$.

65. Since $a = 6$ and $c = 5$, we find $b^2 = a^2 - c^2 = 36 - 25 = 11$. Then the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{36} + \frac{y^2}{11} = 1.$$

66. The midpoint of $(0, 0)$ and $(3, 4)$ is the center which is $(3/2, 2)$. Then the radius r satisfies

$$r^2 = \left(\frac{3}{2}\right)^2 + 2^2 = \frac{25}{4}.$$

The circle is given by

$$\left(x - \frac{3}{2}\right)^2 + (y - 2)^2 = \frac{25}{4}.$$

67. Let $f(x) = x(x - 1) > 0$.

We apply the test-point method.

Note, $f(-1) > 0, f(1/2) < 0, f(2) > 0$.

+	0	−	0	+
-1	0	$\frac{1}{2}$	1	2

The solution set is $(-\infty, 0) \cup (1, \infty)$.

68. Combine the logarithms and solve for x.

$$\begin{aligned} \log\left(\frac{x+1}{x}\right) &= 2 \\ \frac{x+1}{x} &= 10^2 \\ 1+\frac{1}{x} &= 100 \\ \frac{1}{x} &= 99 \end{aligned}$$

The solution set is $\left\{\frac{1}{99}\right\}$.

Thinking Outside the Box

LXXXIX On the xyz-space, the points $F(x,0,0)$, $T(0,\frac{8x}{x-8},8)$, and $C(8,8,\frac{8(x-8)}{x})$ are collinear. The points F and T are the endpoints of an iron pipe where F lies on the floor of one of the corridors, and T lies on the ceiling of the other corridor.

Using the Pythagorean Theorem, the length of the iron pipe (distance from F to T) is

$$d = \sqrt{x^2 + \left(\frac{8x}{x-8}\right)^2 + 8^2}.$$

Using a calculator, the minimum value of d is 24, and this occurs when $x = 16$. Hence, the longest length of an iron pipe that can be transported is 24 feet.

XC Since the circumference is 30 inches, the radius of the basketball is $15/\pi$ inches. For simplicity assume the basketball has radius 1 inch and just multiply the radius of the small ball by $15/\pi$.

Slice the cubic box vertically on the diagonal and we get a rectangle that is $\sqrt{8}$-inch wide by 2-inch high with a hypotenuse of $\sqrt{12}$ inches. The circular cross section of the basketball touches the top and bottom of the rectangle, but not the sides. The circular cross section of the small ball in the upper right corner is tangent to the large circle and the top of the rectangle, but does not touch the right side of this rectangle.

Let x be the radius of the small ball and y be the distance from the small ball to the upper right corner of the rectangle. Of course there is another cirle in the lower left corner of the rectangle that does not touch the left side. Then we have

$$2 + 4x + 2y = \sqrt{12}.$$

The second equation comes from thinking of a cubic box that would be $2x$ on each side and exactly contain the small ball. Its diagonal has length $x\sqrt{12}$. The diagonal is also $2x + 2y$. Then

$$2x + 2y = x\sqrt{12}.$$

Solve simultaneously to get $x = 2 - \sqrt{3}$. Multiplying by $15/\pi$, we obtain the radius of the smaller ball, namely,

$$\frac{15}{\pi}(2-\sqrt{3}) \text{ inches.}$$

10.3 Pop Quiz

1. Since $c^2 = a^2+b^2 = 9+4 = 13$, we find $c = \sqrt{13}$. Thus, the foci are $(\pm\sqrt{13}, 0)$. The asymptotes are $y = \pm\frac{b}{a}x$ or $y = \pm\frac{2}{3}x$.

2. Since the vertices are $(0, \pm 6)$, we have $a = 6$. The hyperbola can be expressed in the form $y^2/a^2 - x^2/b^2 = 1$. Since $y = \pm 3x$ are the asymptotes, we obtain $a/b = 3$ or $6/b = 3$. Then $b = 2$ and the hyperbola is given by

$$\frac{y^2}{36} - \frac{x^2}{4} = 1.$$

10.3 Linking Concepts

a) The asymptotes $y = \pm\frac{b}{a}x$ are perpendicular precisely when $\frac{b}{a}\left(-\frac{b}{a}\right) = -1$ or equivalently when $a = b$. Since $c^2 = a^2+b^2 = 2a^2$, $c = \sqrt{2}a$. The eccentricity is $e = \frac{c}{a} = \frac{\sqrt{2}a}{a} = \sqrt{2}$.

b) The eccentricity of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $e_1 = \frac{c}{a} = \frac{\sqrt{a^2+b^2}}{a}$. Similarly, the eccentricity of $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ is $e_2 = \frac{\sqrt{a^2+b^2}}{b}$. Then

$$\begin{aligned} e_1^2 e_2^2 &= \frac{(a^2+b^2)^2}{a^2b^2} \\ &= \frac{a^4+2a^2b^2+b^4}{a^2b^2} \\ &= \frac{b^2(a^2+b^2)}{a^2b^2} + \frac{a^2(a^2+b^2)}{a^2b^2} \\ &= \frac{a^2+b^2}{a^2} + \frac{a^2+b^2}{b^2} \\ e_1^2 e_2^2 &= e_1^2 + e_2^2. \end{aligned}$$

c) One can start off by rewriting y_1^2, i.e.,

$$\begin{aligned} y_1^2 &= b^2\left(\frac{x_1^2}{a^2} - 1\right) \\ &= (c^2 - a^2)\left(\frac{x_1^2}{a^2} - 1\right) \\ &= ((ae)^2 - a^2)\left(\frac{x_1^2}{a^2} - 1\right) \\ &= (e^2-1)(x_1^2 - a^2) \\ y_1^2 &= e^2x_1^2 - e^2a^2 - x_1^2 + a^2. \end{aligned}$$

Note, the foci are $(\pm ae, 0)$. By using the expression above for y_1^2, one finds that the shorter foci (the distance between (x_1, y_1) and $(ae, 0)$) is given by

$$\begin{aligned} &= \sqrt{(x_1 - ae)^2 + y_1^2} \\ &= \sqrt{x_1^2 - 2x_1ae + a^2e^2 + y_1^2} \\ &= \sqrt{x_1^2 - 2x_1ae + a^2e^2 + e^2x_1^2 - e^2a^2 - x_1^2 + a^2} \\ &= \sqrt{a^2 - 2x_1ae + e^2x_1^2} \\ &= \sqrt{(a - ex_1)^2} \\ &= ex_1 - a \end{aligned}$$

since $x_1 > a$ and $e > 1$. Thus, the shorter focal radius is $ex_1 - a$. Since two focal radii differ by $2a$, the longer focal radius is given by $2a + (ex_1 - a)$, or equivalently by $ex_1 + a$.

For Thought

1. False, the graph is an ellipse **2.** True

3. True, since the equation can be rewritten as $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

4. False, the $x'y'$ coordinates are given by
$$(x', y') = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \cos 45^o & -\sin 45^o \\ \sin 45^o & \cos 45^o \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \end{bmatrix}.$$

5. True, the $x'y'$ coordinates are given by
$$(x', y') = \begin{bmatrix} \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} \cos 30^o & -\sin 30^o \\ \sin 30^o & \cos 30^o \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \end{bmatrix}.$$

6. True, the xy coordinates are given by
$$(x, y) = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} \cos 60^o & \sin 60^o \\ -\sin 60^o & \cos 60^o \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} & 1 \end{bmatrix}.$$

7. True **8.** False, $\theta = 30°$

9. True, see Example 5. **10.** True

10.4 Exercises

1. Completing the square,

$$\begin{aligned} x^2 + 6x + y^2 - 3y &= -1 \\ (x+3)^2 + \left(y - \frac{3}{2}\right)^2 &= -1 + 9 + \frac{9}{4} \\ (x+3)^2 + \left(y - \frac{3}{2}\right)^2 &= \frac{41}{4} \end{aligned}$$

one gets a circle.

2. Completing the square,

$$\begin{aligned} x^2 + 8x - (y^2 - 6y) &= -2 \\ (x+4)^2 - (y-3)^2 &= -2 + 16 - 9 \\ (x+4)^2 - (y-3)^2 &= 5 \end{aligned}$$

one obtains a hyperbola.

3. Completing the square, one obtains

$$\begin{aligned} 2(x^2+2x)+6(y^2-2y) &= 3 \\ 2(x+1)^2+6(y-1)^2 &= 3+2+6 \\ \frac{(x+1)^2}{11/2}+\frac{(y-1)^2}{11/6} &= 1. \end{aligned}$$

We get an ellipse.

4. Completing the square, we get

$$\begin{aligned} 3(x^2+4x)-9 &= 3y \\ 3(x+2)^2-9-12 &= 3y \\ (x+2)^2-7 &= y. \end{aligned}$$

A parabola.

5. Completing the square, one obtains

$$\begin{aligned} 3x &= -6(y^2+3y)+8 \\ 3x &= -6\left(y+\frac{3}{2}\right)^2+8+\frac{27}{2} \\ 3x &= -6\left(y+\frac{3}{2}\right)^2+\frac{43}{2} \\ x &= -2\left(y+\frac{3}{2}\right)^2+\frac{43}{6} \end{aligned}$$

A parabola.

6. Completing the square, we find

$$\begin{aligned} 4\left(x^2+\frac{1}{2}x\right)+5\left(y^2-\frac{1}{5}y\right) &= 1 \\ 4\left(x+\frac{1}{4}\right)^2+5\left(y-\frac{1}{10}\right)^2 &= 1+\frac{1}{4}+\frac{1}{20} \\ 4\left(x+\frac{1}{4}\right)^2+5\left(y-\frac{1}{10}\right)^2 &= \frac{26}{20} \\ \frac{(x+1/4)^2}{13/40}+\frac{(y-1/10)^2}{13/50} &= 1 \end{aligned}$$

An ellipse.

7. An ellipse since $AC=(4)(3)>0$.

8. An ellipse since $AC=(-3)(-8)>0$.

9. A parabola since $AC=(2)(0)=0$.

10. A parabola since $AC=(0)(5)=0$.

11. A hyperbola since $AC=(9)(-5)<0$.

12. A hyperbola since $AC=(-8)(1)<0$.

13. Since $[x',y'] = \begin{bmatrix} 3 & 3 \end{bmatrix}\begin{bmatrix} \cos 45^o & -\sin 45^o \\ \sin 45^o & \cos 45^o \end{bmatrix}$ $= \begin{bmatrix} 3 & 3 \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} & 0 \end{bmatrix}$, the $x'y'$-coordinates are $(3\sqrt{2},0)$.

14. Since $[x',y'] = \begin{bmatrix} 0 & 5 \end{bmatrix}\begin{bmatrix} \cos 45^o & -\sin 45^o \\ \sin 45^o & \cos 45^o \end{bmatrix}$ $= \begin{bmatrix} 0 & 5 \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$ $= \begin{bmatrix} 5\sqrt{2}/2 & 5\sqrt{2}/2 \end{bmatrix}$, the $x'y'$-coordinates are $(5\sqrt{2},5\sqrt{2}/2)$.

15. Since $[x',y'] = \begin{bmatrix} \sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} \cos 45^o & -\sin 45^o \\ \sin 45^o & \cos 45^o \end{bmatrix}$ $= \begin{bmatrix} \sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$, the $x'y'$-coordinates are $(1,-1)$.

16. Since $[x',y'] =$ $\begin{bmatrix} -\sqrt{2} & \sqrt{2} \end{bmatrix}\begin{bmatrix} \cos 45^o & -\sin 45^o \\ \sin 45^o & \cos 45^o \end{bmatrix}$ $=\begin{bmatrix} -\sqrt{2} & \sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}=$ $\begin{bmatrix} 0 & 2 \end{bmatrix}$, the $x'y'$-coordinates are $(0,2)$.

17. Since $[x',y'] =$ $\begin{bmatrix} 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} \cos 60^o & -\sin 60^o \\ \sin 60^o & \cos 60^o \end{bmatrix}$ $=\begin{bmatrix} 1 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}=$ $\begin{bmatrix} 2 & 0 \end{bmatrix}$, the $x'y'$-coordinates are $(2,0)$.

18. Since $[x',y'] =$ $\begin{bmatrix} -\sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} \cos 60^o & -\sin 60^o \\ \sin 60^o & \cos 60^o \end{bmatrix}$ $=\begin{bmatrix} -\sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}=$ $\begin{bmatrix} 0 & 2 \end{bmatrix}$, the $x'y'$-coordinates are $(0,2)$.

19. Since $[x',y'] =$ $\begin{bmatrix} 2 & 5 \end{bmatrix}\begin{bmatrix} \cos 60^o & -\sin 60^o \\ \sin 60^o & \cos 60^o \end{bmatrix}$

$=\begin{bmatrix} 2 & 5 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}=$
$\begin{bmatrix} \frac{2+5\sqrt{3}}{2} & \frac{5-2\sqrt{3}}{2} \end{bmatrix}$, the
$x'y'$-coordinates are $\left(\frac{2+5\sqrt{3}}{2}, \frac{5-2\sqrt{3}}{2}\right)$.

20. Since $[x', y'] =$
$\begin{bmatrix} -1 & 2 \end{bmatrix}\begin{bmatrix} \cos 60^o & -\sin 60^o \\ \sin 60^o & \cos 60^o \end{bmatrix}$
$=\begin{bmatrix} -1 & 2 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}=$
$\begin{bmatrix} \frac{2\sqrt{3}-1}{2} & \frac{\sqrt{3}+2}{2} \end{bmatrix}$, the
$x'y'$-coordinates are $\left(\frac{2\sqrt{3}-1}{2}, \frac{\sqrt{3}+2}{2}\right)$.

21. Since $\cot 2\theta = 0$, the smallest positive angle θ satisfies $2\theta = 90^o$. Thus, $\theta = 45^o$.

22. Since $\cot 2\theta = \frac{1}{\sqrt{3}}$, the smallest positive angle θ
satisfies $2\theta = 60^o$. Thus, $\theta = 30^o$.

23. Since $\cot 2\theta = -\frac{1}{\sqrt{3}}$, the smallest positive angle θ satisfies $2\theta = 120^o$. Thus, $\theta = 60^o$.

24. Since $\cot 2\theta = \sqrt{3}$, the smallest positive angle θ satisfies $2\theta = 30^o$. Thus, $\theta = 15^o$.

25. Note, $x = x'\cos(\pi/4) - y'\sin(\pi/4) = \frac{x'-y'}{\sqrt{2}}$ and $y = x'\sin(\pi/4) + y'\cos(\pi/4) = \frac{x'+y'}{\sqrt{2}}$. Substituting into $y = x$, we get $\frac{x'-y'}{\sqrt{2}} = \frac{x'+y'}{\sqrt{2}}$. Then $x' - y' = x' + y'$. Solving for y', we have $y' = 0$.

26. Note, $x = x'\cos(\pi/4) - y'\sin(\pi/4) = \frac{x'-y'}{\sqrt{2}}$ and $y = x'\sin(\pi/4) + y'\cos(\pi/4) = \frac{x'+y'}{\sqrt{2}}$. Substituting into $y = -x$, we obtain $\frac{x'-y'}{\sqrt{2}} = -\frac{x'+y'}{\sqrt{2}}$. Then $x' - y' = -x' - y'$ or $x' = 0$.

27. Note, $x = x'\cos(\pi/3) - y'\sin(\pi/3) = \frac{x' - y'\sqrt{3}}{2}$ and $y = x'\sin(\pi/3) + y'\cos(\pi/3) = \frac{x'\sqrt{3}+y'}{2}$. Substituting into $y = 3x$, we obtain

$$\frac{x'\sqrt{3}+y'}{2} = 3\frac{x'-y'\sqrt{3}}{2}.$$

Then $x'\sqrt{3} + y' = 3(x' - y'\sqrt{3})$. Solving for y', $y' + 3\sqrt{3}y' = 3x' - x'\sqrt{3}$. Thus, $y' = \frac{3-\sqrt{3}}{1+3\sqrt{3}}x'$.

28. Note, $x = x'\cos(\pi/6) - y'\sin(\pi/6) = \frac{\sqrt{3}x'-y'}{2}$ and $y = x'\sin(\pi/6) + y'\cos(\pi/6) = \frac{x'+\sqrt{3}y'}{2}$. Substituting into $y = 2x$, we get

$$\frac{x'+\sqrt{3}y'}{2} = 2\frac{\sqrt{3}x'-y'}{2}.$$

Then $y'(\sqrt{3}+2) = (2\sqrt{3}-1)x'$. Solving for y', we obtain $y' = \frac{2\sqrt{3}-1}{\sqrt{3}+2}x'$.

29. Note, $x = x'\cos(\pi/4) - y'\sin(\pi/4) = \frac{x'-y'}{\sqrt{2}}$ and
$y = x'\sin(\pi/4) + y'\cos(\pi/4) = \frac{x'+y'}{\sqrt{2}}$. Substituting into $xy = 2$, we get $\frac{(x')^2-(y')^2}{2} = 2$. Then $(x')^2 - (y')^2 = 4$.

30. Note, $x = x'\cos(\pi/4) - y'\sin(\pi/4) = \frac{x'-y'}{\sqrt{2}}$ and
$y = x'\sin(\pi/4) + y'\cos(\pi/4) = \frac{x'+y'}{\sqrt{2}}$. Substituting
into $xy = -2$, we get $\frac{(x')^2-(y')^2}{2} = -2$. Thus, $(y')^2 - (x')^2 = 4$.

31. Note, $x = x'\cos(\pi/6) - y'\sin(\pi/6) = \frac{\sqrt{3}x'-y'}{2}$ and $y = x'\sin(\pi/6) + y'\cos(\pi/6) = \frac{x'+\sqrt{3}y'}{2}$.
Substituting into $x^2 + y^2 = 4$, we get

$\dfrac{3(x')^2 - 2\sqrt{3}x'y' + (y')^2}{4} + \dfrac{(x')^2 + 2\sqrt{3}x'y' + 3(y')^2}{4} = 4.$

Then $\dfrac{4(x')^2 + 4(y')^2}{4} = 4$ or $(x')^2 + (y')^2 = 4$.

32. Note, $x = x'\cos(\pi/4) - y'\sin(\pi/4) = \dfrac{x'-y'}{\sqrt{2}}$ and $y = x'\sin(\pi/4) + y'\cos(\pi/4) = \dfrac{x'+y'}{\sqrt{2}}$. Substituting into $x^2 - xy + y^2 = 1$, we get $\left(\dfrac{x'-y'}{\sqrt{2}}\right)^2 - \left(\dfrac{x'-y'}{\sqrt{2}}\right)\left(\dfrac{x'+y'}{\sqrt{2}}\right) + \left(\dfrac{x'+y'}{\sqrt{2}}\right)^2 = 1$. Multiplying the equation by 2, we get

$$(x'-y')^2 - ((x')^2 - (y')^2) + (x'+y')^2 = 2.$$

So, $(x')^2 + 3(y')^2 = 2$.

33. Note, $x = x'\cos(\pi/4) - y'\sin(\pi/4) = \dfrac{x'-y'}{\sqrt{2}}$ and $y = x'\sin(\pi/4) + y'\cos(\pi/4) = \dfrac{x'+y'}{\sqrt{2}}$. Substituting into $x^2 - 2xy + y^2 - \sqrt{2}x - \sqrt{2}y = 0$, we get $\left(\dfrac{x'-y'}{\sqrt{2}}\right)^2 - 2\left(\dfrac{x'-y'}{\sqrt{2}}\right)\left(\dfrac{x'+y'}{\sqrt{2}}\right) + \left(\dfrac{x'+y'}{\sqrt{2}}\right)^2 - \sqrt{2}\left(\dfrac{x'-y'}{\sqrt{2}}\right) - \sqrt{2}\left(\dfrac{x'+y'}{\sqrt{2}}\right) = 0$. Multiplying by 2, we obtain $(x'-y')^2 - 2((x')^2 - (y')^2) + (x'+y')^2 - 2(x'-y') - 2(x'+y') = 0$. This can be simplified to $x' = (y')^2$.

34. Note, $x = x'\cos(\pi/6) - y'\sin(\pi/6) = \dfrac{\sqrt{3}x'-y'}{2}$ and $y = x'\sin(\pi/6) + y'\cos(\pi/6) = \dfrac{x'+\sqrt{3}y'}{2}$. Substituting into the original equation, we get $2\left(\dfrac{\sqrt{3}x'-y'}{2}\right)^2 + \sqrt{3}\left(\dfrac{\sqrt{3}x'-y'}{2}\right)\left(\dfrac{x'+\sqrt{3}y'}{2}\right) + \left(\dfrac{x'+\sqrt{3}y'}{2}\right)^2 - 2\left(\dfrac{\sqrt{3}x'-y'}{2}\right) - 2\left(\dfrac{x'+\sqrt{3}y'}{2}\right) = 0$. Multiplying by 4, we get $2(\sqrt{3}x'-y')^2 + \sqrt{3}(\sqrt{3}x'-y')(x'+\sqrt{3}y') + (x'+\sqrt{3}y') - 4(\sqrt{3}x'-y') + 4(x'+\sqrt{3}y') = 0$. This can be simplified to $5(x')^2 - (2\sqrt{3}+2)x' + (y')^2 + (2-2\sqrt{3})y' = 0$.

35. Note, $\cot 2\theta = \dfrac{A-C}{B} = 0$. So, $\theta = \pi/4$. Then $x = x'\cos(\pi/4) - y'\sin(\pi/4) = \dfrac{x'-y'}{\sqrt{2}}$ and $y = x'\sin(\pi/4) + y'\cos(\pi/4) = \dfrac{x'+y'}{\sqrt{2}}$. Substitute into $xy - 6 = 0$. So, $\left(\dfrac{x'-y'}{\sqrt{2}}\right)\left(\dfrac{x'+y'}{\sqrt{2}}\right) - 6 =$. Simplifying, we obtain $(x')^2 - (y')^2 = 12$. This is a hyperbola.

36. Note, $\cot 2\theta = \dfrac{A-C}{B} = 0$. So, $\theta = \pi/4$. Then $x = x'\cos(\pi/4) - y'\sin(\pi/4) = \dfrac{x'-y'}{\sqrt{2}}$ and $y = x'\sin(\pi/4) + y'\cos(\pi/4) = \dfrac{x'+y'}{\sqrt{2}}$. Substituting into $xy + 3 = 0$, we get $\left(\dfrac{x'-y'}{\sqrt{2}}\right)\left(\dfrac{x'+y'}{\sqrt{2}}\right) + 3 = 0$. Simplifying, we obtain $(x')^2 - (y')^2 = -6$. This is a hyperbola.

37. Note, $\cot 2\theta = \dfrac{13-7}{6\sqrt{3}} = \dfrac{1}{\sqrt{3}}$. So, $\theta = \pi/6$, $x = x'\cos(\pi/6) - y'\sin(\pi/6) = \dfrac{\sqrt{3}x'-y'}{2}$, and $y = x'\sin(\pi/6) + y'\cos(\pi/6) = \dfrac{x'+\sqrt{3}y'}{2}$. Substituting into the original equation, we get $13\left(\dfrac{\sqrt{3}x'-y'}{2}\right)^2 + 6\sqrt{3}\left(\dfrac{\sqrt{3}x'-y'}{2}\right)\left(\dfrac{x'+\sqrt{3}y'}{2}\right) + 7\left(\dfrac{x'+\sqrt{3}y'}{2}\right)^2 - 16 = 0$. Multiplying by 4, we get $13(3(x')^2 - 2\sqrt{3}x'y' + (y')^2) + 6\sqrt{3}(\sqrt{3}(x')^2 + 2x'y' - \sqrt{3}(y')^2) + 7((x')^2 + 2\sqrt{3}x'y' + 3(y')^2) - 64 = 0$.
This simplifies to $64(x')^2 + 16(y')^2 - 64 = 0$. Thus, $4(x')^2 + (y')^2 = 4$. An ellipse.

38. Note, $\cot 2\theta = \dfrac{1-11}{10\sqrt{3}} = \dfrac{-1}{\sqrt{3}}$. So, $\theta = \pi/3$, $x = x'\cos(\pi/3) - y'\sin(\pi/3) = \dfrac{x' - y'\sqrt{3}}{2}$, and $y = x'\sin(\pi/3) + y'\cos(\pi/3) = \dfrac{x'\sqrt{3} + y'}{2}$. Substituting into the original equation, we get $\left(\dfrac{x' - y'\sqrt{3}}{2}\right)^2 +$ $10\sqrt{3}\left(\dfrac{x' - y'\sqrt{3}}{2}\right)\left(\dfrac{x'\sqrt{3} + y'}{2}\right) +$ $11\left(\dfrac{x'\sqrt{3} + y'}{2}\right)^2 + 16 = 0$. Multiplying by 4, we get $(x' - y'\sqrt{3})^2 + 10\sqrt{3}(x' - y'\sqrt{3})(x'\sqrt{3} + y') + 11(x'\sqrt{3} + y')^2 + 64 = 0$. Note, there must be no $x'y'$ term. This reduces to $64(x')^2 - 16(y')^2 + 64 = 0$ and can be rewritten as $(y')^2 - 4(x')^2 = 4$. A hyperbola.

39. Note, $\cot 2\theta = \dfrac{A-C}{B} = \dfrac{3-3}{-10} = 0$. So, $\theta = \pi/4$. Then $x = x'\cos(\pi/4) - y'\sin(\pi/4) = \dfrac{x' - y'}{\sqrt{2}}$ and $y = x'\sin(\pi/4) + y'\cos(\pi/4) = \dfrac{x' + y'}{\sqrt{2}}$. Substitute into the original equation. So, $3\left(\dfrac{x' - y'}{\sqrt{2}}\right)^2 -$ $10\left(\dfrac{x' - y'}{\sqrt{2}}\right)\left(\dfrac{x' + y'}{\sqrt{2}}\right) + 3\left(\dfrac{x' + y'}{\sqrt{2}}\right)^2$ $-16\sqrt{2}\left(\dfrac{x' - y'}{\sqrt{2}}\right) + 16\sqrt{2}\left(\dfrac{x' + y'}{\sqrt{2}}\right) + 12 = 0$. Multiplying by 2, we get $3(x' - y')^2 - 10((x')^2 - (y')^2 + 3(x' + y')^2 - 32(x' - y')^2 + 32(x' + y') + 24 = 0$. Note, there must be no $x'y'$ term. Simplifying, we obtain $-4(x')^2 + 16(y')^2 + 64y' + 24 = 0$. Rewriting, we get $(x')^2 - 4(y')^2 - 16y' - 6 = 0$.
This is a hyperbola.

40. Note, $\cot 2\theta = \dfrac{A-C}{B} = \dfrac{5-5}{-26} = 0$. So, $\theta = \pi/4$. Then $x = x'\cos(\pi/4) - y'\sin(\pi/4) = \dfrac{x' - y'}{\sqrt{2}}$ and $y = x'\sin(\pi/4) + y'\cos(\pi/4) = \dfrac{x' + y'}{\sqrt{2}}$. Substitute into the original equation. So, $5\left(\dfrac{x' - y'}{\sqrt{2}}\right)^2 -$ $26\left(\dfrac{x' - y'}{\sqrt{2}}\right)\left(\dfrac{x' + y'}{\sqrt{2}}\right) + 5\left(\dfrac{x' + y'}{\sqrt{2}}\right)^2$ $+16\sqrt{2}\left(\dfrac{x' - y'}{\sqrt{2}}\right) + 16\sqrt{2}\left(\dfrac{x' + y'}{\sqrt{2}}\right) - 104 = 0$. Multiplying by 2, we get $5(x' - y')^2 - 26((x')^2 - (y')^2) + 5(x' + y')^2 + 32(x' - y') + 32(x' + y') - 208 = 0$. Note, there must be no $x'y'$ term. Simplifying, we obtain $-16(x')^2 + 36(y')^2 + 64x' - 208 = 0$. Rewriting, we get $4(x')^2 - 16x' - 9(y')^2 + 52 = 0$. A hyperbola.

41. Note, $\cot 2\theta = \dfrac{A-C}{B} = \dfrac{0-0}{2} = 0$. So, $\theta = \pi/4$. Then $x = x'\cos(\pi/4) - y'\sin(\pi/4) = \dfrac{x' - y'}{\sqrt{2}}$ and $y = x'\sin(\pi/4) + y'\cos(\pi/4) = \dfrac{x' + y'}{\sqrt{2}}$. Substitute into the original equation. So, $2\left(\dfrac{x' - y'}{\sqrt{2}}\right)\left(\dfrac{x' + y'}{\sqrt{2}}\right) +$ $3\sqrt{2}\left(\dfrac{x' - y'}{\sqrt{2}}\right) + \sqrt{2}\left(\dfrac{x' + y'}{\sqrt{2}}\right) + 4 = 0$.
Simplifying, we obtain

$$((x')^2 - (y')^2) + 3(x' - y') + (x' + y') + 4 = 0.$$

Thus, we have a hyperbola

$$(x')^2 + 4x' - (y')^2 - 2y' + 4 = 0.$$

42. Note, $\cot 2\theta = \dfrac{A-C}{B} = \dfrac{0-0}{4} = 0$. So, $\theta = \pi/4$. Then $x = x'\cos(\pi/4) - y'\sin(\pi/4) = \dfrac{x' - y'}{\sqrt{2}}$ and $y = x'\sin(\pi/4) + y'\cos(\pi/4) = \dfrac{x' + y'}{\sqrt{2}}$. Substitute into the original equation. So, $4\left(\dfrac{x' - y'}{\sqrt{2}}\right)\left(\dfrac{x' + y'}{\sqrt{2}}\right) +$ $-5\sqrt{2}\left(\dfrac{x' - y'}{\sqrt{2}}\right) + 3\sqrt{2}\left(\dfrac{x' + y'}{\sqrt{2}}\right) + 6 = 0$. Simplifying, we obtain $2((x')^2 - (y')^2) - 5(x' - y') + 3(x' + y') + 6 = 0$. Dividing by 2, we get $(x')^2 - x' - (y')^2 + 4y' + 3 = 0$. A hyperbola.

43. Note, $\cot 2\theta = \dfrac{A-C}{B} = \dfrac{3-1}{-2\sqrt{3}} = \dfrac{-1}{\sqrt{3}}$. So, $\theta = \pi/3$. Then $x = x'\cos(\pi/3) - y'\sin(\pi/3) =$

$\dfrac{x' - \sqrt{3}y'}{2}$ and $y = x'\sin(\pi/3) + y'\cos(\pi/3) = \dfrac{\sqrt{3}x' + y'}{2}$. Substitute into the original equation. So, $3\left(\dfrac{x' - \sqrt{3}y'}{2}\right)^2 - 2\sqrt{3}\left(\dfrac{x' - \sqrt{3}y'}{2}\right)\left(\dfrac{\sqrt{3}x' + y'}{2}\right) + \left(\dfrac{\sqrt{3}x' + y'}{2}\right)^2 + (\sqrt{3} - 1)\left(\dfrac{x' - \sqrt{3}y'}{2}\right) - (1 + \sqrt{3})\left(\dfrac{\sqrt{3}x' + y'}{2}\right) - 1 = 0.$

Multiplying by 4, we get $3\left(x' - \sqrt{3}y'\right)^2 - 2\sqrt{3}\left(x' - \sqrt{3}y'\right)\left(\sqrt{3}x' + y'\right) + \left(\sqrt{3}x' + y'\right)^2 + 2(\sqrt{3} - 1)\left(x' - \sqrt{3}y'\right) - 2(1 + \sqrt{3})\left(\sqrt{3}x' + y'\right) - 4 = 0$. Expanding and combining like terms, we obtain $-8x' + 16(y')^2 - 8y' - 4 = 0$. Rewriting, we get $2x' - 4(y')^2 + 2y' + 1 = 0$. A parabola.

44. Note, $\cot 2\theta = \dfrac{A - C}{B} = \dfrac{0 - 0}{2} = 0$. So, $\theta = \pi/4$. Then $x = x'\cos(\pi/4) - y'\sin(\pi/4) = \dfrac{x' - y'}{\sqrt{2}}$ and $y = x'\sin(\pi/4) + y'\cos(\pi/4) = \dfrac{x' + y'}{\sqrt{2}}$. Substitute into the original equation. So, $\left(\dfrac{x' - y'}{\sqrt{2}}\right)^2 + 2\left(\dfrac{x' - y'}{\sqrt{2}}\right)\left(\dfrac{x' + y'}{\sqrt{2}}\right) + \left(\dfrac{x' + y'}{\sqrt{2}}\right)^2 + 3\sqrt{2}\left(\dfrac{x' - y'}{\sqrt{2}}\right) + 5\sqrt{2}\left(\dfrac{x' + y'}{\sqrt{2}}\right) + 2 = 0.$
Multiplying by 2, we get $(x' - y')^2 + 2((x')^2 - (y')^2) + (x' + y')^2 + 6(x' - y') + 10(x' + y') + 4 = 0$. Expanding and combining like terms, we get $4(x')^2 + 16x' + 4y' + 4 = 0$. Simplifying, we obtain $(x')^2 + 4x' + y' + 1 = 0$. A parabola.

45. A hyperbola since $B^2 - 4AC = 3^2 - 4(0)(0) = 9 > 0$.

46. A hyperbola since $B^2 - 4AC = (2\sqrt{2})^2 - 4(1)(1) = 4 > 0$.

47. An ellipse since $B^2 - 4AC = 3^2 - 4(2)(2) = -7 < 0$.

48. An ellipse since $B^2 - 4AC = \sqrt{3}^2 - 4(4)(3) = -45 < 0$.

49. A parabola since $B^2 - 4AC = (6\sqrt{2})^2 - 4(2)(9) = 0$.

50. A parabola since $B^2 - 4AC = (-6)^2 - 4(3)(3) = 0$.

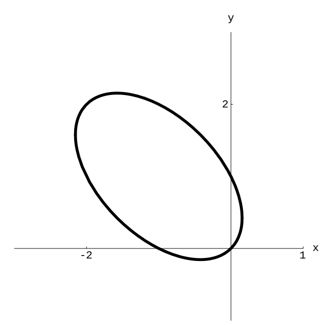
51. By grouping the y's, we obtain $y^2 + (x-1)y + (x^2 + x) = 0$. By the quadratic formula we get

$$y = \frac{-(x-1) \pm \sqrt{(x-1)^2 - 4(x^2 + x)}}{2}.$$

This simplifies to

$$y = \frac{1 - x \pm \sqrt{-3x^2 - 6x + 1}}{2}.$$

The graph is an ellipse as shown.



52. Using the double-angle identities, we get $B' = 2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) = (C - A)\sin 2\theta + B\cos 2\theta$.

Likewise, $2A' = 2(A\cos^2\theta + C\sin^2\theta + B\cos\theta\sin\theta) = 2A\cos^2\theta + 2C\sin^2\theta + B\sin 2\theta$.

Similarly, $2C' = 2A\sin^2\theta + 2C\cos^2\theta - B\sin 2\theta$.

If we expand the product $(2A')(2C')$, we get $4A'C' = 4A^2\cos^2\theta\sin^2\theta + 4C^2\cos^2\theta\sin^2\theta - 2AB\cos^2\theta\sin 2\theta + 2AB\sin^2\theta\sin 2\theta + 4AC\cos^4\theta + 4AC\sin^4\theta + 2BC\cos^2\theta\sin 2\theta - 2BC\sin^2\theta\sin 2\theta - B^2\sin^2 2\theta$.

Furthermore, $4A'C' = A^2\sin^2 2\theta + C^2\sin^2 2\theta - 2AB\cos 2\theta\sin 2\theta + 2BC\sin 2\theta\cos 2\theta +$

$4AC(\cos^4\theta + \sin^4\theta) - B^2\sin^2 2\theta =$
$A^2\sin^2 2\theta + C^2\sin^2 2\theta +$
$2B(C-A)\sin 2\theta\cos 2\theta + 4AC(\cos^4\theta + \sin^4\theta) -$
$B^2\sin^2 2\theta$.

Moreover, $4A'C' = (A^2 + C^2 - 2AC)\sin^2 2\theta +$
$2AC\sin^2 2\theta + 4AC(\cos^2 2\theta + 2\cos^2\theta\sin^2\theta) -$
$B^2(1-\cos^2 2\theta) + 2B(C-A)\sin 2\theta\cos 2\theta$.

Also, $4A'C' = (A-C)^2\sin^2 2\theta +$
$2B(C-A)\sin 2\theta\cos 2\theta + 2AC\sin^2 2\theta +$
$4AC(\cos^2 2\theta + (\sin^2 2\theta)/2) - B^2(1-\cos^2 2\theta)$.

And, $4A'C' = (A-C)^2\sin^2 2\theta +$
$2B(C-A)\sin 2\theta\cos 2\theta +$
$4AC(\sin^2 2\theta + \cos^2 2\theta) + B^2\cos^2 2\theta - B^2$.

Then $4A'C' = (A-C)^2\sin^2 2\theta +$
$2B(C-A)\sin 2\theta\cos 2\theta + B^2\cos^2 2\theta + 4AC - B^2$.

Note, by looking at the earlier expansion of B', we get $4A'C' = (B')^2 + 4AC - B^2$. Hence, $(B')^2 - 4A'C' = B^2 - 4AC$.

53. **a)** When $A = -C \neq 0$ and $D = E = F = 0$, we get $Ax^2 - Ay^2 = 0$. Since $A \neq 0$, $x^2 = y^2$ or $x = \pm y$, which is a pair of lines.

b) When $A = C = 1$ and $D = E = F = 0$, we get $x^2 + y^2 = 0$. So, the graph of $x^2 + y^2 = 0$ is a single point, namely, $(0, 0)$.

54. When $A = C = 1$, $D = E = 0$, and $F = 3$ we get $x^2 + y^2 + 3 = 0$. Since the left side of the latter equation is always positive for all real numbers x and y, the equation has no graph.

55. The parabola opens downward and $p = -\frac{1}{4}$. Since the vertex is at $(0, 3/4)$, the parabola is given by $x^2 = -(y - \frac{3}{4})$ or $y = -x^2 + \frac{3}{4}$.

56. In standard form, write $y = -2(x-1)^2 - 3$ or $(x-1)^2 = -\frac{1}{2}(y+3)$. Then $p = -\frac{1}{8}$ and the parabola opens downward. We find vertex $(1, -3)$, axis of symmetry $x = 1$, focus $(1, -25/8)$, and directrix $y = -23/8$.

57. The major axis is the x-axis, $c = 8$, and $b = 6$. We find $a^2 = b^2 + c^2 = 100$, and the ellipse is given by

$$\frac{x^2}{100} + \frac{y^2}{36} = 1.$$

58. Applying the method of completing the square we find

$$\left(x - \frac{5}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{25}{4} + \frac{1}{4}$$

The center is at $\left(\frac{5}{2}, -\frac{1}{2}\right)$ and with radius $\frac{\sqrt{26}}{2}$.

59. The hyperbola opens up and down, $a = 6$, and $\frac{a}{b} = 2$ since $y = \pm 2x$ are the asymptotes. Then $b = 3$, and the hyperbola is given by

$$\frac{y^2}{36} - \frac{x^2}{9} = 1.$$

60. Rewriting the equation, we find

$$y^2 - (x-3)^2 = 9.$$

The conic is an ellipse.

Thinking Outside the Box XCI

The domain of $N(x) = \sqrt{5x+6} + \sqrt{7x+8}$ is $[-8/7, \infty)$. We note that

$$N(s) < N(t) \text{ for all } -\frac{8}{7} \leq s < t.$$

Thus, the minimum value of N is

$$\begin{aligned} N(-8/7) &= \sqrt{5\left(-\frac{8}{7}\right) + 6} = \sqrt{\frac{2}{7}} \\ &= \frac{\sqrt{14}}{7}. \end{aligned}$$

10.4 Pop Quiz

1. Completing the square, we find

$$\begin{aligned} 2\left(x^2 - 4x\right) + 4\left(y^2 + \frac{1}{4}y\right) &= 1 \\ 2\left(x^2 - 4x + 4\right) + 4\left(y^2 + \frac{1}{4}y + \frac{1}{64}\right) &= 1 + 8 + \frac{1}{16} \\ 2(x-2)^2 + 4\left(y + \frac{1}{8}\right)^2 &= \frac{145}{16} \\ \frac{(x-2)^2}{2} + \left(y + \frac{1}{8}\right)^2 &= \frac{145}{64} \end{aligned}$$

This is an ellipse centered at $(2, -1/8)$.

2. Note, $x = x'\cos(\pi/4) - y'\sin(\pi/4) = \dfrac{x'-y'}{\sqrt{2}}$ and $y = x'\sin(\pi/4) + y'\cos(\pi/4) = \dfrac{x'+y'}{\sqrt{2}}$. Substituting into $y = 3x$, we obtain

$$\frac{x'+y'}{\sqrt{2}} = 3\left(\frac{x'-y'}{\sqrt{2}}\right).$$

Then $x'+y' = 3(x'-y')$ or

$$y' = \frac{1}{2}x'.$$

3. Since $\cot 2\theta = \dfrac{A-C}{B} = 0$, we find $\theta = \pi/4$. Then

$$x = x'\cos(\pi/4) - y'\sin(\pi/4) = \frac{x'-y'}{\sqrt{2}}$$

and

$$y = x'\sin(\pi/4) + y'\cos(\pi/4) = \frac{x'+y'}{\sqrt{2}}.$$

Substitute into $xy = -1$. Then

$$\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) = -1$$

or equivalently $\dfrac{(y')^2}{2} - \dfrac{(x')^2}{2} = 1$.

For Thought

1. True

2. False, rather the eccentricity satisfies $0 < e < 1$.

3. True **4.** True **5.** True

6. False. Since $r = \dfrac{\frac{1}{3}\cdot 4}{1-\frac{1}{3}\cos\theta}$, the directrix is $x = -4$. **7.** True

8. True, for the discriminant $B^2 - 4AC = 0^2 - 4(2)(-3)$ is positive.

9. True, for the discriminant $B^2 - 4AC = 0^2 - 4(2)(0)$ is zero.

10. True, for the discriminant $B^2 - 4AC = 0^2 - 4(2)(5)$ is negative.

10.5 Exercises

1. Since $r = \dfrac{2\cdot 3}{1-2\cos\theta}$, eccentricity is $e = 2$, conic is a hyperbola, and distance is $p = 3$.

2. Since $r = \dfrac{5\cdot\frac{3}{5}}{1-5\sin\theta}$, eccentricity is $e = 5$, conic is a hyperbola, and distance is $p = \dfrac{3}{5}$.

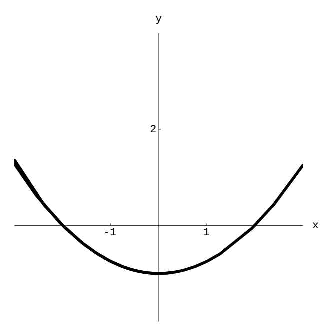
3. Since $r = \dfrac{1\cdot\frac{3}{4}}{1-\sin\theta}$, eccentricity is $e = 1$, conic is a parabola, and distance is $p = \dfrac{3}{4}$.

4. Since $r = \dfrac{1\cdot\frac{6}{5}}{1-\cos\theta}$, eccentricity is $e = 1$, conic is a parabola, and distance is $p = \dfrac{6}{5}$.

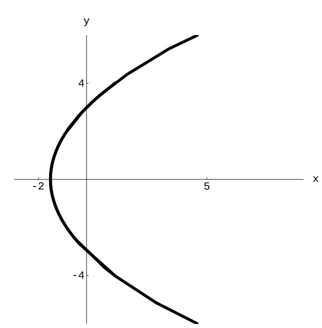
5. Since $r = \dfrac{\frac{4}{3}\cdot\frac{3}{4}}{1+\frac{4}{3}\sin\theta}$, eccentricity is $e = \dfrac{4}{3}$, conic is a hyperbola, and distance is $p = \dfrac{3}{4}$.

6. Since $r = \dfrac{\frac{5}{2}\cdot\frac{6}{5}}{1+\frac{5}{2}\cos\theta}$, eccentricity is $e = \dfrac{5}{2}$, conic is a hyperbola, and distance is $p = \dfrac{6}{5}$.

7. A parabola with directrix $y = -2$

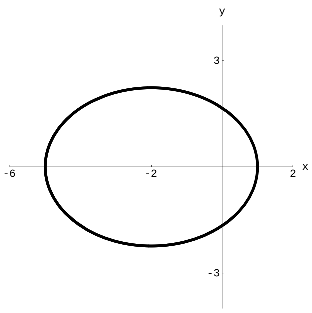


8. A parabola with directrix $x = -3$



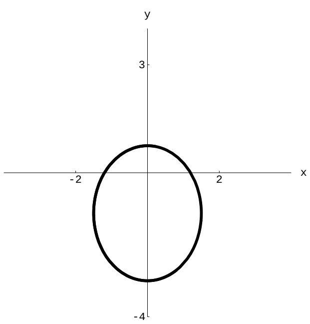
9. An ellipse with directrix $x = \dfrac{5}{2}$ since

$$r = \frac{\frac{2}{3} \cdot \frac{5}{2}}{1 + \frac{2}{3}\cos\theta}$$



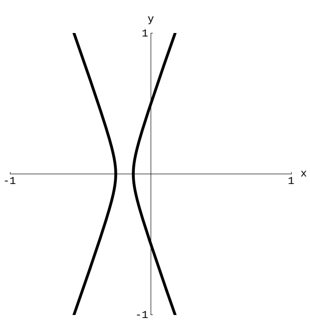
10. An ellipse with directrix $y = 2$ since

$$r = \frac{\frac{3}{5} \cdot 2}{1 + \frac{3}{5}\sin\theta}$$



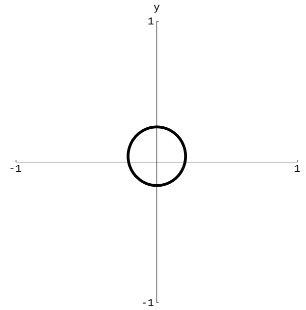
11. A hyperbola with directrix $x = -\dfrac{1}{6}$ since

$$r = \frac{3 \cdot \frac{1}{6}}{1 - 3\cos\theta}$$



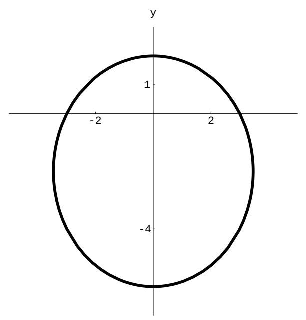
12. An ellipse with directrix $y = -1$ since

$$r = \frac{\frac{1}{5} \cdot 1}{1 - \frac{1}{5}\sin\theta}$$



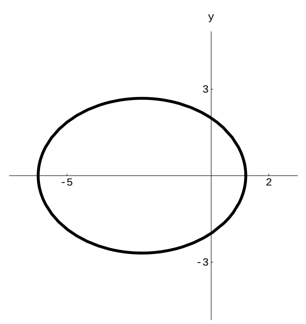
13. An ellipse with directrix $y = 6$ since

$$r = \frac{\frac{1}{2} \cdot 6}{1 + \frac{1}{2}\sin\theta}$$



14. An ellipse with directrix $x = 3$ since

$$r = \frac{\frac{2}{3} \cdot 3}{1 + \frac{2}{3}\cos\theta}$$



15. $r = \dfrac{2}{1 + \sin\theta}$

16. Since $r = \dfrac{1(3)}{1 - \sin\theta}$, we get $r = \dfrac{3}{1 - \sin\theta}$.

17. Since $r = \dfrac{2 \cdot 5}{1 + 2\cos\theta}$, we find $r = \dfrac{10}{1 + 2\cos\theta}$.

18. Since $r = \dfrac{3 \cdot 6}{1 + 3\cos\theta}$, we get $r = \dfrac{18}{1 + 3\cos\theta}$.

19. Note, $r = \dfrac{\frac{1}{2} \cdot 4}{1 + \frac{1}{2}\sin\theta}$. So $r = \dfrac{4}{2 + \sin\theta}$.

20. Note, $r = \dfrac{\frac{1}{3} \cdot 6}{1 + \frac{1}{3} \sin\theta}$. So $r = \dfrac{6}{3 + \sin\theta}$.

21. Note, $r = \dfrac{\frac{3}{4} \cdot 8}{1 - \frac{3}{4} \cos\theta}$. Then $r = \dfrac{24}{4 - 3\cos\theta}$.

22. Note, $r = \dfrac{\frac{1}{5} \cdot 10}{1 - \frac{1}{5} \cos\theta}$. So $r = \dfrac{10}{5 - \cos\theta}$.

23. Multiplying by $1 + \sin\theta$, we obtain

$$\begin{aligned} r + r\sin\theta &= 3 \\ r &= 3 - y \\ x^2 + y^2 &= 9 - 6y + y^2. \end{aligned}$$

Simplifying, we get $x^2 + 6y - 9 = 0$, a parabola.

24. Multiplying by $1 + \cos\theta$, we obtain

$$\begin{aligned} r + r\cos\theta &= 6 \\ r &= 6 - x \\ x^2 + y^2 &= 36 - 12x + x^2. \end{aligned}$$

Thus, we get $y^2 + 12x - 36 = 0$, a parabola.

25. Multiplying by $4 - \cos\theta$, we get

$$\begin{aligned} 4r - r\cos\theta &= 3 \\ 4r &= 3 + x \\ 16(x^2 + y^2) &= 9 + 6x + x^2. \end{aligned}$$

Thus, we obtain $15x^2 + 16y^2 - 6x - 9 = 0$, an ellipse.

26. Multiplying by $4 - 3\sin\theta$, we get

$$\begin{aligned} 4r - 3r\sin\theta &= 9 \\ 4r &= 9 + 3y \\ 16(x^2 + y^2) &= 81 + 54y + 9y^2. \end{aligned}$$

Thus, $16x^2 + 7y^2 - 54y - 81 = 0$. An ellipse.

27. Multiply by $3 + 9\cos\theta$. Then

$$\begin{aligned} 3r + 9r\cos\theta &= 1 \\ 3r &= 1 - 9x \\ 9(x^2 + y^2) &= 1 - 18x + 81x^2. \end{aligned}$$

Thus, we get $72x^2 - 9y^2 - 18x + 1 = 0$.
A hyperbola.

28. Multiply by $5 - 10\sin\theta$. Then

$$\begin{aligned} 5r - 10r\sin\theta &= 1 \\ 5r &= 1 + 10y \\ 25(x^2 + y^2) &= 1 + 20y + 100y^2. \end{aligned}$$

Simplifying, we get $25x^2 - 75y^2 - 20y - 1 = 0$.
A hyperbola.

29. Multiply by $6 - \sin\theta$. Then

$$\begin{aligned} 6r - r\sin\theta &= 2 \\ 6r &= 2 + y \\ 36(x^2 + y^2) &= 4 + 4y + y^2. \end{aligned}$$

Thus, we get $36x^2 + 35y^2 - 4y - 4 = 0$.
An ellipse.

30. Multiply by $3 - 9\cos\theta$. Then

$$\begin{aligned} 3r - 9r\cos\theta &= 6 \\ 3r &= 6 + 9x \\ 9(x^2 + y^2) &= 36 + 108x + 81x^2 \\ x^2 + y^2 &= 4 + 12x + 9x^2 \end{aligned}$$

Simplifying, we get $8x^2 - y^2 + 12x + 4 = 0$.
A hyperbola.

31. Since $r = \dfrac{\frac{1}{3} \cdot 12}{1 + \frac{1}{3}\cos\theta}$, $e = \dfrac{1}{3}$. By letting $\theta = 0, \pi$, we get the vertices in polar coordinates, $(3, 0)$ and $(6, \pi)$. The length of the major axis is $2a = 9$. So $a = \dfrac{9}{2}$. Recall, if $2c$ is the distance between the foci then $e = \dfrac{c}{a}$. So, $\dfrac{1}{3} = \dfrac{c}{9/2}$ and $2c = 3$. Thus, the foci are $(0, 0)$ and $(3, \pi)$.

32. Note, $e = 3$. By letting $\theta = 0, \pi$, we obtain the vertices in polar coordinates $(3, 0)$ and $(6, 0)$. The distance between the vertices is $2a = 3$. So $a = \dfrac{3}{2}$. Recall, if $2c$ is the distance between the foci then $e = \dfrac{c}{a}$. So, $3 = \dfrac{c}{3/2}$ and $2c = 9$. Thus, the foci are $(0, 0)$ and $(9, 0)$.

33. Note, $e = 1$. Since the focus is at the pole, the directrix of the parabola is $x = 12$, and the vertex is midway between the focus and directrix, the polar coordinates of the vertex is $(6, 0)$.

34. Suppose the directrix $x = p$ with $p > 0$, the focus is at the pole, and the eccentricity is e. Let P be a point in the conic. The distance between the P and the pole, F, is r. Note, the distance between P and the directrix is $PD = |p - r\cos\theta|$. Thus,

$$\frac{PF}{PD} = \frac{r}{|p - r\cos\theta|} = e$$

If (r_1, θ_1) satisfies $\dfrac{r}{p - r\cos\theta} = e$, then $(r_1, \theta_1 + \pi)$ satisfies $\dfrac{-r}{p - r\cos\theta} = e$. We might as well assume $\dfrac{r}{p - r\cos\theta} = e$. Solving for r, we get $r = \dfrac{pe}{1 + e\cos\theta}$.

35. Assume the directrix D is $y = p$ with $p > 0$, and a focus F is at the pole. If P is any point on the conic, then the distance between P and F is $PF = r$. By analyzing different cases as to whether P is above or below the directrix, the distance between P and the directrix is $PD = |p - r\sin\theta|$. So,

$$\frac{PF}{PD} = \frac{r}{|p - r\sin\theta|} = e$$

Note, if (r_1, θ_1) satisfies

$$\frac{r}{p - r\sin\theta} = e$$

then $(-r_1, -\theta_1)$ would satisfy

$$\frac{-r}{p - r\sin\theta} = e.$$

We might as well assume that $\dfrac{r}{p - r\sin\theta} = e$. Solving for r, we get

$$r = \frac{ep}{1 + e\sin\theta}.$$

36. Assume the directrix D is $y = -p$ with $p > 0$, and a focus F is at the pole. If P is any point on the conic, then the distance between P and F is $PF = r$. By analyzing different cases as to whether P is above or below the directrix, the distance between P and the directrix is $PD = |p + r\sin\theta|$. So,

$$\frac{PF}{PD} = \frac{r}{|p + r\sin\theta|} = e$$

Note, if (r_1, θ_1) satisfies

$$\frac{r}{p + r\sin\theta} = e$$

then $(-r_1, -\theta_1)$ would satisfy

$$\frac{-r}{p + r\sin\theta} = e.$$

We might as well assume that $\dfrac{r}{p + r\sin\theta} = e$. Solving for r, we get

$$r = \frac{ep}{1 - e\sin\theta}.$$

37. By letting $\theta = 0, \pi$, we obtain the vertices $(2, 0)$ and $(2/3, \pi)$ in polar coordinates of the ellipse. So, one-half the length of the major axis is $a = \dfrac{4}{3}$. Since the distance c between the pole and the vertex $(2/3, \pi)$ is $c = \dfrac{2}{3}$, the eccentricity is

$$e = \frac{c}{a} = \frac{2/3}{4/3} = \frac{1}{2}.$$

38. By letting $\theta = 0, \pi$, we obtain the vertices $(6, \pi)$ and $(2, \pi)$ in polar coordinates of the hyperbola. So, one-half the length a of the transverse axis is $a = 2$. Since the center is $(4, \pi)$, the distance c between the center and the pole is $c = 4$. Thus, the eccentricity is $e = \dfrac{c}{a} = \dfrac{4}{2} = 2$.

39. The parabola opens to the left, vertex $(4, 0)$, and $p = -1$. Since $y^2 = -4(x - 4)$, we obtain

$$x = -\frac{1}{4}y^2 + 4.$$

40. Rewrite equation as

$$(x-5)^2 = -\frac{1}{2}(y-9).$$

The vertex is $(5,9)$, and axis of symmetry is $x=5$. Since $p=-\frac{1}{8}$, the focus is $(5, 71/8)$.

41. Since $a=8$, $b=6$, and center at the origin, the ellipse is given by

$$\frac{x^2}{64}+\frac{y^2}{36}=1.$$

42. The slopes of the asymptotes are $\pm 2/5$. Since the asymptotes pass through the center $(3,0)$, the equations of the asymptotes are $y=\frac{2}{5}x-\frac{6}{5}$ and $y=-\frac{2}{5}x+\frac{6}{5}$.

43. We find $[x', y'] =$
$\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} \cos 45^o & -\sin 45^o \\ \sin 45^o & \cos 45^o \end{bmatrix}$
$= \begin{bmatrix} \sqrt{2} & 0 \end{bmatrix}$

44. We find $[x', y'] =$
$\begin{bmatrix} x & \sqrt{3}x \end{bmatrix}\begin{bmatrix} \cos 60^o & -\sin 60^o \\ \sin 60^o & \cos 60^o \end{bmatrix}$
$= \begin{bmatrix} 2x & 0 \end{bmatrix}$. Thus, $y'=0$.

Thinking Outside the Box XCII

Note, if $a>b>0$ and $n>0$, then

$$\frac{a}{n}+\frac{b}{n+1} > \frac{b}{n}+\frac{a}{n+1}.$$

Thus, the arrangement with the largest sum is

$$\frac{2006}{1}+\frac{2005}{2}+...+\frac{2}{2005}+\frac{1}{2006}.$$

10.5 Pop Quiz

1. Since

$$r=\frac{\frac{1}{3}(3)}{1-\frac{1}{3}\cos\theta}$$

we find $e=1/3$ and the conic is an ellipse. Since $p=3$, the directrix is $x=-3$.

2. Rewriting, we have

$$r=\frac{4\left(\frac{1}{2}\right)}{1+4\sin\theta}.$$

Then $e=4$ and the conic is a hyperbola. The directrix is $y=\frac{1}{2}$.

3. Sine $e=1$, the conic is a parabola. Note, $p=2$ and

$$r=\frac{ep}{1-e\cos\theta}.$$

The polar equation is

$$r=\frac{2}{1-\cos\theta}.$$

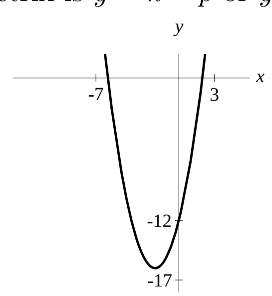
Review Exercises

1. If $y=0$, then by factoring we get $0=(x+6)(x-2)$ and the x-intercepts are $(2,0)$ and $(-6,0)$.

If $x=0$, then $y=-12$ and the y-intercept is $(0,-12)$.

By completing the square, we obtain $y=(x+2)^2-16$, vertex $(h,k)=(-2,-16)$, and the axis of symmetry is $x=-2$.

Since $p=\dfrac{1}{4a}=\dfrac{1}{4}$, the focus is $(h, k+p)=(-2,-63/4)$ and directrix is $y=k-p$ or $y=-65/4$.

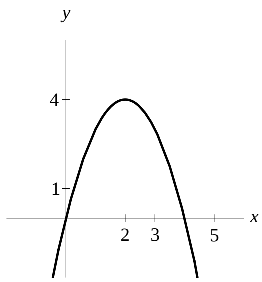


2. If $y=0$, then by factoring we get $0=x(4-x)$ and the x-intercepts are $(0,0)$ & $(4,0)$.

If $x=0$, then $y=0$ and y-intercept is $(0,0)$.

By completing the square, one gets $y=-(x-2)^2+4$, vertex $(h,k)=(2,4)$, and axis of symmetry is $x=2$.

Since $p = \dfrac{1}{4a} = -\dfrac{1}{4}$, the focus is
$(h, k + p) = (2, 15/4)$ and
directrix is $y = k - p$ or $y = 17/4$.

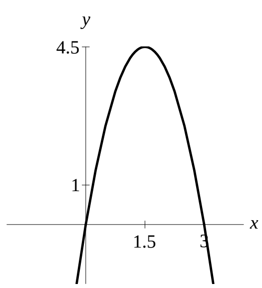


3. If $y = 0$, then by factoring we find $0 = x(6-2x)$ and the x-intercepts are $(0, 0)$ and $(3, 0)$.

If $x = 0$, then $y = 0$ and the y-intercept is $(0, 0)$.

By completing the square, one gets
$y = -2(x - 3/2)^2 + 9/2$, with
vertex $(h, k) = (3/2, 9/2)$,
and axis of symmetry is $x = 3/2$.

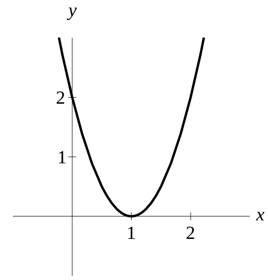
Since $p = \dfrac{1}{4a} = -\dfrac{1}{8}$, the focus is
$(h, k + p) = (3/2, 35/8)$ and
the directrix is $y = k - p$ or $y = 37/8$.



4. If $y = 0$, then by factoring we obtain $0 = 2(x - 1)^2$ and the x-intercept is $(1, 0)$.

If $x = 0$, then $y = 2$ and y-intercept is $(0, 2)$.

Since $y = 2(x - 1)^2$ is of the form
$y = a(x - h)^2 + k$, the vertex is
$(h, k) = (1, 0)$, and the axis of symmetry is
$x = 1$. Since $p = \dfrac{1}{4a} = \dfrac{1}{8}$, the focus is
$(h, k + p) = (1, 1/8)$ and directrix is $y = k - p$ or $y = -1/8$.



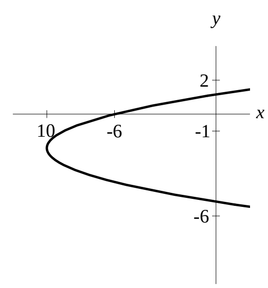
5. By completing the square, we have $x = (y+2)^2 - 10$. If $x = 0$, then $y+2 = \pm\sqrt{10}$ and the y-intercepts are $(0, -2 \pm \sqrt{10})$.

If $y = 0$, then $x = -6$ and the x-intercept is $(-6, 0)$. Since $x = (y + 2)^2 - 10$ is of the form
$x = a(y - h)^2 + k$, the vertex is
$(k, h) = (-10, -2)$, and the axis
of symmetry is $y = -2$.

Since $p = \dfrac{1}{4a} = \dfrac{1}{4}$, the focus is

$$(k + p, h) = (-39/4, -2)$$

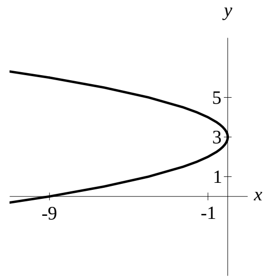
and directrix is $x = k - p$ or
$x = -41/4$.



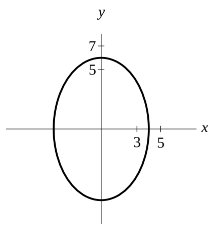
6. By factoring, we find $x = -(y - 3)^2$.
If $x = 0$, then $y = 3$ and y-intercept is $(0, 3)$.
If $y = 0$, then $x = -9$ and x-intercept is $(-9, 0)$. Since $x = -(y - 3)^2$ is of the form $x = a(y - h)^2 + k$, the vertex is $(k, h) = (0, 3)$, and the axis of symmetry is $y = 3$. Since

$$p = \frac{1}{4a} = -\frac{1}{4},$$

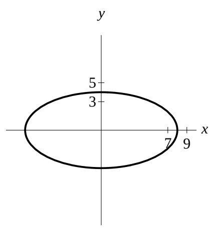
the focus is $(k+p, h) = (-1/4, 3)$ and directrix is $x = k - p$ or $x = 1/4$.



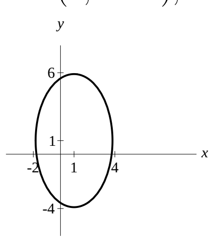
7. Since $c = \sqrt{a^2 - b^2} = \sqrt{36 - 16} = 2\sqrt{5}$, the foci are $(0, \pm 2\sqrt{5})$



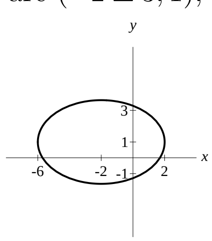
8. Since $c = \sqrt{a^2 - b^2} = \sqrt{64 - 16} = 4\sqrt{3}$, the foci are $(\pm 4\sqrt{3}, 0)$



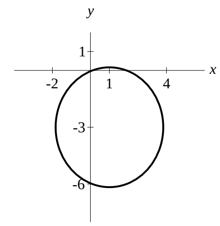
9. Since $c = \sqrt{a^2 - b^2} = \sqrt{24 - 8} = 4$, the foci are $(1, 1 \pm 4)$, or $(1, 5)$ and $(1, -3)$.



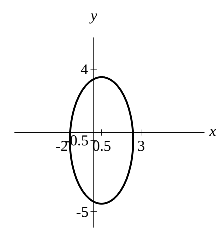
10. Since $c = \sqrt{a^2 - b^2} = \sqrt{16 - 7} = 3$, the foci are $(-2 \pm 3, 1)$, or $(1, 1)$ and $(-5, 1)$.



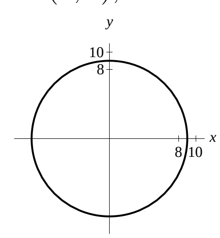
11. Since $c = \sqrt{a^2 - b^2} = \sqrt{10 - 8} = \sqrt{2}$, the foci are $(1, -3 \pm \sqrt{2})$.



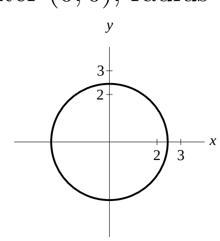
12. Since $c = \sqrt{a^2 - b^2} = \sqrt{16 - 4} = 2\sqrt{3}$, the foci are $\left(\frac{1}{2}, -\frac{1}{2} \pm 2\sqrt{3}\right)$ or $\left(\frac{1}{2}, \frac{-1 \pm 4\sqrt{3}}{2}\right)$



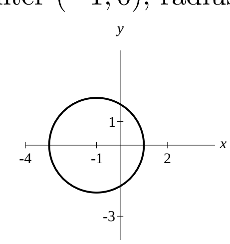
13. center $(0, 0)$, radius 9



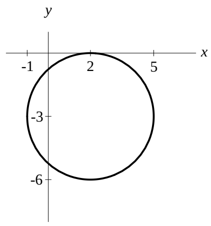
14. Divide by 6, so $x^2 + y^2 = 6$ has center $(0, 0)$, radius $\sqrt{6}$



15. center $(-1, 0)$, radius 2



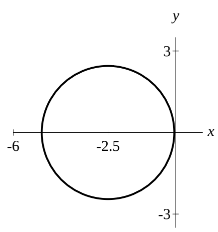
16. center $(2, -3)$, radius 3



17. Completing the square, we have

$$\begin{aligned} x^2 + 5x + \frac{25}{4} + y^2 &= -\frac{1}{4} + \frac{25}{4} \\ \left(x + \frac{5}{2}\right)^2 + y^2 &= 6, \end{aligned}$$

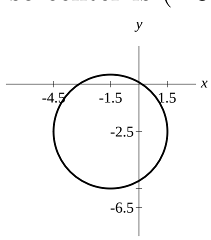
and so center is $(-5/2, 0)$ and radius is $\sqrt{6}$.



18. Completing the square, we find

$$\begin{aligned} x^2 + 3x + \frac{9}{4} + y^2 + 5y + \frac{25}{4} &= \frac{1}{2} + \frac{9}{4} + \frac{25}{4} \\ \left(x + \frac{3}{2}\right)^2 + \left(y + \frac{5}{2}\right)^2 &= 9, \end{aligned}$$

and so center is $(-3/2, -5/2)$ and radius is 3.



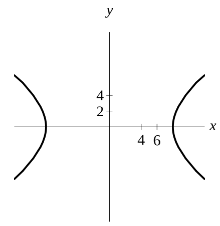
19. $x^2 + (y + 4)^2 = 9$

20. $(x + 2)^2 + (y + 5)^2 = 1$

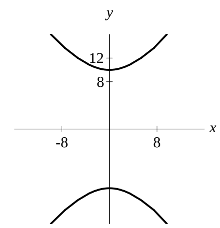
21. $(x + 2)^2 + (y + 7)^2 = 6$

22. $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{4}\right)^2 = \frac{1}{2}$

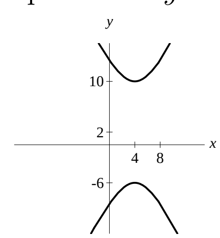
23. Since $c = \sqrt{a^2 + b^2} = \sqrt{8^2 + 6^2} = 10$,
foci $(\pm 10, 0)$, asymptotes $y = \pm\frac{b}{a} = \pm\frac{3}{4}x$



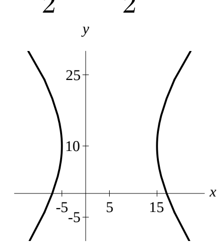
24. Since $c = \sqrt{a^2 + b^2} = \sqrt{10^2 + 8^2} = 2\sqrt{41}$,
foci $(0, \pm 2\sqrt{41})$, asymptotes $y = \pm\frac{a}{b} = \pm\frac{5}{4}x$



25. Since $c = \sqrt{a^2 + b^2} = \sqrt{8^2 + 4^2} = 4\sqrt{5}$,
the foci are $(4, 2 \pm 4\sqrt{5})$. Solving
for y in $y - 2 = \pm\frac{8}{4}(x - 4)$,
asymptotes are $y = 2x - 6$ and $y = -2x + 10$.



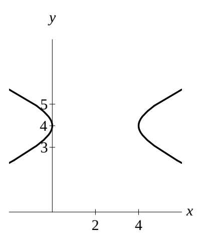
26. Since $c = \sqrt{a^2 + b^2} = \sqrt{10^2 + 15^2} = 5\sqrt{13}$, the
foci are $(5 \pm 5\sqrt{13}, 10)$. Solving for y
in $y - 10 = \pm\frac{15}{10}(x - 5)$, we find that
the asymptotes are $y = \frac{3}{2}x + \frac{5}{2}$ and
$y = -\frac{3}{2}x + \frac{35}{2}$.



27. Completing the square, we have

$$\begin{aligned} x^2-4x-4(y^2-8y) &= 64 \\ x^2-4x+4-4(y^2-8y+16) &= 64+4-64 \\ (x-2)^2-4(y-4)^2 &= 4 \\ \frac{(x-2)^2}{4}-(y-4)^2 &= 1, \end{aligned}$$

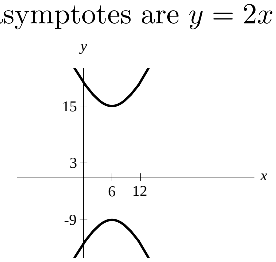
and so $c=\sqrt{a^2+b^2}=\sqrt{2^2+1^2}=\sqrt{5}$, and the foci are $(2\pm\sqrt{5},4)$. Solving for y in $y-4=\pm\frac{1}{2}(x-2)$, we get that the asymptotes are $y=\frac{1}{2}x+3$ and $y=-\frac{1}{2}x+5$.



28. Completing the square, we obtain

$$\begin{aligned} y^2-6y+9-4(x^2-12x+36) &= 144 \\ (y-3)^2-4(x-6)^2 &= 144 \\ \frac{(y-3)^2}{144}-\frac{(x-6)^2}{36} &= 1 \end{aligned}$$

and so $c=\sqrt{a^2+b^2}=\sqrt{12^2+6^2}=6\sqrt{5}$, and the foci are $(6,3\pm6\sqrt{5})$. Solving for y in $y-3=\pm\frac{12}{6}(x-6)$, we find that the asymptotes are $y=2x-9$ and $y=-2x+15$.



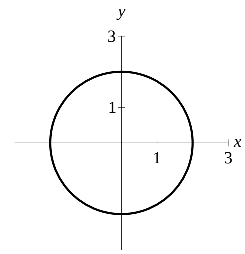
29. Hyperbola **30.** Circle

31. Ellipse **32.** Circle

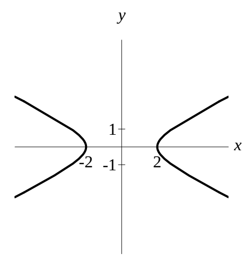
33. Parabola **34.** Parabola

35. Hyperbola **36.** Ellipse

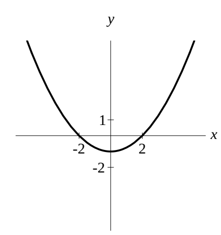
37. $x^2+y^2=4$ is a circle



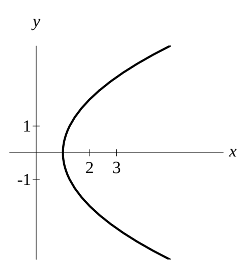
38. Since $x^2-4y^2=4$, $\frac{x^2}{4}-y^2=1$.



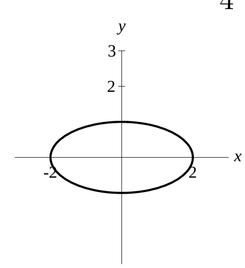
39. Since $4y=x^2-4$, $y=\frac{1}{4}x^2-1$.



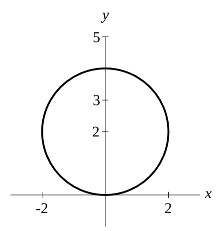
40. Since $4x=y^2+4$, $x=\frac{1}{4}y^2+1$.



41. Since $x^2+4y^2=4$, $\frac{x^2}{4}+y^2=1$.

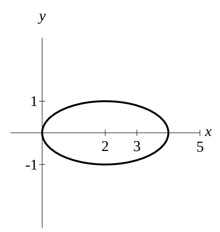


42. Since $x^2 + y^2 - 4y + 4 = 4$, $x^2 + (y-2)^2 = 4$.



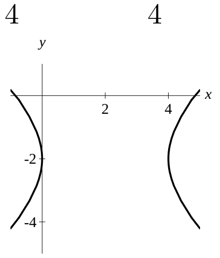
43. Since $x^2 - 4x + 4 + 4y^2 = 4$, we find

$(x-2)^2 + 4y^2 = 4$ and $\dfrac{(x-2)^2}{4} + y^2 = 1$.



44. Since $x^2 - 4x + 4 = (y+2)^2 + 4$, we get

$\dfrac{(x-2)^2}{4} - \dfrac{(y+2)^2}{4} = 1$



45. Since the vertex is midway between the focus $(1,3)$ and directrix $x = \dfrac{1}{2}$, the vertex is $\left(\dfrac{3}{4}, 3\right)$ and $p = \dfrac{1}{4}$. Since $a = \dfrac{1}{4p} = 1$, parabola is given by $x = (y-3)^2 + \dfrac{3}{4}$.

46. Since radius is $\sqrt{2^2 + 8^2} = \sqrt{68}$, the circle is given by $x^2 + y^2 = 68$.

47. From the foci and vertices, we get $c = 4$ and $a = 6$, respectively. Since $b^2 = a^2 - c^2 = 36 - 16 = 20$, the ellipse is given by $\dfrac{x^2}{36} + \dfrac{y^2}{20} = 1$.

48. We use the form $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$.
From the y-intercepts, we find $a = 3$.
From the slope of the asymptotes, we obtain $\dfrac{a}{b} = 3$ and so $b = 1$.
An equation is $\dfrac{y^2}{9} - x^2 = 1$.

49. Radius is $\sqrt{(-1-1)^2 + (-1-3)^2} = \sqrt{20}$.
Equation is $(x-1)^2 + (y-3)^2 = 20$.

50. Midway between the foci is the center $(1,2)$. Distance between center and a focus is $c = 2$. Distance from center to vertex is $a = 4$. Since $b^2 = a^2 - c^2 = 16 - 4 = 12$, an equation is $\dfrac{(x-1)^2}{16} + \dfrac{(y-2)^2}{12} = 1$.

51. From the foci and x-intercepts one gets $c = 3$ and $a = 2$, respectively. Since $b^2 = c^2 - a^2 = 9 - 4 = 5$, the hyperbola is given by $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$.

52. Since the vertex is midway between the focus $(0,3)$ and directrix $y = 1$, vertex is $(0,2)$ and $p = 1$. Since $a = \dfrac{1}{4p} = \dfrac{1}{4}$, parabola is given by $y = \dfrac{1}{4}x^2 + 2$.

53. Since the center of the circle is $(-2,3)$ and the raidus is 3, we have $(x+2)^2 + (y-3)^2 = 9$.

54. Note, the vertex of the parabola is $(2,3)$. An equation we can start with is $y = a(x-2)^2 + 3$. Note, the graph passes through $(1,2)$. Substitute, $x = 1$ and $y = 2$ into $y = a(x-2)^2 + 3$. Then $2 = a + 3$ or $a = -1$. Thus, $y = -1(x-2)^2 + 3$ or $y = -x^2 + 4x - 1$.

55. Note, the center of the ellipse is $(-2,1)$. Using the lengths of the major axis, we get $a = 3$ and $b = 1$. Thus, an equation is

$$\frac{(x+2)^2}{9} + (y-1)^2 = 1.$$

56. Note, the center of the hyperbola is $(0,0)$. Using the fundamental rectangle, we find $a=3$ and $b=3$. Then an equation is

$$\frac{y^2}{9}-\frac{x^2}{9}=1.$$

57. Note, the center of the hyperbola is $(2,1)$. By using the fundamental rectangle, we find $a=3$ and $b=2$. Thus, an equation is

$$\frac{(y-1)^2}{9}-\frac{(x-2)^2}{4}=1.$$

58. Note, the center of the ellipse is $(0,0)$. By using the lengths of the major axis, we get $a=5$ and $b=2$. Then an equation is

$$\frac{x^2}{4}+\frac{y^2}{25}=1.$$

59. Note, $\cot 2\theta=\dfrac{A-C}{B}=\dfrac{3-3}{4}=0.$

Then $\theta=\pi/4$,

$$x=x'\cos(\pi/4)-y'\sin(\pi/4)=\frac{x'-y'}{\sqrt{2}}$$

and

$$y=x'\sin(\pi/4)+y'\cos(\pi/4)=\frac{x'+y'}{\sqrt{2}}.$$

Substitute into the original equation.

$$3\left(\frac{x'-y'}{\sqrt{2}}\right)^2+$$
$$4\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right)+3\left(\frac{x'+y'}{\sqrt{2}}\right)^2+$$
$$\left(\frac{x'-y'}{\sqrt{2}}\right)-\left(\frac{x'+y'}{\sqrt{2}}\right)=0.$$

Multiplying by 2, we get
$3(x'-y')^2+4((x')^2-(y')^2)+$
$3(x'+y')^2+\sqrt{2}(x'-y')-\sqrt{2}(x'+y')=0.$

Expanding and combining like terms, we get

$$10(x')^2+2(y')^2-2\sqrt{2}y'=0.$$

Simplifying, we obtain

$$5(x')^2+(y')^2-\sqrt{2}y'=0$$

which is an ellipse.

60. Note, $\cot 2\theta=\dfrac{A-C}{B}=\dfrac{5-2}{-3\sqrt{3}}=\dfrac{-1}{\sqrt{3}}.$

Then $\theta=\pi/3$,

$$x=x'\cos(\pi/3)-y'\sin(\pi/3)=\frac{x'-\sqrt{3}y'}{2}$$

and

$$y=x'\sin(\pi/3)+y'\cos(\pi/3)=\frac{\sqrt{3}x'+y'}{2}.$$

Substitute into the original equation.

$$5\left(\frac{x'-\sqrt{3}y'}{2}\right)^2-$$
$$3\sqrt{3}\left(\frac{x'-\sqrt{3}y'}{2}\right)\left(\frac{\sqrt{3}x'+y'}{2}\right)+$$
$$2\left(\frac{\sqrt{3}x'+y'}{2}\right)^2+\left(\frac{x'-\sqrt{3}y'}{2}\right)-$$
$$\left(\frac{\sqrt{3}x'+y'}{2}\right)=0.$$

Multiplying by 4, we get

$$5\left(x'-\sqrt{3}y'\right)^2-$$
$$3\sqrt{3}\left(x'-\sqrt{3}y'\right)\left(\sqrt{3}x'+y'\right)+$$
$$2\left(\sqrt{3}x'+y'\right)^2+2\left(x'-\sqrt{3}y'\right)-$$
$$2\left(\sqrt{3}x'+y'\right)=0.$$

Expanding and combining like terms, we get

$$2(x')^2+26(y')^2+2(1-\sqrt{3})x'-2(1+\sqrt{3})y'=0.$$

Rewriting, we get

$$(x')^2+13(y')^2+(1-\sqrt{3})x'-(1+\sqrt{3})y'=0$$

which is an ellipse.

61. Note, $\cot 2\theta=\dfrac{9-1}{8\sqrt{3}}=\dfrac{1}{\sqrt{3}}$. Then $\theta=\pi/6$,

$$x=x'\cos(\pi/6)-y'\sin(\pi/6)=\frac{\sqrt{3}x'-y'}{2}$$

and

$$y=x'\sin(\pi/6)+y'\cos(\pi/6)=\frac{x'+\sqrt{3}y'}{2}.$$

Substitute into the original equation.

$$9\left(\frac{\sqrt{3}x'-y'}{2}\right)^2+$$
$$8\sqrt{3}\left(\frac{\sqrt{3}x'-y'}{2}\right)\left(\frac{x'+\sqrt{3}y'}{2}\right)+$$
$$\left(\frac{x'+\sqrt{3}y'}{2}\right)^2+2\left(\frac{x'+\sqrt{3}y'}{2}\right)=0.$$

Multiplying by 4, we get

$$9(\sqrt{3}x'-y')^2+8\sqrt{3}(\sqrt{3}x'-y')(x'+\sqrt{3}y')+$$
$$(x'+\sqrt{3}y')^2+4(x'+\sqrt{3}y')=0.$$

This simplifies to

$$52(x')^2-12(y')^2+4x'+4\sqrt{3}y'=0.$$

Thus, we obtain the hyperbola

$$13(x')^2+x'-3(y')^2+\sqrt{3}y'=0.$$

62. Note, $\cot 2\theta = \dfrac{A-C}{B} = \dfrac{3-3}{7\sqrt{2}} = 0$. Then $\theta=\pi/4$,

$$x = x'\cos(\pi/4) - y'\sin(\pi/4) = \frac{x'-y'}{\sqrt{2}}$$

and

$$y = x'\sin(\pi/4) + y'\cos(\pi/4) = \frac{x'+y'}{\sqrt{2}}$$

Substitute into the original equation.

$$3\left(\frac{x'-y'}{\sqrt{2}}\right)^2+$$
$$7\sqrt{2}\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right)+3\left(\frac{x'+y'}{\sqrt{2}}\right)^2+$$
$$\left(\frac{x'-y'}{\sqrt{2}}\right)-\left(\frac{x'+y'}{\sqrt{2}}\right)=0.$$

Multiplying by 2, we get

$$3(x'-y')^2+7\sqrt{2}((x')^2-(y')^2)+$$
$$3(x'+y')^2+\sqrt{2}(x'-y')-\sqrt{2}(x'+y')=0.$$

Expanding and combining like terms, we get

$$(6+7\sqrt{2})(x')^2+(6-7\sqrt{2})(y')^2-2\sqrt{2}y'=0$$

which is a hyperbola.

63. Note, $\cot 2\theta = \dfrac{A-C}{B} = \dfrac{4-4}{-6\sqrt{2}} = 0$ and $\theta=\pi/4$. Then

$$x = x'\cos(\pi/4) - y'\sin(\pi/4) = \frac{x'-y'}{\sqrt{2}}$$

and

$$y = x'\sin(\pi/4) + y'\cos(\pi/4) = \frac{x'+y'}{\sqrt{2}}.$$

Substitute into the original equation.

$$4\left(\frac{x'-y'}{\sqrt{2}}\right)^2-$$
$$6\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right)+4\left(\frac{x'+y'}{\sqrt{2}}\right)^2-$$
$$2\left(\frac{x'-y'}{\sqrt{2}}\right)+2\left(\frac{x'+y'}{\sqrt{2}}\right)+1=0.$$

Multiplying by 2, we get

$$4(x'-y')^2-6((x')^2-(y')^2)+$$
$$4(x'+y')^2-2\sqrt{2}(x'-y')+2\sqrt{2}(x'+y')+2=0.$$

Expanding and combining like terms, we get

$$2(x')^2+14(y')^2+4\sqrt{2}y'+2=0.$$

Simplifying, we obtain the ellipse

$$(x')^2+7(y')^2+2\sqrt{2}y'+1=0.$$

64. Note, $\cot 2\theta = \dfrac{A-C}{B} = \dfrac{3-(-9)}{-12\sqrt{3}} = \dfrac{-1}{\sqrt{3}}$ and $\theta=\pi/3$. Then

$$x = x'\cos(\pi/3) - y'\sin(\pi/3) = \frac{x'-\sqrt{3}y'}{2}$$

and

$$y = x'\sin(\pi/3) + y'\cos(\pi/3) = \frac{\sqrt{3}x'+y'}{2}.$$

Substitute into the original equation.

$$3\left(\frac{x'-\sqrt{3}y'}{2}\right)^2-$$
$$-12\sqrt{3}\left(\frac{x'-\sqrt{3}y'}{2}\right)\left(\frac{\sqrt{3}x'+y'}{2}\right)-$$

$$9\left(\frac{\sqrt{3}x' + y'}{2}\right)^2 + 3\left(\frac{x' - \sqrt{3}y'}{2}\right) +$$
$$\left(\frac{\sqrt{3}x' + y'}{2}\right) = 0.$$

Multiplying by 4, we get

$$3\left(x' - \sqrt{3}y'\right)^2 -$$
$$12\sqrt{3}\left(x' - \sqrt{3}y'\right)\left(\sqrt{3}x' + y'\right) -$$
$$9\left(\sqrt{3}x' + y'\right)^2 + 6\left(x' - \sqrt{3}y'\right) +$$
$$2\left(\sqrt{3}x' + y'\right) = 0.$$

Expanding and combining like terms, we obtain

$$-60(x')^2+36(y')^2+(6+2\sqrt{3})x'-(6\sqrt{3}-2)y' = 0.$$

Rewriting, we get the hyperbola

$$-15(x')^2+\left(\frac{\sqrt{3}+3}{2}\right)(x')+9(y')^2-\left(\frac{3\sqrt{3}-1}{2}\right)(y') = 0.$$

65. Hyperbola since

$$B^2 - 4AC = 3^2 - 4(0)(0) = 9 > 0.$$

66. Ellipse since

$$B^2 - 4AC = 1^2 - 4(1)(1) = -3 < 0.$$

67. Parabola since

$$B^2 - 4AC = 20^2 - 4(25)(4) = 0.$$

68. Ellipse since

$$B^2 - 4AC = 3^2 - 4(4)(1) = -7 < 0.$$

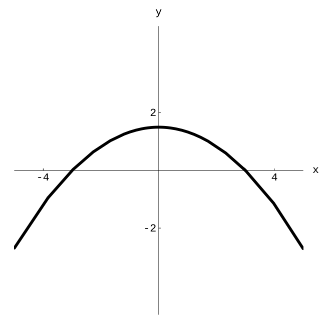
69. Hyperbola since

$$B^2 - 4AC = (12\sqrt{3})^2 - 4(1)(9) > 0.$$

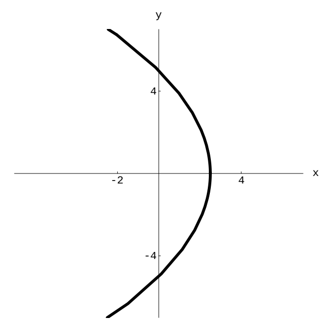
70. Parabola since

$$B^2 - 4AC = (-8)^2 - 4(2)(8) = 0.$$

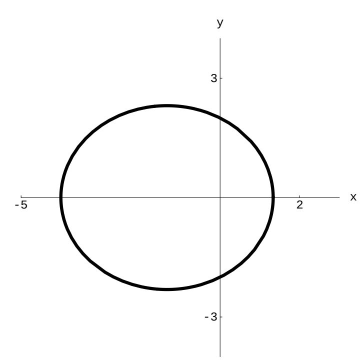
71. Parabola with directrix $y = 3$



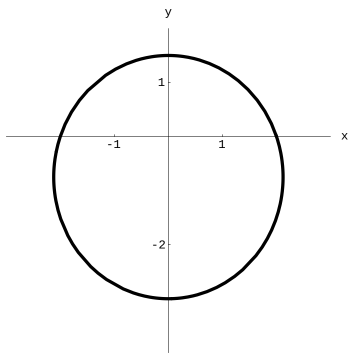
72. A parabola with directrix $x = 5$



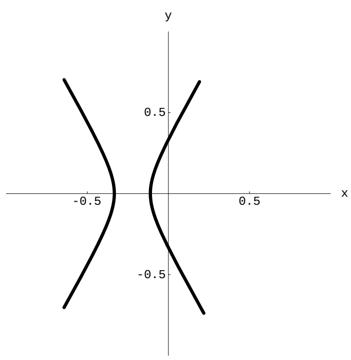
73. Since $r = \dfrac{\frac{1}{2} \cdot 4}{1 + \frac{1}{2}\cos\theta}$, we have an ellipse with directrix $x = 4$



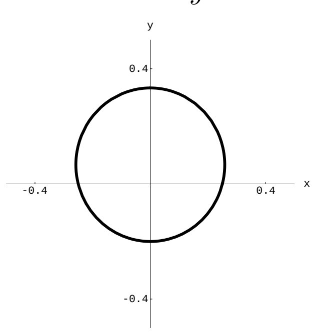
74. Since $r = \dfrac{\frac{1}{3} \cdot 6}{1 + \frac{1}{3}\sin\theta}$, we have an ellipse with directrix $y = 6$



75. Since $r = \dfrac{2 \cdot \frac{1}{6}}{1 - 2\cos\theta}$, we have a hyperbola with directrix $x = -\dfrac{1}{6}$



76. Since $r = \dfrac{\frac{1}{4} \cdot 1}{1 - \frac{1}{4}\sin\theta}$, we have an ellipse with directrix $y = -1$



77. $r = \dfrac{3}{1 + \sin\theta}$ **78.** $r = \dfrac{4}{1 - \sin\theta}$

79. Since $r = \dfrac{3(6)}{1 - 3\cos\theta}$, we find $r = \dfrac{18}{1 - 3\cos\theta}$.

80. Since $r = \dfrac{4 \cdot 8}{1 + 4\cos\theta}$, we get $r = \dfrac{32}{1 + 4\cos\theta}$.

81. Note, $r = \dfrac{\frac{1}{3} \cdot 9}{1 + \frac{1}{3}\sin\theta}$. Then $r = \dfrac{9}{3 + \sin\theta}$.

82. Note, $r = \dfrac{\frac{1}{4} \cdot 12}{1 - \frac{1}{4}\sin\theta}$. So $r = \dfrac{12}{4 - \sin\theta}$.

83. The equation is of the form $\dfrac{x^2}{100^2} - \dfrac{y^2}{b^2} = 1$. Since the graph passes through $(120, 24\sqrt{11})$, we get

$$\begin{aligned}
\frac{120^2}{100^2} - \frac{(24\sqrt{11})^2}{b^2} &= 1 \\
1.44 - \frac{6336}{b^2} &= 1 \\
b^2 &= \frac{6336}{0.44} \\
b^2 &= 120^2.
\end{aligned}$$

Equation is $\dfrac{x^2}{100^2} - \dfrac{y^2}{120^2} = 1$.

84. Since $p = 10$, $a = \dfrac{1}{4p} = \dfrac{1}{40}$ and the parabola is given by $y = \dfrac{1}{40}x^2$.
The thickness at the outside edge is
$y = \dfrac{1}{40}15^2 = 5.625$ inches.

85. Note $c = 30$ and $a = 34$. Then an equation we can use is of the form $\dfrac{x^2}{34^2} + \dfrac{y^2}{b^2} = 1$.
Since $b^2 = a^2 - c^2 = 34^2 - 30^2 = 16^2$, the equation is

$$\frac{x^2}{34^2} + \frac{y^2}{16^2} = 1.$$

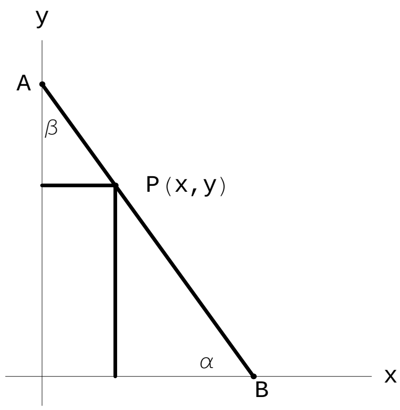
To find h, let $x = 32$.

$$\begin{aligned}
\frac{32^2}{34^2} + \frac{y^2}{16^2} &= 1 \\
y^2 &= \left(1 - \frac{32^2}{34^2}\right)16^2 \\
y &\approx 5.407.
\end{aligned}$$

Thus, $h = 2y \approx 10.81$ feet.

Thinking Outside the Box XCIII

a) Let point $P(x, y)$ represent the coordinates of the paint brush, and let $AP = a$, $BP = b$.



Then we find

$$\sin\beta = \frac{x}{a}, \ \sin\alpha = \frac{y}{b}$$

Since $\sin\beta = \cos\alpha$, we obtain

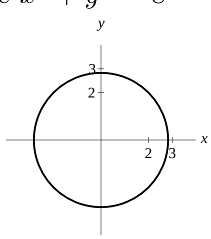
$$\begin{aligned} \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 &= \sin^2\beta + \sin^2\alpha \\ &= \cos^2\alpha + \sin^2\alpha \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1. \end{aligned}$$

Thus, the curve is an ellipse.

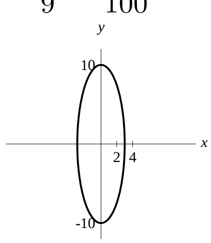
b) An equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, see part a).

Chapter 10 Test

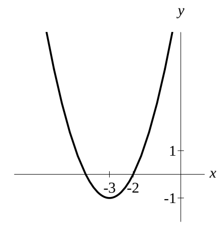
1. Circle $x^2 + y^2 = 8$



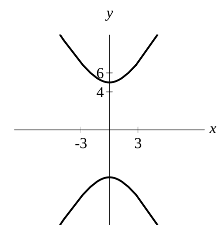
2. Ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{100} = 1$



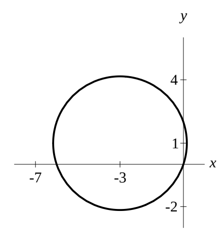
3. Parabola $y = x^2 + 6x + 8$



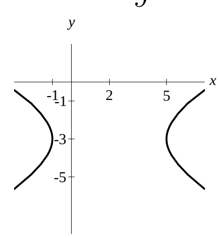
4. Hyperbola $\dfrac{y^2}{25} - \dfrac{x^2}{9} = 1$



5. Circle $(x + 3)^2 + (y - 1)^2 = 10$



6. Hyperbola $\dfrac{(x-2)^2}{9} - \dfrac{(y+3)^2}{4} = 1$



7. By completing the square, the equation can be written as $(x - 4)^2 - 16 = y^2$. This is a hyperbola.

8. By completing the square, we can rewrite the equation as $(x - 4)^2 - 16 = y$. This is a parabola.

9. By completing the square, we can rewrite the equation as $-(x - 4)^2 + 16 = y^2$. This is a circle.

10. By completing the square, the equation can be written as $-(x - 4)^2 + 16 = 8y^2$. This is an ellipse.

11. Since $(2\sqrt{3})^2 = 12$, the circle is given by $(x+3)^2 + (y-4)^2 = 12$.

12. Midway between the focus $(2,0)$ and directrix $x = -2$ is the vertex $(0,0)$. Since $p = 2$, $a = \dfrac{1}{4p} = \dfrac{1}{8}$. Equation is $x = \dfrac{1}{8}y^2$.

13. Equation is of the form $\dfrac{x^2}{2^2} + \dfrac{y^2}{a^2} = 1$.

Since $a^2 = b^2 + c^2 = 2^2 + \sqrt{6}^2 = 10$,

equation is $\dfrac{x^2}{4} + \dfrac{y^2}{10} = 1$.

14. From the foci and vertices one gets $c = 8$ and $a = 6$. Since $b^2 = c^2 - a^2 = 8^2 - 6^2 = 28$,

equation is $\dfrac{x^2}{36} - \dfrac{y^2}{28} = 1$.

15. Complete the square to get $y = (x-2)^2 - 4$.

So vertex is $(h,k) = (2,-4)$, axis of symmetry $x = 2$, $a = 1$, and $p = \dfrac{1}{4a} = \dfrac{1}{4}$.

Focus is $(h, k+p) = (2, -15/4)$ and directrix is $y = k - p = -17/4$.

16. Since $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$, we get

$c = \sqrt{a^2 - b^2} = \sqrt{4^2 - 2^2} = 2\sqrt{3}$.

Foci are $(\pm 2\sqrt{3}, 0)$, length of major axis is $2a = 8$, length of minor axis is $2b = 4$.

17. Since $y^2 - \dfrac{x^2}{16} = 1$, we obtain

$c = \sqrt{a^2 + b^2} = \sqrt{1^2 + 4^2} = \sqrt{17}$.

Foci are $(0, \pm\sqrt{17})$, vertices $(0, \pm 1)$,

asymptotes $y = \pm\dfrac{1}{4}x$, length of transverse axis is $2a = 2$, length of conjugate axis is $2b = 8$.

18. Completing the square, we find

$$\begin{aligned} x^2 + x + \frac{1}{4} + y^2 - 3y + \frac{9}{4} &= -\frac{1}{4} + \frac{1}{4} + \frac{9}{4} \\ \left(x + \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 &= \frac{9}{4}. \end{aligned}$$

The center is $\left(-\dfrac{1}{2}, \dfrac{3}{2}\right)$ and the radius is $\dfrac{3}{2}$.

19. Note, $\cot 2\theta = \dfrac{4-3}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$ and $\theta = \pi/6$. Then we get

$$x = x'\cos(\pi/6) - y'\sin(\pi/6) = \frac{\sqrt{3}x' - y'}{2}$$

and

$$y = x'\sin(\pi/6) + y'\cos(\pi/6) = \frac{x' + \sqrt{3}y'}{2}.$$

Substitute into the original equation.

$$4\left(\frac{\sqrt{3}x' - y'}{2}\right)^2 + \sqrt{3}\left(\frac{\sqrt{3}x' - y'}{2}\right)\left(\frac{x' + \sqrt{3}y'}{2}\right) + 3\left(\frac{x' + \sqrt{3}y'}{2}\right)^2 + \left(\frac{\sqrt{3}x' - y'}{2}\right) - \left(\frac{x' + \sqrt{3}y'}{2}\right) = 0.$$

Multiplying by 4, we get

$$4(\sqrt{3}x' - y')^2 + \sqrt{3}(\sqrt{3}x' - y')(x' + \sqrt{3}y') + 3(x' + \sqrt{3}y')^2 + 2(\sqrt{3}x' - y') - 2(x' + \sqrt{3}y') = 0.$$

This simplifies to

$$18(x')^2 + 10(y')^2 + 2(\sqrt{3} - 1)x' - 2(\sqrt{3} + 1)y' = 0.$$

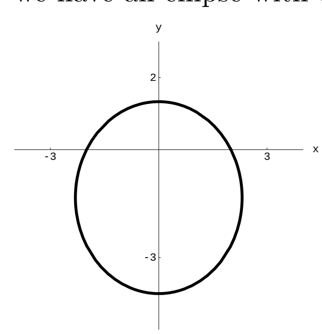
Dividing by 2, we get the ellipse

$$9(x')^2 + 5(y')^2 + (\sqrt{3} - 1)x' - (\sqrt{3} + 1)y' = 0.$$

20. Since

$$r = \frac{\frac{1}{2} \cdot 4}{1 + \frac{1}{2}\sin\theta}$$

we have an ellipse with directrix $y = 4$.



21. Since

$$\frac{x^2}{225} + \frac{y^2}{81} = 1$$

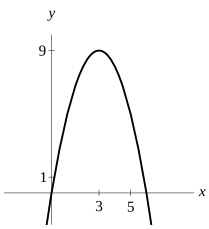
we find $a = 15$ and $b = 9$. Note,

$$c = \sqrt{a^2 - c^2} = \sqrt{15^2 - 9^2} = 12.$$

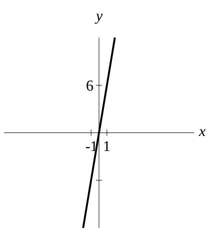
Since the foci are $(\pm 12, 0)$, the distance from the point of generation of the waves to the kidney stones is $2c = 24$ cm.

Tying It All Together

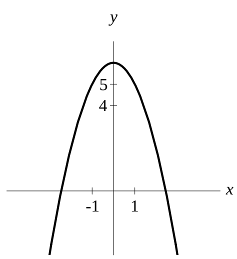
1. Parabola $y = 6x - x^2$



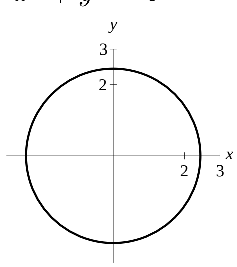
2. Line $y = 6x$



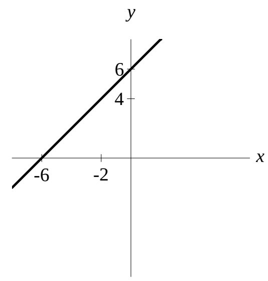
3. Parabola $y = 6 - x^2$



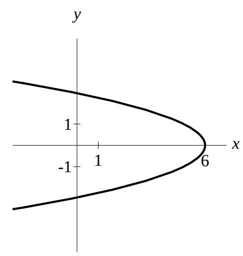
4. Circle $x^2 + y^2 = 6$



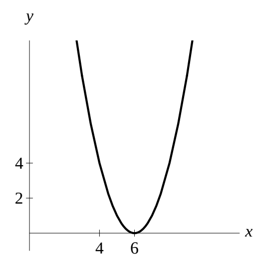
5. Line $y = x + 6$



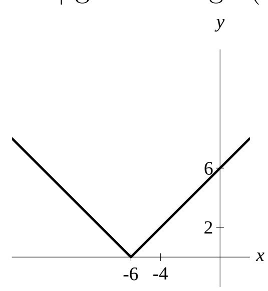
6. Parabola $x = 6 - y^2$



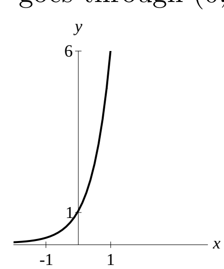
7. Parabola $y = (6 - x)^2 = (x - 6)^2$



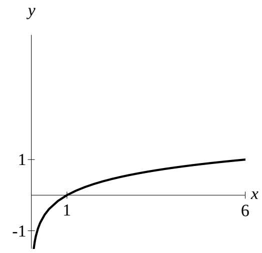
8. $y = |x + 6|$ goes through $(-6, 0)$, $(-5, 1)$



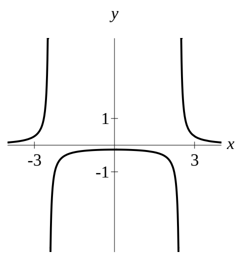
9. $y = 6^x$ goes through $(0, 1)$, $(1, 6)$



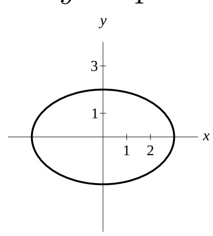
10. $y = \log_6(x)$ goes through $(6, 1)$, $(1/6, -1)$



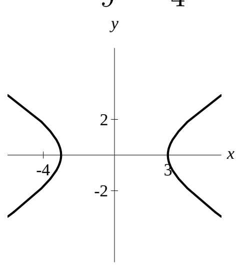
11. $y = \dfrac{1}{x^2 - 6}$ has asymptotes $x = \pm\sqrt{6}$



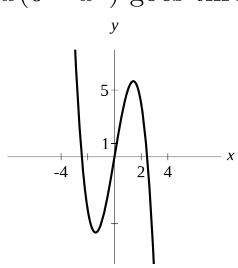
12. Ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$



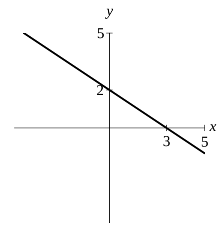
13. Hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$



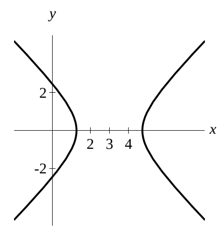
14. $y = x(6 - x^2)$ goes through $(0, 0)$, $(\pm\sqrt{6}, 0)$



15. Line $2x + 3y = 6$



16. By completing the square, the equation may be written as $\dfrac{(x-3)^2}{3} - \dfrac{y^2}{3} = 1$. This is a hyperbola.



17. Solving for x, we get

$$\begin{aligned} 3x - 9 + 5 &= -9 \\ 3x &= -5. \end{aligned}$$

The solution set is $\{-5/3\}$.

18. Solving for x, we find

$$\begin{aligned} 5x - 8x + 12 &= 17 \\ -3x &= 5. \end{aligned}$$

The solution set is $\{-5/3\}$.

19. Solving for x, we obtain

$$\begin{aligned} 2x - \frac{2}{3} - \frac{3}{2} + 3x &= \frac{3}{2} \\ 5x - \frac{13}{6} &= \frac{3}{2} \\ 5x &= \frac{11}{3}. \end{aligned}$$

The solution set is $\{11/15\}$.

20. Solving for x, we have

$$\begin{aligned} -4x + 2 &= 3x + 2 \\ -7x &= 0. \end{aligned}$$

The solution set is $\{0\}$.

21. Solving for x, we find

$$\begin{aligned}\frac{1}{2}x-\frac{1}{4}x &= \frac{3}{2}+\frac{1}{3}\\ \frac{1}{4}x &= \frac{11}{6}\\ x &= \frac{44}{6}.\end{aligned}$$

Solution set is $\{22/3\}$.

22. Multiplying both sides by 100, we obtain

$$\begin{aligned}5(x-20)+2(x+10) &= 270\\ 5x-100+2x+20 &= 270\\ 7x &= 350.\end{aligned}$$

The solution set is $\{50\}$.

23. By using the quadratic formula to solve $2x^2+31x-51=0$, one obtains

$$\begin{aligned}x &= \frac{-31\pm\sqrt{31^2-4(2)(-51)}}{4}\\ &= \frac{-31\pm\sqrt{1369}}{4}\\ &= \frac{-31\pm 37}{4}\\ &= \frac{3}{2}, -17.\end{aligned}$$

The solution set is $\left\{\frac{3}{2}, -17\right\}$.

24. Since $x(2x+31)=0$, the solution set is $\left\{0, -\frac{31}{2}\right\}$.

25. By using the quadratic formula to solve $x^2-34x+286=0$, one obtains

$$\begin{aligned}x &= \frac{34\pm\sqrt{34^2-4(1)(286)}}{2}\\ x &= \frac{34\pm\sqrt{12}}{2}\\ x &= \frac{34\pm 2\sqrt{3}}{2}\\ x &= 17\pm\sqrt{3}.\end{aligned}$$

The solution set is $\left\{17\pm\sqrt{3}\right\}$.

26. In solving $x^2-34x+290=0$, one can use the quadratic formula. Then

$$\begin{aligned}x &= \frac{34\pm\sqrt{34^2-4(1)(290)}}{2}\\ x &= \frac{34\pm\sqrt{-4}}{2}\\ x &= \frac{34\pm 2i}{2}\\ x &= 17\pm i.\end{aligned}$$

The solution set is $\{17\pm i\}$.

27. identity

28. scalar

29. $m\times p$

30. identity

31. determinant

32. determinant

33. Cramer's rule

34. parabola

35. axis of symmetry

36. vertex

37. ellipse

38. circle

39. hyperbola

Concepts of Calculus

1. The quadratic equation

$$x^2-mx-b=0$$

has exactly one solution exactly when the discriminant is zero. Recall, the discriminant is

$$(-m)^2-4(1)(-b) \text{ or } m^2+4b.$$

Thus, the quadratic equation has only one solution exactly when

$$m^2+4b=0$$

or equivalently when

$$b=-\frac{1}{4}m^2.$$

2. Substitute $y = x^2$ into $y - y_1 = m(x - x_1)$ and solve for m. Then

$$\begin{aligned} x^2 - y_1 &= m(x - x_1) \\ x^2 - mx + (mx_1 - y_1) &= 0 \\ x^2 - mx + (mx_1 - x_1^2) &= 0. \end{aligned}$$

If there is exactly one solution for x, then the discriminant $b^2 - 4ac$ must be zero. Thus,

$$\begin{aligned} m^2 - 4(mx_1 - x_1^2) &= 0 \\ 4x_1^2 - 4mx_1 + m^2 &= 0 \\ (2x_1 - m)^2 &= 0. \end{aligned}$$

The value of the slope at (x_1, y_1) is $m = 2x_1$.

3. Based on the answer to Exericse 2, we obtain that the slope m of the tangent line at $(3, 9)$ is

$$m = 2x_1 = 2(3) = 6.$$

Now, substitute $x = 3$ and $y = 9$ into the slope-intercept form

$$y = 6x + b$$

of the tangent line.

$$\begin{aligned} 9 &= 6(3) + b \\ 9 &= 18 + b \\ b &= -9 \end{aligned}$$

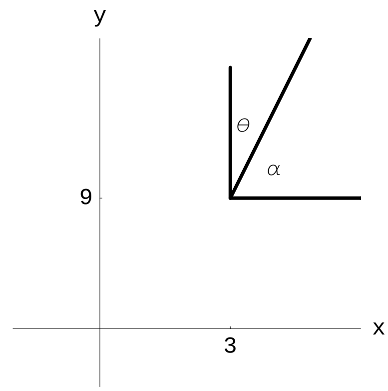
The tangent line is given by

$$y = 6x - 9.$$

4. Note, by using Exercise 2, we obtain that the slope m of the tangent line through $(3, 9)$ is

$$m = 6.$$

If α is the angle made by this tangent line with the horizontal, as shown below,



then

$$\tan\alpha = 6.$$

Moreover, it follows from

$$\theta = \frac{\pi}{2} - \alpha$$

that

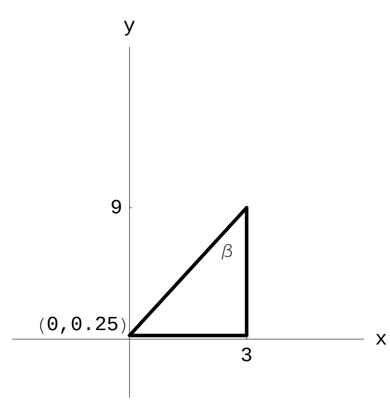
$$\tan\theta = \cot\alpha = \frac{1}{6}.$$

Thus, $\tan\theta = \frac{1}{6}$.

5. From Exercise 4, we obtained $\tan\theta = \frac{1}{6}$. Then by using a double angle identity we find

$$\begin{aligned} \tan(2\theta) &= \frac{2\tan(\theta)}{1 - \tan^2(\theta)} \\ &= \frac{2(1/6)}{1 - 1/36} \\ &= \frac{1/3}{35/36} \\ \tan(2\theta) &= \frac{12}{35}. \end{aligned}$$

6. In the right triangle below, the endpoints



of the hypotenuse are $(3,9)$ and the focus $(0,0.25)$ of the parabola $y=x^2$. Next,we calculate $\tan\beta$.

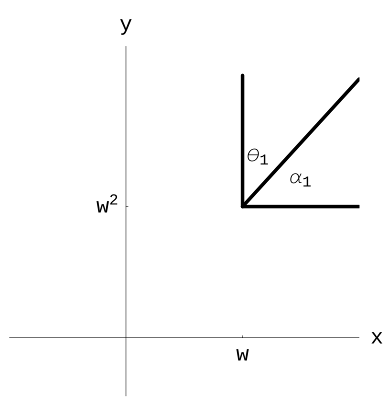
$$\begin{aligned}\tan\beta &= \frac{\text{opposite}}{\text{adjacent}}\\ &= \frac{3}{9-0.25}\\ &= \frac{3}{8.75}\\ \tan\beta &= \frac{12}{35}\end{aligned}$$

Since $\tan\beta = \tan(2\theta) = \dfrac{12}{35}$ by Exercise 5, it follows that the light ray that reflects off the point $(3,9)$ passes through the focus $(0,0.25)$ of the parabola.

7. Note, by using Exercise 2, we obtain that the slope m of the tangent line at (w,w^2) is

$$m = 2w.$$

If α_1 is the angle made by this tangent line with the horizontal, as shown below,



then

$$\tan\alpha_1 = 2w.$$

Let θ_1 be the angle of incidence at (w,w^2). Then

$$\theta_1 = \frac{\pi}{2} - \alpha_1$$

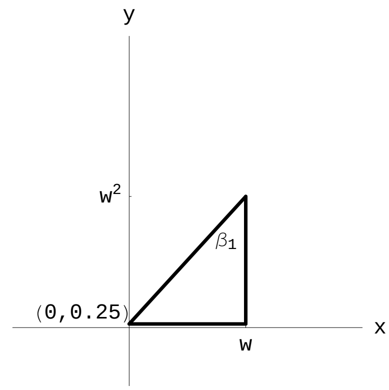
and

$$\tan\theta_1 = \cot\alpha_1 = \frac{1}{2w}.$$

And, by using a double angle identity we obtain

$$\tan(2\theta_1) = \frac{4w}{4w^2-1}.$$

Next, consider the right triangle below where the endpoints



of the hypotenuse are (w,w^2) and the focus $(0,0.25)$ of the parabola $y=x^2$. We calculate $\tan\beta_1$.

$$\begin{aligned}\tan\beta_1 &= \frac{\text{opposite}}{\text{adjacent}}\\ &= \frac{w}{w^2-1/4}\\ \tan\beta_1 &= \frac{4w}{4w^2-1}\end{aligned}$$

Thus, we obtain

$$\tan\beta_1 = \tan(2\theta_1).$$

Hence, the light ray that reflects off the point (w,w^2) passes through the focus $(0,0.25)$ of the parabola.