SECTION 9.2 – Estimating a Population Mean

Point Estimate

A **point estimate** is the value of a statistic that estimates the value of a parameter.

Point estimate of a population mean, $\mu \rightarrow \bar{x}$.

Confidence-Interval Estimate

The confidence interval estimates for the population mean are of the form

$$\text{Point estimate} \pm \text{margin of error} \quad \text{or} \quad \bar{x} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

Note: This model above presents a problem because we need to know the population standard deviation, σ, to construct this interval. In some cases, statisticians use previous known population standard deviations from past data sets. Sometimes for certain variables, the mean changes whereas the standard deviations may not, such as gas prices from week to week. We will assume that we don't know the population standard deviation, σ. Hence, we will need another logical option. So, we will use the sample standard deviation, s, as an estimate of σ, which will now give us the following confidence interval instead,

$$\bar{x} \pm z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}.$$

Unfortunately, there is a problem with this approach as well. The sample standard deviation, s, is a statistic and therefore will vary from sample to sample. Using the normal model to determine the critical value, $z_{\frac{\alpha}{2}}$, in the margin of error does not take into account the additional variability introduced by using s in place of σ. This is not much of a problem for large samples because the variability in the sample standard deviation decreases as the sample size increases (Law of Large Numbers), but for small samples, we have a real problem. Put another way, the z-score of $\bar{x}$, $\dfrac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$, is normally distributed with mean 0 and standard deviation 1 (provided $\bar{x}$ is normally distributed). However, $\dfrac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$ is *not* normally distributed with mean 0 and standard deviation 1. So, a new model must be used to determine the margin of error that accounts for this additional variability. This leads to the story of William Gosset.

In the early 1900s, William Gosset a graduate from Oxford worked for the Guinness brewery in Dublin, Ireland. Gosset was in charge of conducting experiments at the brewery to identify the best barley variety. When working with beer, Gosset was limited to small data sets. At the time, the model used for constructing confidence intervals about a mean was the normal mode, regardless of whether the population standard deviation was known. Gosset did not know the population standard deviation, so he simply substituted the sample standard deviation for the population standard deviation as suggested by $\bar{x} \pm z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$ from above. While doing this, he was finding that his confidence intervals did not include the population mean at the rate expected. This led Gosset in 1908 to develop a model that accounts for the additional variability introduced by using s in place of σ when determining the margin of error. Guinness would not allow Gosset to publish his results under his real name (Guinness was very secretive about its brewing practices), but did allow the results to be published under a pseudonym. Gosset chose Student. So, we have the Student's *t*-distribution.

SECTION 9.2 – Estimating a Population Mean

Student's *t*-Distribution

Suppose that a simple random sample of size n is taken from a population. If the population from which the sample is drawn follows a normal distribution, the distribution of

$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}$$

follows a Student's t-distribution with $n - 1$ degrees of freedom, where $\bar{x}$ is the sample mean and s is the sample standard deviation.

Properties of the *t*-Distribution

1. The t-distribution is different for different degrees of freedom.
2. The t-distribution is centered at 0 and is symmetric about 0.
3. The area under the curve is 1. The area under the curve to the right of 0 equals the area under the curve to the left of 0, which equals $\frac{1}{2}$.
4. As t increases or decreases without bound, the graph approaches, but never equals, zero.
5. The area in the tails of the t-distribution is a little greater than the area in the tails of the standard normal distribution, because we are using s as an estimate of σ, thereby introducing further variability into the t-statistic.
6. As the sample size n increases, the density curve of t gets closer to the standard normal density curve. This result occurs because, as the sample size increases, the values of s get closer to the value of σ, by the Law of Large Numbers.

An example of *t*-curves with different degrees of freedom and all with a 95% confidence level

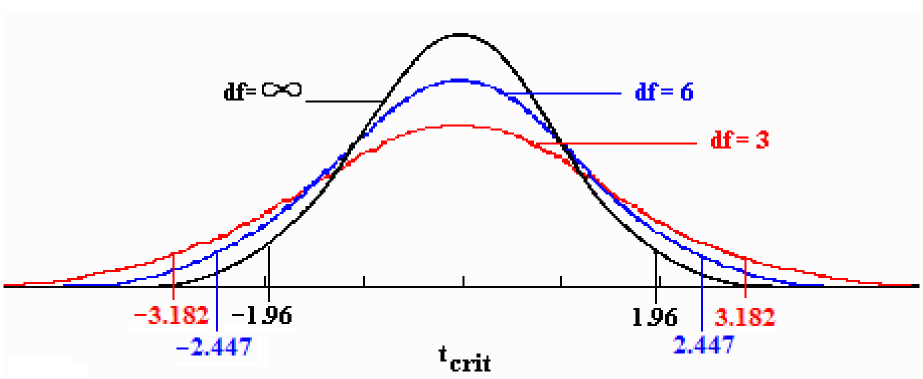
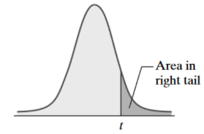


Table VII - Critical Values of Student's *t*-Distribution



df	Amount of α in one tail											
	0.25	0.20	0.15	0.10	0.05	0.025	0.02	0.01	0.005	0.0025	0.001	0.0005
1	1.000	1.376	1.963	3.078	6.314	12.706	15.894	31.821	63.657	127.232	318.309	636.619
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.089	22.327	31.599
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.215	12.924
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.767
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
31	0.682	0.853	1.054	1.309	1.696	2.040	2.144	2.453	2.744	3.022	3.375	3.633
32	0.682	0.853	1.054	1.309	1.694	2.037	2.141	2.449	2.738	3.015	3.365	3.622
33	0.682	0.853	1.053	1.308	1.692	2.035	2.139	2.445	2.733	3.008	3.356	3.611
34	0.682	0.852	1.052	1.307	1.691	2.032	2.136	2.441	2.728	3.002	3.348	3.601
35	0.681	0.852	1.052	1.306	1.690	2.030	2.133	2.438	2.724	2.996	3.340	3.591
36	0.681	0.852	1.052	1.306	1.688	2.028	2.131	2.434	2.719	2.990	3.333	3.582
37	0.681	0.851	1.051	1.305	1.687	2.026	2.129	2.431	2.715	2.985	3.326	3.574
38	0.681	0.851	1.051	1.304	1.686	2.024	2.127	2.429	2.712	2.980	3.319	3.566
39	0.681	0.851	1.050	1.304	1.685	2.023	2.125	2.426	2.708	2.976	3.313	3.558
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
70	0.678	0.847	1.044	1.294	1.667	1.994	2.093	2.381	2.648	2.899	3.211	3.435
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
90	0.677	0.846	1.042	1.291	1.662	1.987	2.084	2.368	2.632	2.878	3.183	3.402
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
200	0.676	0.843	1.039	1.286	1.653	1.972	2.067	2.345	2.601	2.839	3.131	3.340
300	0.675	0.843	1.038	1.284	1.650	1.968	2.063	2.339	2.592	2.828	3.118	3.323
400	0.675	0.843	1.038	1.284	1.649	1.966	2.060	2.336	2.588	2.823	3.111	3.315
500	0.675	0.842	1.038	1.283	1.648	1.965	2.059	2.334	2.586	2.820	3.107	3.310
600	0.675	0.842	1.037	1.283	1.647	1.964	2.058	2.333	2.584	2.817	3.104	3.307
700	0.675	0.842	1.037	1.283	1.647	1.963	2.058	2.332	2.583	2.816	3.102	3.304
800	0.675	0.842	1.037	1.283	1.647	1.963	2.057	2.331	2.582	2.815	3.100	3.303
900	0.675	0.842	1.037	1.282	1.647	1.963	2.057	2.330	2.581	2.814	3.099	3.301
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
2000	0.675	0.842	1.037	1.282	1.646	1.961	2.055	2.328	2.578	2.810	3.094	3.295
Z*	0.674	0.842	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.090	3.291
	50%	60%	70%	80%	*90%*	*95%*	96%	98%	*99%*	99.5%	99.8%	99.9%
	Confidence Level											

SECTION 9.2 – Estimating a Population Mean

Constructing a $(1-\alpha)\cdot 100\%$ Confidence Interval for a Population Mean, μ

Provided

- sample data come from a simple random sample or randomized experiment
- sample size is small relative to the population size ($n \le 0.05N$), and
- the data come from a population that is normally distributed, or the sample size is large.

A $(1-\alpha)\cdot 100\%$ confidence interval for μ is given by:

Lower bound: $\bar{x} - t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$ Upper bound: $\bar{x} + t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$

or simply,

$$\bar{x} \pm t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$$

where $t_{\frac{\alpha}{2}}$ is the critical value with $n - 1$ degrees of freedom.

Margin of Error for the Estimate of μ

The **margin of error, *E***, in a $(1-\alpha)\cdot 100\%$ confidence interval for a population mean is given by

$$E = t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$$

Note: This is simply half the length of the confidence interval.

Sample Size Needed for Estimating the Population Mean μ

The sample size required to estimate the population mean, μ, with a level of confidence $(1-\alpha)\cdot 100\%$ within a specified margin of error of at most E is given by

$$n = \left(\frac{z_{\frac{\alpha}{2}} \cdot s}{E} \right)^2$$

where n is *rounded up* to the nearest whole number.

Note: If we are trying to solve for n, we cannot use $t_{\frac{\alpha}{2}}$, because we would need the degrees of freedom, $n - 1$. Since we don't know n, we can't find $n - 1$. So, we will use $z_{\frac{\alpha}{2}}$ instead, since the t-distribution approaches the standard normal z-distribution as the sample size increases.

Confidence Intervals involving Normal Distributions vs. Approximately Normal Distributions

- If the variable x is normally distributed, then the confidence intervals calculated are exactly correct.
- If the variable x is not normal, yet the sample size is large, then the confidence intervals calculated are approximately correct.

SECTION 9.2 – Estimating a Population Mean

☺ **Exercises:**

1) Using Table VII, find the *t*-value such that the area in the right tail is 0.02 with 19 degrees of freedom.

2) Using Table VII, find the *t*-value such that the area left of the *t*-value is 0.05 with 6 degrees of freedom.

3) For a sample size of 36 and a confidence level of 90%, use Table VII to find $t_{\alpha/2}$.

4) For a sample size of 75 and a confidence level of 95%, use Table VII to find $t_{\alpha/2}$.

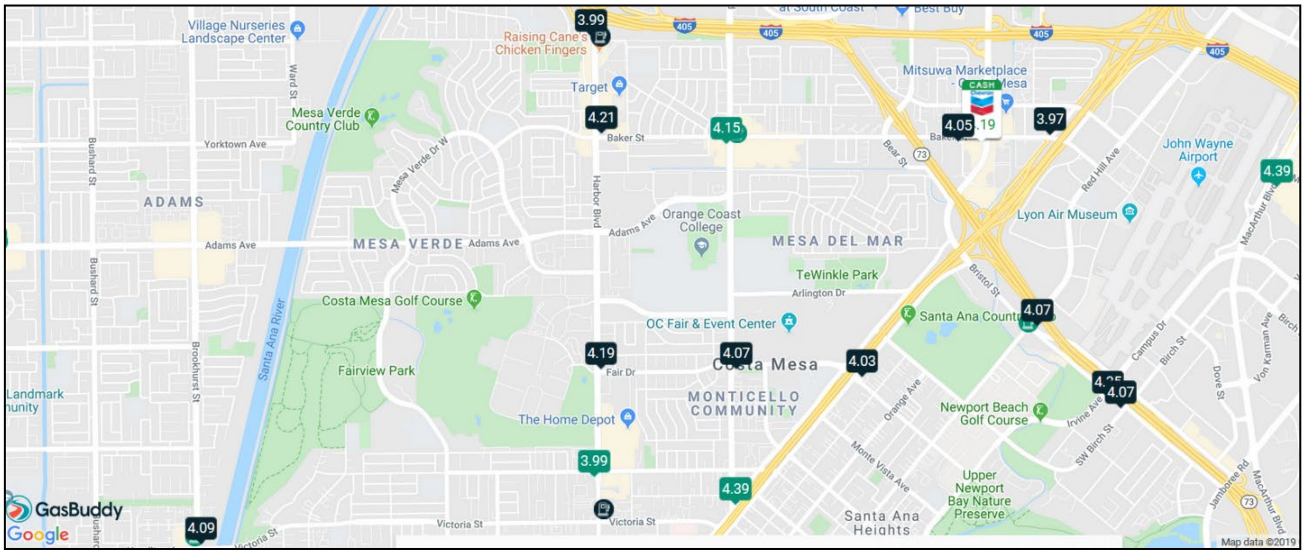
5) Determine the point estimate of the population mean and margin of error for the confidence interval that has a lower bound of 20 and an upper bound of 30.

6) Determine the point estimate of the population mean and margin of error for the confidence interval that has a lower bound of 16 and an upper bound of 34

SECTION 9.2 – Estimating a Population Mean

☺ **Exercises:**

7) A snapshot shown below from Gasbuddy.com (or the Gas Buddy App) displays some of the gas prices around Costa Mesa taken on October 17th, 2019. A random sample of gas prices from 16 gas stations recorded a mean cost of \$4.14 and a standard deviation of \$0.14. Create a 95% confidence interval for the true mean price per gallon of gas of all gas stations in Costa Mesa and interpret your results. Assume that gas prices follow a normal distribution.



8) Among a sample of 65 students selected at random from one college, the mean number of siblings is 1.3 with a standard deviation of 1.1. Find a 90% confidence interval for the mean number of siblings for all students at this college and interpret your results. Round the nearest hundredths.

SECTION 9.2 – Estimating a Population Mean

☺ **Exercises:**

9) How much fat do reduced fat cookies typically have? You take a random sample of 51 reduced-fat cookies and test them in a lab, finding a mean fat content of 3.2 grams and a standard deviation of 1.1 grams of fat. Find a 99% confidence interval for the mean grams of fat and interpret your results.

10) Determine the sample size required to estimate the mean score on a standardized test within 4 points of the true mean with 90% confidence. Assume that $s = 15$ based on earlier studies.

11) A doctor at a local hospital is interested in estimating the birth weight of infants. How large a sample must she select if she desires to be 99% confident that her estimate is within 2 ounces of the true mean? Assume that $s = 7$ ounces based on earlier studies.

SECTION 9.2 – Estimating a Population Mean

☺ **Exercises:**

12) A simple random sample of size n is drawn from a population that is normally distributed. The sample mean, $\bar{x}$, is found to be 50, and the sample standard deviation, s, is found to be 8.

a) Construct a 98% confidence interval for μ if the sample size, n, is 20.

b) Construct a 98% confidence interval for μ if the sample size, n, is 15. How does decreasing the sample size affect the margin of error, E?

c) Construct a 95% confidence interval for μ if the sample size, n, is 20. Compare the results to those obtained in part **a)**. How does decreasing the level of confidence affect the margin of error, E?

d) Could we have computed the confidence intervals in parts **a)–c)** if the population had not been normally distributed? Why?

SECTION 9.2 – Estimating a Population Mean

☺ **Exercises:**

13) **Caffeinated Sports Drinks.** Researchers conducted an experiment to determine the effectiveness of a commercial caffeinated carbohydrate–electrolyte sports drink compared with a placebo. Sixteen highly trained cyclists each completed two trials of prolonged cycling in a warm environment, one while receiving the sports drink and another while receiving a placebo. For a given trial, one beverage treatment was administered throughout a 2-hour variable-intensity cycling bout followed by a 15-minute performance ride. Total work (in kilojoules) performed during the final 15 minutes was used to measure performance. The beverage order for individual subjects was randomly assigned with a period of at least five days separating the trials. Assume that the researchers verified the normality of the population of total work performed for each treatment.

Source: Kirk J. Cureton, Gordon L. Warren, et al., "Caffeinated Sports Drink: Ergogenic Effects and Possible Mechanisms." International Journal of Sport Nutrition and Exercise Metabolism 17(1):35–55, 2007

a) Why do you think the sample size was small ($n = 16$) for this experiment?

b) For the sports-drink treatment, the mean total work performed during the performance ride for the $n = 16$ riders was 218 kilojoules, with standard deviation 31 kilojoules. Construct and interpret a 95% confidence interval for the population mean total work performed.

c) Is it possible for the population mean total work performed for the sports-drink treatment to be less than 198 kilojoules? Do you think this is likely?

d) For the placebo treatment, the mean total work performed during the performance ride for the $n = 16$ riders was 178 kilojoules, with standard deviation 31 kilojoules. Construct and interpret a 95% confidence interval for the population mean total work performed.

e) Is it possible for the population mean total work performed for the placebo treatment to be more than 198 kilojoules? Do you think this is likely?

f) The researchers concluded that the caffeinated carbohydrate–electrolyte sports drink substantially enhanced physical performance during prolonged exercise compared with the placebo. Do your findings in parts **b)** and **d)** support the researchers' conclusion? Explain.