Chapter 4

LINEAR PROGRAMMING: THE SIMPLEX METHOD

4.1 Slack Variables and the Pivot

Your Turn 1

The two new equations are:

$$300x_1 + 60x_2 + 180x_3 \le 20{,}000$$
$$5x_1 + 10x_2 + 15x_3 \le 900$$

The new answer tableau is:

x_1	x_2	x_3	s_1	s_2	s_3	z	
1	1	1	1	0	0	0	100
15	3	9	0	1	0	0	1000
1	2	3	0	0	1	0	180
−120	−40	−60	0	0	0	1	0

Your Turn 2

Pivot around the indicated 6.

x_1	x_2	x_3	s_1	s_2	s_3	z	
3	[6]	2	1	0	0	0	60
8	5	4	0	1	0	0	80
3	6	7	0	0	1	0	120
−30	−50	−15	0	0	0	1	0

The result is:

	x_1	x_2	x_3	s_1	s_2	s_3	z	
	3	[6]	2	1	0	0	0	60
$-5R_1 + 6R_2 \to R_2$	33	0	14	−5	6	0	0	180
$-R_1 + R_3 \to R_3$	0	0	5	−1	0	1	0	60
$25R_1 + 3R_4 \to R_4$	−15	0	5	25	0	0	3	1500

	x_1	x_2	x_3	s_1	s_2	s_3	z	
$\frac{1}{6}R_1 \to R_1$	1/2	1	1/3	1/6	0	0	0	10
$\frac{1}{6}R_2 \to R_2$	11	0	7/3	−5/6	1	0	0	30
	0	0	5	−1	0	1	0	60
$\frac{1}{3}R_4 \to R_4$	−5	0	5/3	25/3	0	0	1	500

The solution given by this tableau is:

$$x_1 = 0, x_2 = 10, x_3 = 0, s_1 = 0,$$
$$s_2 = 30, s_3 = 60, z = 500$$

4.1 Exercises

1. $x_1 + 2x_2 \le 6$

Add s_1 to the given inequality to obtain

$$x_1 + 2x_2 + s_1 = 6.$$

2. $6x_1 + 2x_2 \le 50$

Add s_1 to the given inequality to obtain

$$6x_1 + 2x_2 + s_1 = 50.$$

3. $2.3x_1 + 5.7x_2 + 1.8x_3 \le 17$

Add s_1 to the given inequality to obtain

$$2.3x_1 + 5.7x_2 + 1.8x_3 + s_1 = 17.$$

4. $8x_1 + 6x_2 + 5x_3 \le 250$

Add s_1 to the given inequality to obtain

$$8x_1 + 6x_2 + 5x_3 + s_1 = 250.$$

5. (a) Since there are three constraints to be converted into equations we need three slack variables.

(b) We use s_1, s_2, and s_3 for the slack variables.

(c) The equations are

$$\begin{aligned} 2x_1 + 3x_2 + s_1 \qquad\qquad &= 15 \\ 4x_1 + 5x_2 \qquad + s_2 \qquad &= 35 \\ x_1 + 6x_2 \qquad\qquad + s_3 &= 20. \end{aligned}$$

6. Maximize $z = 1.2x_1 + 3.5x_2$

subject to: $2.4x_1 + 1.5x_2 \le 10$
$1.7x_1 + 1.9x_2 \le 15$

with $x_1 \ge 0, x_2 \ge 0.$

(a) We need one slack variable for each inequality. Thus, 2 are needed.

(b) We will use s_1 and s_2 for the slack variables.

(c) $2.4x_1 + 1.5x_2 \le 10$ becomes

$$2.4x_1 + 1.5x_2 + s_1 = 10.$$

$1.7x_1 + 1.9x_2 \le 15$ becomes

$$1.7x_1 + 1.9x_2 + s_2 = 15.$$

7. **(a)** There are two constraints to be converted into equations, so we must introduce two slack variables.

(b) Call the slack variables s_1 and s_2.

(c) The equations are

$$\begin{aligned} 7x_1 + 6x_2 + 8x_3 + s_1 \qquad &= 118 \\ 4x_1 + 5x_2 + 10x_3 \qquad + s_2 &= 220. \end{aligned}$$

8. Maximize $z = 12x_1 + 15x_2 + 10x_3$

subject to: $2x_1 + 2x_2 + x_3 \leq 8$

$x_1 + 4x_2 + 3x_3 \leq 12$

with $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

(a) There are two inequalities, so 2 slack variables are needed.

(b) Use s_1 and s_2 for the slack variables.

(c)
$$\begin{aligned} 2x_1 + 2x_2 + x_3 + s_1 \qquad &= 8 \\ x_1 + 4x_2 + 3x_3 \qquad + s_2 &= 12 \end{aligned}$$

9. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$\begin{aligned} 4x_1 + 2x_2 &\leq 5 \\ x_1 + 2x_2 &\leq 4 \end{aligned}$$

and $z = 7x_1 + x_2$ is maximized.

We need two slack variables, s_1 and s_2. Then the problem can be restated as:

Find $x_1 \geq 0, x_2 \geq 0, s_1 \geq 0,$ and $s_2 \geq 0$ such that

$$\begin{aligned} 4x_1 + 2x_2 + s_1 \qquad &= 5 \\ x_1 + 2x_2 \qquad + s_2 &= 4. \end{aligned}$$

and $z = 7x_1 + x_2$ is maximized.

Rewrite the objective function as

$$-7x_1 - x_2 + z = 0.$$

The initial simplex tableau is

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 4 & 2 & 1 & 0 & 0 & 5 \\ 1 & 2 & 0 & 1 & 0 & 4 \\ \hline -7 & -1 & 0 & 0 & 1 & 0 \end{array}.$$

10. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$\begin{aligned} 2x_1 + 3x_2 &\leq 100 \\ 5x_1 + 4x_2 &\leq 200 \end{aligned}$$

and $z = x_1 + 3x_2$ is maximized. We need two slack variables. Add s_1 and s_2 to get the system

$$\begin{aligned} 2x_1 + 3x_2 + s_1 \qquad\qquad &= 100 \\ 5x_1 + 4x_2 \qquad + s_2 \qquad &= 200 \\ -x_1 - 3x_2 \qquad\qquad + z &= 0. \end{aligned}$$

The initial simplex tableau is

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 2 & 3 & 1 & 0 & 0 & 100 \\ 5 & 4 & 0 & 1 & 0 & 200 \\ \hline -1 & -3 & 0 & 0 & 1 & 0 \end{array}.$$

11. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$\begin{aligned} x_1 + x_2 &\leq 10 \\ 5x_1 + 2x_2 &\leq 20 \\ x_1 + 2x_2 &\leq 36 \end{aligned}$$

and $z = x_1 + 3x_2$ is maximized.

Using slack variables $s_1, s_2,$ and s_3, the problem can be restated as:

Find $x_1 \geq 0, x_2 \geq 0, s_1 \geq 0,$ $s_2 \geq 0,$ and $s_3 \geq 0$ such that

$$\begin{aligned} x_1 + x_2 + s_1 \qquad\qquad &= 10 \\ 5x_1 + 2x_2 \qquad + s_2 \qquad &= 20 \\ x_1 + 2x_2 \qquad\qquad + s_3 &= 36 \end{aligned}$$

and $z = x_1 + 3x_2$ is maximized.

Rewrite the objective function as

$$-x_1 - 3x_2 + z = 0.$$

The initial simplex tableau is

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ 5 & 2 & 0 & 1 & 0 & 0 & 20 \\ 1 & 2 & 0 & 0 & 1 & 0 & 36 \\ \hline -1 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

12. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$\begin{aligned} x_1 + x_2 &\leq 25 \\ 4x_1 + 3x_2 &\leq 48 \end{aligned}$$

and $z = 5x_1 + 3x_2$ is maximized.

We add s_1 and s_2 to get the system

$$\begin{aligned} x_1 + x_2 + s_1 \qquad\qquad &= 25 \\ 4x_1 + 3x_2 \qquad + s_2 \qquad &= 48 \\ -5x_1 - 3x_2 \qquad\qquad + z &= 0. \end{aligned}$$

The initial simplex tableau is

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \end{array}$$

$$\left[\begin{array}{rrrrr|r} 1 & 1 & 1 & 0 & 0 & 25 \\ 4 & 3 & 0 & 1 & 0 & 48 \\ \hline -5 & -3 & 0 & 0 & 1 & 0 \end{array}\right].$$

13. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$3x_1 + x_2 \leq 12$$
$$x_1 + x_2 \leq 15$$

and $z = 2x_1 + x_2$ is maximized.

Using slack variables s_1 and s_2, the problem can be restated as:

Find $x_1 \geq 0, x_2 \geq 0, s_1 \geq 0,$ and $s_2 \geq 0$ such that

$$\begin{aligned} 3x_1 + x_2 + s_1 &= 12 \\ x_1 + x_2 + s_2 &= 15 \end{aligned}$$

and $z = 2x_1 + x_2$ is maximized.

Rewrite the objective function as

$$-2x_1 - x_2 + z = 0.$$

The initial simplex tableau is

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \end{array}$$

$$\left[\begin{array}{rrrrr|r} 3 & 1 & 1 & 0 & 0 & 12 \\ 1 & 1 & 0 & 1 & 0 & 15 \\ \hline -2 & -1 & 0 & 0 & 1 & 0 \end{array}\right].$$

14. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$10x_1 + 4x_2 \leq 100$$
$$20x_1 + 10x_2 \leq 150$$

and $z = 4x_1 + 5x_2$ is maximized.

Add s_1 and s_2 to get

$$\begin{aligned} 10x_1 + 4x_2 + s_1 &= 100 \\ 20x_1 + 10x_2 + s_2 &= 150 \\ -4x_1 - 5x_2 + z &= 0. \end{aligned}$$

The initial simplex tableau is

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \end{array}$$

$$\left[\begin{array}{rrrrr|r} 10 & 4 & 1 & 0 & 0 & 100 \\ 20 & 10 & 0 & 1 & 0 & 150 \\ \hline -4 & -5 & 0 & 0 & 1 & 0 \end{array}\right].$$

15.

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \end{array}$$

$$\left[\begin{array}{rrrrrr|r} 1 & 0 & 4 & 5 & 1 & 0 & 8 \\ 3 & 1 & 1 & 2 & 0 & 0 & 4 \\ \hline -2 & 0 & 2 & 3 & 0 & 1 & 28 \end{array}\right]$$

The variables x_2 and s_2 are basic variables, because the columns for these variables have all zeros except for one nonzero entry. If the remaining variables x_1, x_3, and s_1 are zero, then $x_2 = 4$ and $s_2 = 8$. From the bottom row, $z = 28$. The basic feasible solution is $x_1 = 0$, $x_2 = 4$, $x_3 = 0$, $s_1 = 0$, $s_2 = 8$, and $z = 28$.

16.

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \end{array}$$

$$\left[\begin{array}{rrrrrr|r} 1 & 5 & 0 & 1 & 2 & 0 & 6 \\ 0 & 2 & 1 & 2 & 3 & 0 & 15 \\ \hline 0 & 4 & 0 & 1 & -2 & 1 & 64 \end{array}\right]$$

x_1 and x_3 are the basic variables. The solution is $x_1 = 6$, $x_2 = 0$, $x_3 = 15$, $s_1 = 0$, $s_2 = 0$, and $z = 64$.

17.

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \end{array}$$

$$\left[\begin{array}{rrrrrrr|r} 6 & 2 & 2 & 3 & 0 & 0 & 0 & 16 \\ 2 & 2 & 0 & 1 & 0 & 5 & 0 & 35 \\ 2 & 1 & 0 & 3 & 1 & 0 & 0 & 6 \\ \hline -3 & -2 & 0 & 2 & 0 & 0 & 3 & 36 \end{array}\right]$$

The basic variables are x_3, s_2, and s_3. If x_1, x_2, and s_1 are zero, then $2x_3 = 16$, so $x_3 = 8$. Similarly, $s_2 = 6$ and $5s_3 = 35$, so $s_3 = 7$. From the bottom row, $3z = 36$, so $z = 12$. The basic feasible solution is $x_1 = 0, x_2 = 0, x_3 = 8, s_1 = 0,$ $s_2 = 6, s_3 = 7,$ and $z = 12$.

18.

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \end{array}$$

$$\left[\begin{array}{rrrrrrr|r} 0 & 2 & 0 & 5 & 2 & 2 & 0 & 15 \\ 0 & 3 & 1 & 0 & 1 & 2 & 0 & 2 \\ 7 & 4 & 0 & 0 & 3 & 5 & 0 & 35 \\ \hline 0 & -4 & 0 & 0 & 4 & 3 & 2 & 40 \end{array}\right]$$

$x_1, x_3,$ and s_1 are the basic variables. The solution is $x_1 = 5, x_2 = 0, x_3 = 2, s_1 = 3, s_2 = 0, s_3 = 0,$ and $z = 20$.

19.

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \end{array}$$

$$\left[\begin{array}{rrrrrr|r} 1 & 2 & 4 & 1 & 0 & 0 & 56 \\ 2 & \boxed{2} & 1 & 0 & 1 & 0 & 40 \\ \hline -1 & -3 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

Clear the x_2 column.

	x_1	x_2	x_3	s_1	s_2	z	
$-R_2 + R_1 \to R_1$	−1	0	3	1	−1	0	16
	2	$\boxed{2}$	1	0	1	0	40
	−1	−3	−2	0	0	1	0

	x_1	x_2	x_3	s_1	s_2	z	
	−1	0	3	1	−1	0	16
	2	2	1	0	1	0	40
$3R_2 + 2R_3 \to R_3$	4	0	−1	0	3	2	120

x_2 and s_1 are now basic. The solution is $x_1 = 0$, $x_2 = 20, x_3 = 0, s_1 = 16, s_2 = 0$, and $z = 60$.

20.

x_1	x_2	x_3	s_1	s_2	z	
2	3	4	1	0	0	18
6	$\boxed{3}$	2	0	1	0	15
−1	−6	−2	0	0	1	0

Clear the x_2 column.

	x_1	x_2	x_3	s_1	s_2	z	
$-R_2 + R_1 \to R_1$	−4	0	2	1	−1	0	3
	6	$\boxed{3}$	2	0	1	0	15
$2R_2 + R_3 \to R_3$	11	0	2	0	2	1	30

x_2 and s_1 are now basic. Thus, the solution is $x_1 = 0$, $x_2 = 5$, $x_3 = 0$, $s_1 = 3$, $s_2 = 0$, and $z = 30$.

21.

x_1	x_2	x_3	s_1	s_2	s_3	z	
2	2	$\boxed{1}$	1	0	0	0	12
1	2	3	0	1	0	0	45
3	1	1	0	0	1	0	20
−2	−1	−3	0	0	0	1	0

Clear the x_3 column.

	x_1	x_2	x_3	s_1	s_2	s_3	z	
	2	2	1	1	0	0	0	12
$-3R_1 + R_2 \to R_2$	−5	−4	0	−3	1	0	0	9
$-R_1 + R_3 \to R_3$	1	−1	0	−1	0	1	0	8
$3R_1 + R_4 \to R_4$	4	5	0	3	0	0	1	36

x_3, s_2, and s_3 are now basic. The solution is $x_1 = 0$, $x_2 = 0$, $x_3 = 12$, $s_1 = 0$, $s_2 = 9$, $s_3 = 8$, and $z = 36$.

22.

x_1	x_2	x_3	s_1	s_2	s_3	z	
4	2	3	1	0	0	0	22
2	2	$\boxed{5}$	0	1	0	0	28
1	3	2	0	0	1	0	45
−3	−2	−4	0	0	0	1	0

Clear the x_3 column.

	x_1	x_2	x_3	s_1	s_2	s_3	z	
$-3R_2 + 5R_1 \to R_1$	14	4	0	5	−3	0	0	26
	2	2	$\boxed{5}$	0	1	0	0	28
$-2R_2 + 5R_3 \to R_3$	1	11	0	0	−2	5	0	169
$4R_2 + 5R_4 \to R_4$	−7	−2	0	0	4	0	5	112

x_3, s_1, and s_3 are now basic. Thus, the solution is $x_1 = 0$, $x_2 = 0$, $x_3 = \frac{28}{5}$, $s_1 = \frac{26}{5}$, $s_2 = 0$, $s_3 = \frac{169}{5}$, and $z = \frac{112}{5}$.

23.

x_1	x_2	x_3	s_1	s_2	s_3	z	
2	$\boxed{2}$	3	1	0	0	0	500
4	1	1	0	1	0	0	300
7	2	4	0	0	1	0	700
−3	−4	−2	0	0	0	1	0

Clear the x_2 column.

	x_1	x_2	x_3	s_1	s_2	s_3	z	
	2	$\boxed{2}$	3	1	0	0	0	500
$-R_1 + 2R_2 \to R_2$	6	0	−1	−1	2	0	0	100
$-R_1 + R_3 \to R_3$	5	0	1	−1	0	1	0	200
$2R_1 + R_4 \to R_4$	1	0	4	2	0	0	1	1000

x_2, s_2, and s_3 are now basic. Thus, the solution is $x_1 = 0$, $x_2 = 250$, $x_3 = 0$, $s_1 = 0$, $s_2 = 50$, $s_3 = 200$, and $z = 1000$.

24.

x_1	x_2	x_3	x_4	s_1	s_2	s_3	z	
1	2	3	1	1	0	0	0	115
2	1	8	5	0	1	0	0	200
$\boxed{1}$	0	1	0	0	0	1	0	50
−2	−1	−1	−1	0	0	0	1	0

Clear the x_1 column.

$$\begin{array}{r} -R_3+R_1 \to R_1 \\ -2R_3+R_2 \to R_2 \\ \\ 2R_3+R_4 \to R_4 \end{array} \left[\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ 0 & 2 & 2 & 1 & 1 & 0 & -1 & 0 & 65 \\ 0 & 1 & 6 & 5 & 0 & 1 & -2 & 0 & 100 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 50 \\ \hline 0 & -1 & 1 & -1 & 0 & 0 & 2 & 1 & 100 \end{array}\right]$$

$x_1, s_1,$ and s_2 are now basic. Thus, the solution is $x_1 = 50,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0,\ s_1 = 65,$ $s_2 = 100,\ s_3 = 0,$ and $z = 100.$

25. A slack variable (a nonnegative quantity), converts a linear inequality into a linear equation by adding the amount needed in an expression to be equal to a specific value.

26. The number of slack variables must be the same as the number of constraints in a linear programming problem.

27. Let x_1 represent the number of simple figures, x_2 the number of figures with additions, and x_3 the number of computer-drawn sketches. Organize the information in a table.

	Simple Figures	Figures with Additions	Computer -Drawn Sketches	Maximum Allowed
Cost	20	35	60	2200
Royalties	95	200	325	

The cost constraint is

$$20x_1 + 35x_2 + 60x_3 \le 2200.$$

The limit of 400 figures leads to the constraint

$$x_1 + x_2 + x_3 \le 400.$$

The other stated constraints are

$$x_3 \le x_1 + x_2 \text{ and } x_1 \ge 2x_2,$$

and these can be rewritten in standard form as

$$-x_1 - x_2 + x_3 \le 0 \text{ and } -x_1 + 2x_2 \le 0$$

respectively. The problem may be stated as:

Find $x_1 \ge 0,\ x_2 \ge 0,$ and $x_3 \ge 0$ such that

$$\begin{aligned} 20x_1 + 35x_2 + 60x_3 &\le 2200 \\ x_1 + x_2 + x_3 &\le 400 \\ -x_1 - x_2 + x_3 &\le 0 \\ -x_1 + 2x_2 &\le 0 \end{aligned}$$

and $z = 95x_1 + 200x_2 + 325x_3$ is maximized.

Introduce slack variables s_1, s_2, s_3, and s_4, and the problem can be restated as:

Find $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ s_1 \ge 0,\ s_2 \ge 0,$ $s_3 \ge 0,$ and $s_4 \ge 0$ such that

$$\begin{aligned} 20x_1 + 35x_2 + 60x_3 + s_1 &= 2200 \\ x_1 + x_2 + x_3 + s_2 &= 400 \\ -x_1 - x_2 + x_3 + s_3 &= 0 \\ -x_1 + 2x_2 + s_4 &= 0 \end{aligned}$$

and $z = 95x_1 + 200x_2 + 325x_3$ is maximized.

Rewrite the objective function as

$$-95x_1 - 200x_2 - 325x_3 + z = 0.$$

The initial simlpex tableau is

$$\left[\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z & \\ 20 & 35 & 60 & 1 & 0 & 0 & 0 & 0 & 2200 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 400 \\ -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline -95 & -200 & -325 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

28. Let x_1 represent the number of racing bicycles, x_2 the number of touring bicycles, and x_3 the number of mountain bicycles. Organize the information in a table.

	Racing	Touring	Mountain	Amount Available
Steel	17	27	34	91,800
Aluminum	12	21	15	42,000
Profit	\$8	\$12	\$22	

Using this information, the problem may be stated as:

Find $x_1 \ge 0,\ x_2 \ge 0,$ and $x_3 \ge 0$ such that

$$\begin{aligned} 17x_1 + 27x_2 + 34x_3 &\le 91{,}800 \\ 12x_1 + 21x_2 + 15x_3 &\le 42{,}000 \end{aligned}$$

and $z = 8x_1 + 12x_2 + 22x_3$ is maximized.

Introduce slack variables s_1 and s_2, and the problem can be restated as:

Find $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ s_1 \ge 0,$ and $s_2 \ge 0$ such that

$$\begin{aligned} 17x_1 + 27x_2 + 34x_3 + s_1 &= 91{,}800 \\ 12x_1 + 21x_2 + 15x_3 + s_2 &= 42{,}000 \end{aligned}$$

and $z = 8x_1 + 12x_2 + 22x_3$ is maximized.

Rewrite the objective function as

$$-8x_1 - 12x_2 - 22x_3 + z = 0.$$

The initial simplex tableau is

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 17 & 27 & 34 & 1 & 0 & 0 & 91{,}000 \\ 12 & 21 & 15 & 0 & 1 & 0 & 42{,}000 \\ \hline -8 & -12 & -22 & 0 & 0 & 1 & 0 \end{array}.$$

29. Let x_1, x_2, and x_3 represent the number produced of each of the three styles of jackets. Organize the information in a table.

	Style 1	Style 2	Style 3	Sq ft or $ available
Nylon used	6	4	3	300 sq ft
Fleece used	2	3	5	150 sq ft
Cost	$20	$18	$17	$600

The limit of 300 sq ft for nylon leads to the constraint

$$6x_1 + 4x_2 + 3x_3 \le 300.$$

The limit of 150 sq ft for fleece leads to the constraint

$$2x_1 + 3x_2 + 5x_3 \le 150.$$

The cost constraint is

$$20x_1 + 18x_2 + 17x_3 \le 600.$$

The problem may be stated as:

Find $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$ such that

$$\begin{aligned} 6x_1 + 4x_2 + 3x_3 &\le 300 \\ 2x_1 + 3x_2 + 5x_3 &\le 150 \\ 20x_1 + 18x_2 + 17x_3 &\le 600 \end{aligned}$$

and $z = x_1 + x_2 + x_3$ is maximized.

Introduce slack variables s_1, s_2, and s_3, and the problem can be restated as:

Find $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$, $s_1 \ge 0$, $s_2 \ge 0$, and $s_3 \ge 0$ such that

$$\begin{aligned} 6x_1 + 4x_2 + 3x_3 + s_1 \qquad\qquad &= 300 \\ 2x_1 + 3x_2 + 5x_3 \qquad + s_2 \qquad &= 150 \\ 20x_1 + 18x_2 + 17x_3 \qquad\qquad + s_3 &= 600 \end{aligned}$$

and $z = x_1 + x_2 + x_3$ is maximized.

Rewrite the objective function as

$$-x_1 - x_2 - x_3 + z = 0.$$

The initial simplex tableau is

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 6 & 4 & 3 & 1 & 0 & 0 & 0 & 300 \\ 2 & 3 & 5 & 0 & 1 & 0 & 0 & 150 \\ 20 & 18 & 17 & 0 & 0 & 1 & 0 & 600 \\ \hline -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

30. Let x_1 represent the number of Basic sets, x_2 the number of Regular sets, and x_3 the number of Deluxe sets. Organize the information in a table.

	Basic Set	Regular Set	Deluxe Set	Number Available
Utility Knife	2	2	3	800
Chef's Knife	1	1	1	400
Slicer	0	1	1	200
Profit	$30	$40	$60	

Using this information, the problem may be stated as:

Find $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$ such that

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 &\le 800 \\ x_1 + x_2 + x_3 &\le 400 \\ x_2 + x_3 &\le 200 \end{aligned}$$

and $z = 30x_1 + 40x_2 + 60x_3$ is maximized.

Introduce slack variables s_1, s_2, and s_3, and the problem can be restated as:

Find $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$, $s_1 \ge 0$, $s_2 \ge 0$, and $s_3 \ge 0$ such that

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 + s_1 \qquad\qquad &= 800 \\ x_1 + x_2 + x_3 \qquad + s_2 \qquad &= 400 \\ x_2 + x_3 \qquad\qquad + s_3 &= 200 \end{aligned}$$

and $z = 30x_1 + 40x_2 + 60x_3$ is maximized.

Rewrite the objective function as

$$-30x_1 - 40x_2 - 60x_3 + z = 0.$$

The initial simplex tableau is

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 2 & 2 & 3 & 1 & 0 & 0 & 0 & 800 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 400 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 200 \\ \hline -30 & -40 & -60 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

31. Let x_1 = the number of newspaper ads,
x_2 = the number of internet banners,
and x_3 = the number of TV ads.

Organize the information in a table.

	Newspaper Ads	Internet Banners	TV Ads
Cost per Ad	400	20	2000
Maximum Number	30	60	10
Women Seeing Ad	4000	3000	10,000

The cost constraint is

$$400x_1 + 20x_2 + 2000x_3 \le 8000$$

The constraints on the numbers of ads is

$$x_1 \le 30$$
$$x_2 \le 60$$
$$x_3 \le 10.$$

The problem may be stated as:

Find $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$ such that

$$\begin{aligned} 400x_1 + 20x_2 + 2000x_3 &\le 8000 \\ x_1 \qquad\qquad &\le 30 \\ x_2 \qquad &\le 60 \\ x_3 &\le 10 \end{aligned}$$

and $z = 4000x_1 + 3000x_2 + 10,000x_3$ is maximized.

Introduce slack variables s_1, s_2, s_3 and s_4, and the problem can be restated as:

Find $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$, $s_1 \ge 0$, $s_2 \ge 0$, $s_3 \ge 0$, and $s_4 \ge 0$ such that

$$\begin{aligned} 400x_1 + 20x_2 + 2000x_3 + s_1 &= 8000 \\ x_1 + s_2 &= 30 \\ x_2 + s_3 &= 60 \\ x_3 + s_4 &= 10 \end{aligned}$$

and $z = 4000x_1 + 3000x_2 + 10,000x_3$ is maximized.

Rewrite the objective function as

$$-4000x_1 - 3000x_2 - 10,000x_3 + z = 0.$$

The initial simplex tableau is

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z \end{matrix} \\ \left[\begin{array}{cccccccc|c} 400 & 20 & 2000 & 1 & 0 & 0 & 0 & 0 & 8000 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 60 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 10 \\ \hline -4000 & -3000 & -10,000 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

4.2 Maximization Problems

Your Turn 1

Example 2 of Section 3.3 yields the following linear programming problem, where we have renamed x and y as x_1 and x_2.

Maximize $z = 12x_1 + 40x_2$

subject to:
$$\begin{aligned} x_1 + x_2 &\le 16 \\ x_1 + 3x_2 &\le 36 \\ x_1 &\le 10 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

Add a slack variable to each of the first three constraints:

$$\begin{aligned} x_1 + x_2 + s_1 &\le 16 \\ x_1 + 3x_2 + s_2 &\le 36 \\ x_1 + s_3 &\le 10 \end{aligned}$$

with $x_1 \ge 0$, $x_2 \ge 0$, $s_1 \ge 0$, $s_2 \ge 0$, $s_3 \ge 0$

The corresponding initial tableau is

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{matrix} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 16 \\ 1 & \boxed{3} & 0 & 1 & 0 & 0 & 36 \\ 1 & 0 & 0 & 0 & 1 & 0 & 10 \\ \hline -12 & -40 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Since the most negative indicator is -40 and the quotient 36/3 is smaller than 16/1, we pivot on the 3 in column 2:

$$\begin{array}{r} -R_2 + 3R_1 \rightarrow R_1 \\ \\ \\ 40R_2 + 3R_4 \rightarrow R_4 \end{array} \left[\begin{array}{cccccc|c} 2 & 0 & 3 & -1 & 0 & 0 & 12 \\ 1 & 3 & 0 & 1 & 0 & 0 & 36 \\ 1 & 0 & 0 & 0 & 1 & 0 & 10 \\ \hline 4 & 0 & 0 & 40 & 0 & 3 & 1440 \end{array}\right]$$

(columns: x_1, x_2, s_1, s_2, s_3, z)

There are now no negative indicators, so we can read the solution:

$$x_1 = 0,\ x_2 = \frac{36}{3} = 12,\ z = \frac{1440}{3} = 480$$

Your Turn 2

Pivot on the 4 in column 2 of the following tableau.

x_1	x_2	s_1	s_2	s_3	z	
1	−2	1	0	0	0	100
3	[4]	0	1	0	0	200
5	0	0	0	1	0	150
−10	−25	0	0	0	1	0

	x_1	x_2	s_1	s_2	s_3	z	
$R_2 + 2R_1 \rightarrow R_1$	5	0	2	1	0	0	400
	3	4	0	1	0	0	200
	5	0	0	0	1	0	150
$25R_2 + 4R_4 \rightarrow R_4$	35	0	0	25	0	4	5000

There are no negative indicators, so the optimal solution is:

$$x_1 = 0,\ x_2 = \frac{200}{4} = 50,\ s_1 = 200,$$
$$s_2 = 0,\ s_3 = 150,\ z = \frac{5000}{4} = 1250$$

4.2 Warmup Exercises

W1. Maximize $z = 3x_1 + 2x_2$

subject to:
$$3x_1 + 4x_2 \le 12$$
$$2x_1 + 5x_2 \le 10$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$

Introduce slack variables to change the constraints into equalities.

$$3x_1 + 4x_2 + s_1 = 12$$
$$2x_1 + 5x_2 + s_2 = 10$$

Write the objective function with all variables on the left side of the equals sign.

$$z - 3x_1 - 2x_2 = 0$$

The initial simplex tableau is

x_1	x_2	s_1	s_2	z	
3	4	1	0	0	12
2	5	0	1	0	10
−3	−2	0	0	1	0

W2. Maximize $z = 14x_1 + 16x_2 + 12x_3$

subject to:
$$3x_1 + 8x_2 + 2x_3 \le 24$$
$$5x_1 + 6x_2 + 3x_3 \le 30$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$
$$x_3 \ge 0$$

Introduce slack variables to change the constraints into equalities.

$$3x_1 + 8x_2 + 2x_3 + s_1 = 24$$
$$5x_1 + 6x_2 + 3x_3 + s_2 = 30$$

Write the objective function with all variables on the left side of the equals sign.

$$z - 14x_1 - 16x_2 - 12x_3 = 0$$

The initial simplex tableau is

x_1	x_2	x_3	s_1	s_2	z	
3	8	2	1	0	0	24
5	6	3	0	1	0	30
−14	−16	−12	0	0	1	0

W3. The initial tableau from W1 is

x_1	x_2	s_1	s_2	z	
3	4	1	0	0	12
2	5	0	1	0	10
−3	−2	0	0	1	0

	x_1	x_2	s_1	s_2	z	
	3	4	1	0	0	12
$(-2)R_1 + 3R_2 \rightarrow R_2$	0	7	−2	3	0	6
$R_1 + R_3 \rightarrow R_3$	0	2	1	0	1	12

The basic variables are x_1 and s_2, while x_2 and s_1 are 0.

$$x_1 = \frac{12}{3} = 4$$
$$s_2 = \frac{6}{3} = 2$$

For these values of the basic variables, $z = 12$.

W4. The initial tableau from W2 is

x_1	x_2	x_3	s_1	s_2	z	
3	8	2	1	0	0	24
5	6	3	0	1	0	30
−14	−16	−12	0	0	1	0

	x_1	x_2	x_3	s_1	s_2	z	
	3	8	2	1	0	0	24
$(-3)R_1 + 4R_2 \rightarrow R_2$	11	0	6	−3	4	0	48
$R_1 + R_3 \rightarrow R_3$	−8	0	−8	2	0	1	48

The basic variables are x_2 and s_2, while x_1, x_3, and s_1 are 0.

$$x_2 = \frac{24}{8} = 3$$
$$s_2 = \frac{48}{4} = 12$$

For these values of the basic variables, $z = 48$.

4.2 Exercises

1.

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 1 & 4 & 4 & 1 & 0 & 0 & 16 \\ 2 & 1 & 5 & 0 & 1 & 0 & 20 \\ \hline -3 & -1 & -2 & 0 & 0 & 1 & 0 \end{array}$$

The most negative indicator is -3, in the first column. Find the quotients $\frac{16}{1} = 16$ and $\frac{20}{2} = 10$; since 10 is the smaller quotient, 2 in row 2, column 1 is the pivot.

$$\begin{array}{r} \frac{16}{1} = 16 \\ \frac{20}{2} = 10 \\ \\ \end{array} \left[\begin{array}{cccccc|c} 1 & 4 & 4 & 1 & 0 & 0 & 16 \\ \boxed{2} & 1 & 5 & 0 & 1 & 0 & 20 \\ \hline -3 & -1 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

Performing row transformations, we get the following tableau.

$$\begin{array}{r} -R_2 + 2R_1 \to R_1 \\ \\ 3R_2 + 2R_3 \to R_3 \end{array} \left[\begin{array}{cccccc|c} 0 & 7 & 3 & 2 & -1 & 0 & 12 \\ 2 & 1 & 5 & 0 & 1 & 0 & 20 \\ \hline 0 & 1 & 11 & 0 & 3 & 2 & 60 \end{array}\right]$$

All of the numbers in the last row are nonnegative, so we are finished pivoting. Create a 1 in the columns corresponding to x_1, s_1 and z.

$$\begin{array}{r} \frac{1}{2}R_1 \to R_1 \\ \frac{1}{2}R_2 \to R_2 \\ \frac{1}{2}R_3 \to R_3 \end{array} \left[\begin{array}{cccccc|c} 0 & \frac{7}{2} & \frac{3}{2} & 1 & -\frac{1}{2} & 0 & 6 \\ 1 & \frac{1}{2} & \frac{5}{2} & 0 & \frac{1}{2} & 0 & 10 \\ \hline 0 & \frac{1}{2} & \frac{11}{2} & 0 & \frac{3}{2} & 1 & 30 \end{array}\right]$$

The maximum value is 30 and occurs when $x_1 = 10$, $x_2 = 0$, $x_3 = 0$, $s_1 = 6$, and $s_2 = 0$.

2.

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 3 & 3 & 2 & 1 & 0 & 0 & 18 \\ 2 & 2 & 3 & 0 & 1 & 0 & 16 \\ \hline -4 & -6 & -2 & 0 & 0 & 1 & 0 \end{array}$$

The most negative indicator is -6, in the second column. Find the quotients $\frac{18}{3} = 6$ and $\frac{16}{2} = 8$; since 6 is the smaller quotient, 3 in row 1, column 2 is the pivot.

$$\begin{array}{r} \frac{18}{3} = 6 \\ \frac{16}{2} = 8 \\ \\ \end{array} \left[\begin{array}{cccccc|c} 3 & \boxed{3} & 2 & 1 & 0 & 0 & 18 \\ 2 & 2 & 3 & 0 & 1 & 0 & 16 \\ \hline -4 & -6 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + 3R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{cccccc|c} 3 & 3 & 2 & 1 & 0 & 0 & 18 \\ 0 & 0 & 5 & -2 & 3 & 0 & 12 \\ \hline 2 & 0 & 2 & 2 & 0 & 1 & 36 \end{array}\right]$$

All of the numbers in the last row are nonnegative, so we are finished pivoting. Create a 1 in the columns corresponding to x_2 and s_2.

$$\begin{array}{r} \frac{1}{3}R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{cccccc|c} 1 & 1 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 6 \\ 0 & 0 & \frac{5}{3} & -\frac{2}{3} & 1 & 0 & 4 \\ \hline 2 & 0 & 2 & 2 & 0 & 1 & 36 \end{array}\right]$$

The maximum value is 36 when $x_1 = 0$, $x_2 = 6$, $x_3 = 0$, $s_1 = 0$, and $s_2 = 4$.

3.

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 1 & 3 & 1 & 0 & 0 & 0 & 12 \\ 2 & 1 & 0 & 1 & 0 & 0 & 10 \\ 1 & 1 & 0 & 0 & 1 & 0 & 4 \\ \hline -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}$$

The most negative indicator is -2, in the first column. Find the quotients $\frac{12}{1} = 12$, $\frac{10}{2} = 5$, and $\frac{4}{1} = 4$; since 4 is the smallest quotient, 1 in row 3, column 1 is the pivot.

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 1 & 3 & 1 & 0 & 0 & 0 & 12 \\ 2 & 1 & 0 & 1 & 0 & 0 & 10 \\ \boxed{1} & 1 & 0 & 0 & 1 & 0 & 4 \\ \hline -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}$$

	x_1	x_2	s_1	s_2	s_3	z	
$-R_3 + R_1 \to R_1$	0	2	1	0	−1	0	8
$-2R_3 + R_2 \to R_2$	0	−1	0	1	−2	0	2
	1	1	0	0	1	0	4
$2R_3 + R_4 \to R_4$	0	1	0	0	2	1	8

This is a final tableau since all of the numbers in the last row are nonnegative. The maximum value is 8 when $x_1 = 4$, $x_2 = 0$, $s_1 = 8$, $s_2 = 2$, and $s_3 = 0$.

4.

x_1	x_2	x_3	s_1	s_2	s_3	z	
2	1	2	1	0	0	0	25
4	3	2	0	1	0	0	40
3	1	6	0	0	1	0	60
−4	−2	−3	0	0	0	1	0

The most negative indicator is -4, in the first column. Find the quotients $\frac{25}{2} = 12.5$, $\frac{40}{4} = 10$, and $\frac{60}{3} = 20$; since 10 is the smallest quotient, 4 in row 2, column 1 is the pivot.

x_1	x_2	x_3	s_1	s_2	s_3	z	
2	1	2	1	0	0	0	25
[4]	3	2	0	1	0	0	40
3	1	6	0	0	1	0	60
−4	−2	−3	0	0	0	1	0

Performing row transformations, we get the following tableau.

	x_1	x_2	x_3	s_1	s_2	s_3	z	
$-R_2 + 2R_1 \to R_1$	0	−1	2	2	−1	0	0	10
	4	3	2	0	1	0	0	40
$-3R_2 + 4R_3 \to R_3$	0	−5	18	0	−3	4	0	120
$R_2 + R_4 \to R_4$	0	1	−1	0	1	0	1	40

Since there is still a negative indicator, we must repeat the process. Find the quotients $\frac{10}{2} = 5$, $\frac{40}{2} = 20$, and $\frac{120}{18} \approx 6.7$; since 5 is the smallest quotient, 2 in row 1, column 3 is the pivot.

x_1	x_2	x_3	s_1	s_2	s_3	z	
0	−1	[2]	2	−1	0	0	10
4	3	2	0	1	0	0	40
0	−5	18	0	−3	4	0	120
0	1	−1	0	1	0	1	40

Performing row transformations, we get the following tableau.

	x_1	x_2	x_3	s_1	s_2	s_3	z	
	0	−1	2	2	−1	0	0	10
$-R_1 + R_2 \to R_2$	4	4	0	−2	2	0	0	30
$-9R_1 + R_3 \to R_3$	0	4	0	−18	6	4	0	30
$R_1 + 2R_4 \to R_4$	0	1	0	2	1	0	2	90

All of the numbers in the last row are nonnegative, so we are finished pivoting. Create a 1 in the columns corresponding to x_1, x_3, s_3, and z.

	x_1	x_2	x_3	s_1	s_2	s_3	z	
$\frac{1}{2}R_1 \to R_1$	0	$-\frac{1}{2}$	1	1	$-\frac{1}{2}$	0	0	5
$\frac{1}{4}R_2 \to R_2$	1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{15}{2}$
$\frac{1}{4}R_3 \to R_3$	0	1	0	$-\frac{9}{2}$	$\frac{3}{2}$	1	0	$\frac{15}{2}$
$\frac{1}{2}R_4 \to R_4$	0	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	1	45

The maximum is 45 when $x_1 = \frac{15}{2}$, $x_2 = 0$, $x_3 = 5$, $s_1 = 0$, $s_2 = 0$, and $s_3 = \frac{15}{2}$.

5.

x_1	x_2	x_3	s_1	s_2	s_3	z	
2	2	8	1	0	0	0	40
4	−5	6	0	1	0	0	60
2	−2	6	0	0	1	0	24
−14	−10	−12	0	0	0	1	0

The most negative indicator is -14, in the first column. Find the quotients $\frac{40}{2} = 20$, $\frac{60}{4} = 15$, and $\frac{24}{2} = 12$; since 12 is the smallest quotient, 2 in row 3, column 1 is the pivot.

x_1	x_2	x_3	s_1	s_2	s_3	z	
2	2	8	1	0	0	0	40
4	−5	6	0	1	0	0	60
[2]	−2	6	0	0	1	0	24
−14	−10	−12	0	0	0	1	0

Performing row transformations, we get the following tableau.

	x_1	x_2	x_3	s_1	s_2	s_3	z	
$-R_3 + R_1 \to R_1$	0	[4]	2	1	0	−1	0	16
$-2R_3 + R_2 \to R_2$	0	−1	−6	0	1	−2	0	12
	2	−2	6	0	0	1	0	24
$7R_3 + R_4 \to R_4$	0	−24	30	0	0	7	1	168

Since there is still a negative indicator, we must repeat the process. The second pivot is the 4 in column 2, since $\frac{16}{4}$ is the only nonnegative quotient in the only column with a negative indicator. Performing row transformations again, we get the following tableau.

$$\begin{array}{r} \\ R_1 + 4R_2 \to R_2 \\ R_1 + 2R_3 \to R_3 \\ 6R_1 + R_4 \to R_4 \end{array}\begin{array}{c} x_1 \; x_2 \; x_3 \; s_1 \; s_2 \; s_3 \; z \\ \left[\begin{array}{ccccccc|c} 0 & 4 & 2 & 1 & 0 & -1 & 0 & 16 \\ 0 & 0 & -22 & 1 & 4 & -9 & 0 & 64 \\ 4 & 0 & 14 & 1 & 0 & 1 & 0 & 64 \\ \hline 0 & 0 & 42 & 6 & 0 & 1 & 1 & 264 \end{array}\right]\end{array}$$

All of the numbers in the last row are nonnegative, so we are finished pivoting. Create a 1 in the columns corresponding to $x_1, x_2,$ and s_2.

$$\begin{array}{r} \frac{1}{4}R_1 \to R_1 \\ \frac{1}{4}R_2 \to R_2 \\ \frac{1}{4}R_3 \to R_3 \\ \\ \end{array}\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} & 0 & -\frac{1}{4} & 0 & 4 \\ 0 & 0 & -\frac{11}{2} & \frac{1}{4} & 1 & -\frac{9}{4} & 0 & 16 \\ 1 & 0 & \frac{7}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 16 \\ \hline 0 & 0 & 42 & 6 & 0 & 1 & 1 & 264 \end{array}\right]$$

The maximum value is 264 and occurs when $x_1 = 16,\ x_2 = 4,\ x_3 = 0,\ s_1 = 0,\ s_2 = 16,$ and $s_3 = 0.$

6.

$$\begin{array}{c} x_1 \quad x_2 \quad x_3 \quad s_1 \quad s_2 \quad z \\ \left[\begin{array}{cccccc|c} 3 & 2 & 4 & 1 & 0 & 0 & 18 \\ 2 & \boxed{1} & 5 & 0 & 1 & 0 & 8 \\ \hline -1 & -4 & -2 & 0 & 0 & 1 & 0 \end{array}\right]\end{array}$$

The most negative indicator is $-4.$ Of the quotients $\frac{18}{2} = 9$ and $\frac{8}{1} = 8,$ the smallest is 8, so pivot on the 1 in row 2, column 2

$$\begin{array}{r} -2R_2 + R_1 \to R_1 \\ \\ 4R_2 + R_3 \to R_3 \end{array}\begin{array}{c} x_1 \quad x_2 \quad x_3 \quad s_1 \quad s_2 \quad z \\ \left[\begin{array}{cccccc|c} -1 & 0 & -6 & 1 & -2 & 0 & 2 \\ 2 & 1 & 5 & 0 & 1 & 0 & 8 \\ \hline 7 & 0 & 18 & 0 & 4 & 1 & 32 \end{array}\right]\end{array}$$

This solution is optimal. The basic variables are x_2 and s_1. The maximum is 32 when $x_1 = 0,$ $x_2 = 8,\ x_3 = 0,\ s_1 = 2,$ and $s_2 = 0.$

7. Maximize $z = 3x_1 + 5x_2$

subject to: $4x_1 + x_2 \le 25$

$2x_1 + 3x_2 \le 15$

with $x_1 \ge 0, x_2 \ge 0.$

Two slack variables, s_1 and $s_2,$ need to be introduced. The problem can be restated as:

Maximize $z = 3x_1 + 5x_2$

subject to: $4x_1 + x_2 + s_1 = 25$

$2x_1 + 3x_2 + s_2 = 15$

with $x_1 \ge 0, x_2 \ge 0, s_1 \ge 0, s_2 \ge 0.$

Rewrite the objective function as

$$-3x_1 - 5x_2 + z = 0.$$

The initial simplex tableau follows.

$$\begin{array}{c} x_1 \quad x_2 \quad s_1 \quad s_2 \quad z \\ \left[\begin{array}{ccccc|c} 4 & 1 & 1 & 0 & 0 & 25 \\ 2 & 3 & 0 & 1 & 0 & 15 \\ \hline -3 & -5 & 0 & 0 & 1 & 0 \end{array}\right]\end{array}$$

The most negative indicator is $-5,$ in the second column. To select the pivot from column 2, find the quotients $\frac{25}{1} = 25$ and $\frac{15}{3} = 5.$ The smaller quotient is 5, so 3 is the pivot.

$$\begin{array}{c} x_1 \quad x_2 \quad s_1 \quad s_2 \quad z \\ \left[\begin{array}{ccccc|c} 4 & 1 & 1 & 0 & 0 & 25 \\ 2 & \boxed{3} & 0 & 1 & 0 & 15 \\ \hline -3 & -5 & 0 & 0 & 1 & 0 \end{array}\right]\end{array}$$

$$\begin{array}{r} -R_2 + 3R_1 \to R_1 \\ \\ 5R_2 + 3R_3 \to R_3 \end{array}\begin{array}{c} x_1 \quad x_2 \quad s_1 \quad s_2 \quad z \\ \left[\begin{array}{ccccc|c} 10 & 0 & 3 & -1 & 0 & 60 \\ 2 & 3 & 0 & 1 & 0 & 15 \\ \hline 1 & 0 & 0 & 5 & 3 & 75 \end{array}\right]\end{array}$$

All of the indicators are nonnegative. Create a 1in the columns corresponding to $x_2, s_1,$ and z.

$$\begin{array}{r} \\ \frac{1}{3}R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \\ \frac{1}{3}R_3 \to R_3 \end{array}\begin{array}{c} x_1 \quad x_2 \quad s_1 \quad s_2 \quad z \\ \left[\begin{array}{ccccc|c} \frac{10}{3} & 0 & 1 & -\frac{1}{3} & 0 & 20 \\ \frac{2}{3} & 1 & 0 & \frac{1}{3} & 0 & 5 \\ \hline \frac{1}{3} & 0 & 0 & \frac{5}{3} & 1 & 25 \end{array}\right]\end{array}$$

The maximum value is 25 when $x_1 = 0,\ x_2 = 5,$ $s_1 = 20,$ and $s_2 = 0.$

8. Maximize $z = 5x_1 + 2x_2$

subject to: $2x_1 + 4x_2 \le 15$

$3x_1 + x_2 \le 10$

with $x_1 \ge 0, x_2 \ge 0.$

Two slack variables s_1 and s_2 need to be introduced. The problem can be restated as:

Maximize $z = 5x_1 + 2x_2$

subject to: $2x_1 + 4x_2 + s_1 = 15$
$3x_1 + x_2 + s_2 = 10$

with $x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0.$

Rewrite the objective function as

$$-5x_1 - 2x_2 + z = 0.$$

The initial simplex tableau as follows.

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{rrrrr|r} x_1 & x_2 & s_1 & s_2 & z & \\ 2 & 4 & 1 & 0 & 0 & 15 \\ 3 & 1 & 0 & 1 & 0 & 10 \\ \hline -5 & -2 & 0 & 0 & 1 & 0 \end{array}$$

The most negative indicator is -5 in the first column. To select the pivot from column 1, find the quotients $\frac{15}{2}$ and $\frac{10}{3}$, so 3 is the pivot.

$$\begin{array}{rrrrr|r} x_1 & x_2 & s_1 & s_2 & z & \\ 2 & 4 & 1 & 0 & 0 & 15 \\ \boxed{3} & 1 & 0 & 1 & 0 & 10 \\ \hline -5 & -2 & 0 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{r} \\ -2R_2 + 3R_1 \to R_1 \\ \\ 5R_2 + 3R_3 \to R_3 \end{array}\begin{array}{rrrrr|r} x_1 & x_2 & s_1 & s_2 & z & \\ 0 & 10 & 3 & -2 & 0 & 25 \\ 3 & 1 & 0 & 1 & 0 & 10 \\ \hline 0 & -1 & 0 & 5 & 3 & 50 \end{array}$$

Since there is still a negative indicator, we must repeat the process. Find the quotients $\frac{25}{10}$ and $\frac{10}{1}$. Since the smaller quotient is $\frac{25}{10}$, the 10 in row 1, column 2 is the pivot.

$$\begin{array}{rrrrr|r} x_1 & x_2 & s_1 & s_2 & z & \\ 0 & \boxed{10} & 3 & -2 & 0 & 25 \\ 3 & 1 & 0 & 1 & 0 & 10 \\ \hline 0 & -1 & 0 & 5 & 3 & 50 \end{array}$$

$$\begin{array}{r} \\ \\ -R_1 + 10R_2 \to R_2 \\ R_1 + 10R_3 \to R_3 \end{array}\begin{array}{rrrrr|r} x_1 & x_2 & s_1 & s_2 & z & \\ 0 & 10 & 3 & -2 & 0 & 25 \\ 30 & 0 & -3 & 12 & 0 & 75 \\ \hline 0 & 0 & 3 & 48 & 30 & 525 \end{array}$$

All of the indicators are nonnegative. Create a 1in the columns corresponding to $x_1, x_2,$ and z.

$$\begin{array}{r} \\ \frac{1}{10}R_1 \to R_1 \\ \frac{1}{30}R_2 \to R_2 \\ \frac{1}{30}R_3 \to R_3 \end{array}\begin{array}{rrrrr|r} x_1 & x_2 & s_1 & s_2 & z & \\ 0 & 1 & \frac{3}{10} & -\frac{1}{5} & 0 & \frac{5}{2} \\ 1 & 0 & -\frac{1}{10} & \frac{2}{5} & 0 & \frac{5}{2} \\ \hline 0 & 0 & \frac{1}{10} & \frac{8}{5} & 1 & \frac{35}{2} \end{array}$$

The maximum value is 17.5 when $x_1 = 2.5,$ $x_2 = 2.5, s_1 = 0,$ and $s_2 = 0.$

9. Maximize $z = 10x_1 + 12x_2$

subject to: $4x_1 + 2x_2 \leq 20$
$5x_1 + x_2 \leq 50$
$2x_1 + 2x_2 \leq 24$

with $x_1 \geq 0, x_2 \geq 0.$

Three slack variables, $s_1, s_2,$ and $s_3,$ need to be introduced. The initial tableau is as follows.

$$\begin{array}{rrrrrr|r} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 4 & 2 & 1 & 0 & 0 & 0 & 20 \\ 5 & 1 & 0 & 1 & 0 & 0 & 50 \\ 2 & 2 & 0 & 0 & 1 & 0 & 24 \\ \hline -10 & -12 & 0 & 0 & 0 & 1 & 0 \end{array}$$

The most negative indicator is $-12,$ in column 2.

The quotients are $\frac{20}{2} = 10, \frac{50}{1} = 50,$ and $\frac{24}{2} = 12;$ the smallest is 10, so 2 in row 1, column 2 is the pivot.

$$\begin{array}{rrrrrr|r} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 4 & \boxed{2} & 1 & 0 & 0 & 0 & 20 \\ 5 & 1 & 0 & 1 & 0 & 0 & 50 \\ 2 & 2 & 0 & 0 & 1 & 0 & 24 \\ \hline -10 & -12 & 0 & 0 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{r} \\ \\ -R_1 + 2R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \\ 6R_1 + R_4 \to R_4 \end{array}\begin{array}{rrrrrr|r} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 4 & 2 & 1 & 0 & 0 & 0 & 20 \\ 6 & 0 & -1 & 2 & 0 & 0 & 80 \\ -2 & 0 & -1 & 0 & 1 & 0 & 4 \\ \hline 14 & 0 & 6 & 0 & 0 & 1 & 120 \end{array}$$

All of the indicators are nonnegative, so we are finished pivoting. Create a 1 in the columns corresponding to x_2 and s_2.

$$\begin{array}{r} \\ \frac{1}{2}R_1 \to R_1 \\ \frac{1}{2}R_2 \to R_2 \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{cccccc|c} 2 & 1 & \frac{1}{2} & 0 & 0 & 0 & 10 \\ 3 & 0 & -\frac{1}{2} & 1 & 0 & 0 & 40 \\ -2 & 0 & -1 & 0 & 1 & 0 & 4 \\ \hline 14 & 0 & 6 & 0 & 0 & 1 & 120 \end{array}\right] \end{array}$$

The maximum value is 120 when $x_1 = 0$, $x_2 = 10$, $s_1 = 0$, $s_2 = 40$, and $s_3 = 4$.

10. Maximize $z = 1.5x_1 + 4.2x_2$

subject to: $2.8x_1 + 3.4x_2 \le 21$
$1.4x_1 + 2.2x_2 \le 11$

with $x_1 \ge 0, x_2 \ge 0$.

The initial tableau follows.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 2.8 & 3.4 & 1 & 0 & 0 & 21 \\ 1.4 & \boxed{2.2} & 0 & 1 & 0 & 11 \\ \hline -1.5 & -4.2 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 2.2 in row 2, column 2.

$$\begin{array}{r} \\ -1.7R_2 + 1.1R_1 \to R_1 \\ \\ 2.1R_2 + 1.1R_3 \to R_3 \end{array}\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 0.7 & 0 & 1.1 & -1.7 & 0 & 4.4 \\ 1.4 & 2.2 & 0 & 1 & 0 & 11 \\ \hline 1.29 & 0 & 0 & 2.1 & 1.1 & 23.1 \end{array}\right] \end{array}$$

All of the indicators are nonnegative. Create a 1 in the columns corresponding to $x_2, s_2,$ and z.

$$\begin{array}{r} \\ \frac{1}{1.1}R_1 \to R_1 \\ \frac{1}{2.2}R_2 \to R_2 \\ \frac{1}{1.1}R_3 \to R_3 \end{array}\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} \frac{7}{11} & 0 & 1 & -\frac{17}{11} & 0 & 4 \\ \frac{7}{11} & 1 & 0 & \frac{5}{11} & 0 & 5 \\ \hline \frac{129}{110} & 0 & 0 & \frac{21}{11} & 1 & 21 \end{array}\right] \end{array}$$

The maximum is 21 when $x_1 = 0$, $x_2 = 5$, $s_1 = 4$, and $s_2 = 0$.

11. Maximize $z = 8x_1 + 3x_2 + x_3$

subject to: $x_1 + 6x_2 + 8x_3 \le 118$
$x_1 + 5x_2 + 10x_3 \le 220$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$.

Two slack variables, s_1 and s_2, need to be introduced. The initial simplex tableau is as follows.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} \boxed{1} & 6 & 8 & 1 & 0 & 0 & 118 \\ 1 & 5 & 10 & 0 & 1 & 0 & 220 \\ \hline -8 & -3 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The most negative indicator is -8, in the first column. The quotients are $\frac{118}{1} = 118$ and $\frac{220}{1} = 220$; since 118 is the smaller, 1 in row 1, column 1 is the pivot. Performing row transformations, we get the following tableau.

$$\begin{array}{r} \\ \\ -R_1 + R_2 \to R_2 \\ 8R_1 + R_3 \to R_3 \end{array}\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 6 & 8 & 1 & 0 & 0 & 118 \\ 0 & -1 & 2 & -1 & 1 & 0 & 102 \\ \hline 0 & 45 & 63 & 8 & 0 & 1 & 944 \end{array}\right] \end{array}$$

All of the indicators are nonnegative, so we are finished pivoting. The maximum value is 944 when $x_1 = 118$, $x_2 = 0$, $x_3 = 0$, $s_1 = 0$, and $s_2 = 102$.

12. Maximize $z = 8x_1 + 10x_2 + 7x_3$

subject to: $x_1 + 3x_2 + 2x_3 \le 10$
$x_1 + 5x_2 + x_3 \le 8$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$.

The initial tableau is as follows.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 3 & 2 & 1 & 0 & 0 & 10 \\ 1 & \boxed{5} & 1 & 0 & 1 & 0 & 8 \\ \hline -8 & -10 & -7 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 5 in row 2, column 2.

$$\begin{array}{r} \\ -3R_2 + 5R_1 \to R_1 \\ \\ 2R_2 + R_3 \to R_3 \end{array}\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 2 & 0 & 7 & 5 & -3 & 0 & 26 \\ \boxed{1} & 5 & 1 & 0 & 1 & 0 & 8 \\ \hline -6 & 0 & -5 & 0 & 2 & 1 & 16 \end{array}\right] \end{array}$$

Pivot on the 1 in row 2, column 1.

$$\begin{array}{r} \\ -2R_2 + R_1 \to R_1 \\ \\ 6R_2 + R_3 \to R_3 \end{array}\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 0 & -10 & 5 & 5 & -5 & 0 & 10 \\ 1 & 5 & 1 & 0 & 1 & 0 & 8 \\ \hline 0 & 30 & 1 & 0 & 8 & 1 & 64 \end{array}\right] \end{array}$$

Create a 1 in the column corresponding to s_1.

$$\begin{array}{r} \\ \frac{1}{5}R_1 \to R_1 \\ \\ \\ \end{array} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline 0 & -2 & 1 & 1 & -1 & 0 & 2 \\ 1 & 5 & 1 & 0 & 1 & 0 & 8 \\ \hline 1 & 30 & 1 & 0 & 8 & 1 & 64 \end{array}$$

The maximum value is 64 when $x_1 = 8$, $x_2 = 0$, $x_3 = 0$, $s_1 = 2$, and $s_2 = 0$.

13. Maximize $z = 10x_1 + 15x_2 + 10x_3 + 5x_4$

subject to:
$$x_1 + x_2 + x_3 + x_4 \le 300$$
$$x_1 + 2x_2 + 3x_3 + x_4 \le 360$$

with $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ x_4 \ge 0.$

The initial tableau is as follows.

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\ \hline 1 & 1 & 1 & 1 & 1 & 0 & 0 & 300 \\ 1 & \boxed{2} & 3 & 1 & 0 & 1 & 0 & 360 \\ \hline -10 & -15 & -10 & -5 & 0 & 0 & 1 & 0 \end{array}$$

In the column with the most negative indicator, -15, the quotients are $\frac{300}{1} = 300$ and $\frac{360}{2} = 180$. The smaller quotient is 180, so the 2 in row 2, column 2, is the pivot.

$$\begin{array}{r} \\ -R_2 + 2R_1 \to R_1 \\ \\ 15R_2 + 2R_3 \to R_3 \end{array} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\ \hline \boxed{1} & 0 & -1 & 1 & 2 & -1 & 0 & 240 \\ 1 & 2 & 3 & 1 & 0 & 1 & 0 & 360 \\ \hline -5 & 0 & 25 & 5 & 0 & 15 & 2 & 5400 \end{array}$$

Pivot on the 1 in row 1, column 1.

$$\begin{array}{r} \\ \\ -R_1 + R_2 \to R_2 \\ 5R_1 + R_3 \to R_3 \end{array} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\ \hline 1 & 0 & -1 & 1 & 2 & -1 & 0 & 240 \\ 0 & 2 & 4 & 0 & -2 & 2 & 0 & 120 \\ \hline 0 & 0 & 20 & 10 & 10 & 10 & 2 & 6600 \end{array}$$

Create a 1 in the columns corresponding to x_2 and z.

$$\begin{array}{r} \\ \\ \frac{1}{2}R_2 \to R_2 \\ \frac{1}{2}R_3 \to R_3 \end{array} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z & \\ \hline 1 & 0 & -1 & 1 & 2 & -1 & 0 & 240 \\ 0 & 1 & 2 & 0 & -1 & 1 & 0 & 60 \\ \hline 0 & 0 & 10 & 5 & 5 & 5 & 1 & 3300 \end{array}$$

The maximum value is 3300 when $x_1 = 240$, $x_2 = 60$, $x_3 = 0$, $x_4 = 0$, $s_1 = 0$, and $s_2 = 0$.

14. Maximize $z = x_1 + x_2 + 4x_3 + 5x_4$

subject to:
$$x_1 + 2x_2 + 3x_3 + x_4 \le 115$$
$$2x_1 + x_2 + 8x_3 + 5x_4 \le 200$$
$$x_1 + x_3 \le 50$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0.$

$$\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ \hline 1 & 2 & 3 & 1 & 1 & 0 & 0 & 0 & 115 \\ 2 & 1 & 8 & \boxed{5} & 0 & 1 & 0 & 0 & 200 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 50 \\ \hline -1 & -1 & -4 & -5 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 5 in row 2, column 4

$$\begin{array}{r} \\ -R_2 + 5R_1 \to R_1 \\ \\ \\ R_2 + R_4 \to R_4 \end{array} \begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ \hline 3 & 9 & 7 & 0 & 5 & -1 & 0 & 0 & 375 \\ 2 & 1 & 8 & 5 & 0 & 1 & 0 & 0 & 200 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 50 \\ \hline 1 & 0 & 4 & 0 & 0 & 1 & 0 & 1 & 200 \end{array}$$

Create a 1 in the columns corresponding to x_4 and s_1.

$$\begin{array}{r} \\ \frac{1}{5}R_1 \to R_1 \\ \frac{1}{5}R_2 \to R_2 \\ \\ \\ \end{array} \begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ \hline \frac{3}{5} & \frac{9}{5} & \frac{7}{5} & 0 & 1 & -\frac{1}{5} & 0 & 0 & 75 \\ \frac{2}{5} & \frac{1}{5} & \frac{8}{5} & 1 & 0 & \frac{1}{5} & 0 & 0 & 40 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 50 \\ \hline 1 & 0 & 4 & 0 & 0 & 1 & 0 & 1 & 200 \end{array}$$

This solution is optimal. The maximum is 200 when $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 40$, $s_1 = 75$, $s_2 = 0$, and $s_3 = 50$.

15. Maximize $z = 4x_1 + 6x_2$

subject to:
$$x_1 - 5x_2 \le 25$$
$$4x_1 - 3x_2 \le 12$$

with $x_1 \ge 0, x_2 \ge 0.$

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline 1 & -5 & 1 & 0 & 0 & 25 \\ 4 & -3 & 0 & 1 & 0 & 12 \\ \hline -4 & -6 & 0 & 0 & 1 & 0 \end{array}$$

The most negative indicator is -6. The negative quotients $25/(-5)$ and $12/(-3)$ indicate an unbounded feasible region, so there is no unique optimum solution.

16. Maximize $z = 2x_1 + 5x_2 + x_3$

subject to:
$$x_1 - 5x_2 + 2x_3 \le 30$$
$$4x_1 - 3x_2 + 6x_3 \le 72$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline 1 & -5 & 2 & 1 & 0 & 0 & 30 \\ 4 & -3 & 6 & 0 & 1 & 0 & 72 \\ \hline -2 & -5 & -1 & 0 & 0 & 1 & 0 \end{array}$$

The most negative indicator is -5. The negative quotients $30/(-5)$ and $72/(-3)$ indicate an unbounded feasible region, so there is no unique optimum solution.

17. Maximize
$z = 37x_1 + 34x_2 + 36x_3 + 30x_4 + 35x_5$

subject to:

$$16x_1 + 19x_2 + 23x_3 + 15x_4 + 21x_5 \le 42{,}000$$
$$15x_1 + 10x_2 + 19x_3 + 23x_4 + 10x_5 \le 25{,}000$$
$$9x_1 + 16x_2 + 14x_3 + 12x_4 + 11x_5 \le 23{,}000$$
$$18x_1 + 20x_2 + 15x_3 + 17x_4 + 19x_5 \le 36{,}000$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0.$

Four slack variables, s_1, s_2, s_3, and s_4, need to be introduced. The initial simplex tableau follows.

$$\begin{array}{cccccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & s_1 & s_2 & s_3 & s_4 & z & \\ \hline 16 & 19 & 23 & 15 & 21 & 1 & 0 & 0 & 0 & 0 & 42{,}000 \\ 15 & 10 & 19 & 23 & 10 & 0 & 1 & 0 & 0 & 0 & 25{,}000 \\ 9 & 16 & 14 & 12 & 11 & 0 & 0 & 1 & 0 & 0 & 23{,}000 \\ 18 & 20 & 15 & 17 & 19 & 0 & 0 & 0 & 1 & 0 & 36{,}000 \\ \hline -37 & -34 & -36 & -30 & -35 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Using a graphing calculator or computer program, the maximum value is found to be 70,818.18 when $x_1 = 181.82$, $x_2 = 0$, $x_3 = 454.55$, $x_4 = 0$, $x_5 = 1363.64$, $s_1 = 0$, $s_2 = 0$, $s_3 = 0$, and $s_4 = 0$.

18. Maximize
$z = 2.0x_1 + 1.7x_2 + 2.1x_3 + 2.4x_4 + 2.2x_5$

subject to:

$$12x_1 + 10x_2 + 11x_3 + 12x_4 + 13x_5 \le 4250$$
$$8x_1 + 8x_2 + 7x_3 + 18x_4 + 5x_5 \le 4130$$
$$9x_1 + 10x_2 + 12x_3 + 11x_4 + 8x_5 \le 3500$$
$$5x_1 + 3x_2 + 4x_3 + 5x_4 + 4x_5 \le 1600$$

with
$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0.$

Four slack variables, s_1, s_2, s_3, and s_4, need to be introduced. The initial simplex tableau follows.

$$\begin{array}{cccccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & s_1 & s_2 & s_3 & s_4 & z & \\ \hline 12 & 10 & 11 & 12 & 13 & 1 & 0 & 0 & 0 & 0 & 4250 \\ 8 & 8 & 7 & 18 & 5 & 0 & 1 & 0 & 0 & 0 & 4130 \\ 9 & 10 & 12 & 11 & 8 & 0 & 0 & 1 & 0 & 0 & 3500 \\ 5 & 3 & 4 & 5 & 4 & 0 & 0 & 0 & 1 & 0 & 1600 \\ \hline -2.0 & -1.7 & -2.1 & -2.4 & -2.2 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Using a graphing calculator or computer program, the maximum value is found to be 795.68 when $x_1 = 0$, $x_2 = 0$, $x_3 = 46.97$, $x_4 = 176.72$, $x_5 = 124.05$, $s_1 = 0$, $s_2 = 0$, $s_3 = 0$, and $s_4 = 32.31$.

23. Organize the information in a table.

	Church Group	Labor Union	Maximum Time Available
Letter Writing	2	2	16
Follow-up	1	3	12
Money Raised	\$100	\$200	

Let x_1 and x_2 be the number of church groups and labor unions contacted respectively. We need two slack variables, s_1 and s_2.

Maximize $z = 100x_1 + 200x_2$

subject to:
$$2x_1 + 2x_2 + s_1 = 16$$
$$x_1 + 3x_2 + s_2 = 12$$

with $x_1 \ge 0, x_2 \ge 0, s_1 \ge 0, s_2 \ge 0.$

The initial simplex tableau is as follows.

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline 2 & 2 & 1 & 0 & 0 & 16 \\ 1 & \boxed{3} & 0 & 1 & 0 & 12 \\ \hline -100 & -200 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 3 in row 2, column 2.

$$\begin{array}{r} \\ -2R_2 + 3R_1 \to R_1 \\ \\ 200R_2 + 3R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} \boxed{4} & 0 & 3 & -2 & 0 & 24 \\ 1 & 3 & 0 & 1 & 0 & 12 \\ \hline -100 & 0 & 0 & 200 & 3 & 2400 \end{array}\right] \end{array}$$

Pivot on the 4 in row 1, column 1.

$$\begin{array}{r} \\ -R_1 + 4R_2 \to R_2 \\ 25R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 4 & 0 & 3 & -2 & 0 & 24 \\ 0 & 12 & -3 & 6 & 0 & 24 \\ \hline 0 & 0 & 75 & 150 & 3 & 3000 \end{array}\right]$$

This is a final tableau, since all of the indicators are nonnegative. Create a 1 in the columns corresponding to $x_1, x_2,$ and z.

$$\begin{array}{r} \frac{1}{4}R_1 \to R_1 \\ \frac{1}{12}R_2 \to R_2 \\ \frac{1}{3}R_3 \to R_3 \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & \frac{3}{4} & -\frac{1}{2} & 0 & 6 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & 0 & 2 \\ \hline 0 & 0 & 25 & 50 & 1 & 1000 \end{array}\right]$$

(columns: x_1, x_2, s_1, s_2, z)

The maximum amount of money raised is \$1000/mo when $x_1 = 6$ and $x_2 = 2$, that is, when 6 churches and 2 labor unions are contacted.

24. **(a)** Let x_1 be the number of Flexscan sets and x_2 be the number of Panoramic I sets. The problem can be stated as follows.

Maximize $z = 350x_1 + 500x_2$

subject to:

$$\begin{aligned} 5x_1 + 7x_2 &\le 3600 \\ x_1 + 2x_2 &\le 900 \\ 4x_1 + 4x_2 &\le 2600 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0.$

Since there are three constraints, introduce slack variables $s_1, s_2,$ and s_3 and set up the initial tableau.

$$\left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 5 & 7 & 1 & 0 & 0 & 0 & 3600 \\ 1 & \boxed{2} & 0 & 1 & 0 & 0 & 900 \\ 4 & 4 & 0 & 0 & 1 & 0 & 2600 \\ \hline -350 & -500 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Pivot on the 2 in row 2, column 2.

$$\begin{array}{r} \\ -7R_2 + 2R_1 \to R_1 \\ \\ -2R_2 + R_3 \to R_3 \\ 250R_2 + R_4 \to R_4 \end{array} \left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ \boxed{3} & 0 & 2 & -7 & 0 & 0 & 900 \\ 1 & 2 & 0 & 1 & 0 & 0 & 900 \\ 2 & 0 & 0 & -2 & 1 & 0 & 800 \\ \hline -100 & 0 & 0 & 250 & 0 & 1 & 225{,}000 \end{array}\right]$$

Pivot on the 3 in row 1, column 1.

$$\begin{array}{r} \\ \\ -R_1 + 3R_2 \to R_2 \\ -2R_1 + 3R_3 \to R_3 \\ 100R_1 + 3R_4 \to R_4 \end{array} \left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 3 & 0 & 2 & -7 & 0 & 0 & 900 \\ 0 & 6 & -2 & 10 & 0 & 0 & 1800 \\ 0 & 0 & -4 & 8 & 3 & 0 & 600 \\ \hline 0 & 0 & 200 & 50 & 0 & 3 & 765{,}000 \end{array}\right]$$

Create a 1 in the columns corresponding to $x_1, x_2,$ and z.

$$\begin{array}{r} \frac{1}{3}R_1 \to R_1 \\ \frac{1}{6}R_2 \to R_2 \\ \\ \frac{1}{3}R_4 \to R_4 \end{array} \left[\begin{array}{cccccc|c} 1 & 0 & \frac{2}{3} & -\frac{7}{3} & 0 & 0 & 300 \\ 0 & 1 & -\frac{2}{3} & \frac{5}{3} & 0 & 0 & 300 \\ 0 & 0 & -4 & 8 & 3 & 0 & 600 \\ \hline 0 & 0 & \frac{200}{3} & \frac{50}{3} & 0 & 1 & 255{,}000 \end{array}\right]$$

(columns: x_1, x_2, s_1, s_2, s_3, z)

The optimal solution is \$255,000 when 300 Flexscan and 300 Panoramic I sets are produced. (This agrees with the graphical solution found in Exercise 10 of Section 3.3.)

(b) Since $5x_1 + 7x_2 + s_1 = 3600$, let $x_1 = 300$ and $x_2 = 300$ and solve for s_1.

$$\begin{aligned} 5(300) + 7(300) + s_1 &= 3600 \\ s_1 &= 0 \end{aligned}$$

Similarly, find s_2 and s_3.

$$\begin{aligned} x_1 + 2x_2 + s_2 &= 900 \\ 300 + 2(300) + s_2 &= 900 \\ s_2 &= 0 \end{aligned}$$

and

$$\begin{aligned} 4x_1 + 4x_2 + s_3 &= 2600 \\ 4(300) + 4(300) + s_3 &= 2600 \\ s_3 &= 200. \end{aligned}$$

There are 200 leftover hours in the testing and packing department.

25. **(a)** Let x_1 be the number of Royal Flush poker sets, x_2 be the number of Deluxe Diamond sets, and x_3 be the number of Full House sets. The problem can be stated as follows.

Maximize $z = 38x_1 + 22x_2 + 12x_3$

subject to:

$$\begin{aligned} 1000x_1 + 600x_2 + 300x_3 &\le 2{,}800{,}000 \\ 4x_1 + 2x_2 + 2x_3 &\le 10{,}000 \\ 10x_1 + 5x_2 + 5x_3 &\le 25{,}000 \\ 2x_1 + x_2 + x_3 &\le 6000 \end{aligned}$$

with $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

Since there are four constraints, introduce slack variables, $s_1, s_2, s_3,$ and s_4 and set up the initial simplex tableau.

$$\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z & \\ 1000 & 600 & 300 & 1 & 0 & 0 & 0 & 0 & 2{,}800{,}000 \\ 4 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 10{,}000 \\ 10 & 5 & 5 & 0 & 0 & 1 & 0 & 0 & 25{,}000 \\ 2 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 6000 \\ \hline -38 & -22 & -12 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Using a graphing calculator or computer program, the maximum profit is \$104,000 and is obtained when 1000 Royal Flush poker sets, 3000 Deluxe Diamond poker sets, and no Full House poker sets are assembled.

(b) According to the poker chip constraint:

$$1000(1000) + 600(3000) + 300(0) + s_2 = 2{,}800{,}000 \quad s_1 = 0.$$

So all of the poker chips are used. Checking the card constraint:

$$4(1000) + 2(3000) + 2(0) + s_2 = 10{,}000$$
$$s_2 = 0.$$

So all of the cards are used. Checking the dice constraint:

$$10(1000) + 5(3000) + 5(0) + s_3 = 25{,}000$$
$$s_3 = 0.$$

So all of the dice are used. Finally, checking the dealer button constraint:

$$2(1000) + 3000 + 0 + s_4 = 6000$$
$$s_4 = 1000.$$

This means there are 1000 unused dealer buttons.

26. **(a)** Let x_1 be the number of loaves of raisin bread and x_2 be the number of raisin cakes.

Then $x_1 + 5x_2 \leq 150$

$x_1 + 2x_2 \leq 90$

and $2x_1 + x_2 \leq 150.$

To maximize $z = 1.75x_1 + 4x_2$, add s_1, s_2, and s_3 as slack variables. The initial tableau will be as follows.

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 1 & \boxed{5} & 1 & 0 & 0 & 0 & 150 \\ 1 & 2 & 0 & 1 & 0 & 0 & 90 \\ 2 & 1 & 0 & 0 & 1 & 0 & 150 \\ \hline -\frac{7}{4} & -4 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 5 in row 1, column 2.

$$\begin{array}{r} \\ \\ -2R_1 + 5R_2 \to R_2 \\ -R_1 + 5R_3 \to R_3 \\ 4R_1 + 5R_4 \to R_4 \end{array} \begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 1 & 5 & 1 & 0 & 0 & 0 & 150 \\ \boxed{3} & 0 & -2 & 5 & 0 & 0 & 150 \\ 9 & 0 & -1 & 0 & 5 & 0 & 600 \\ \hline -\frac{19}{4} & 0 & 4 & 0 & 0 & 5 & 600 \end{array}$$

Pivot on the 3 in row 2, column 1.

$$\begin{array}{r} \\ -R_2 + 3R_1 \to R_1 \\ \\ -3R_2 + R_3 \to R_3 \\ \frac{19}{4}R_2 + 3R_4 \to R_4 \end{array} \begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 0 & 15 & 5 & -5 & 0 & 0 & 300 \\ 3 & 0 & -2 & 5 & 0 & 0 & 150 \\ 0 & 0 & 5 & -15 & 5 & 0 & 150 \\ \hline 0 & 0 & \frac{5}{2} & \frac{95}{4} & 0 & 15 & \frac{5025}{2} \end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, s_3, and z.

$$\begin{array}{r} \\ \frac{1}{15}R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \\ \frac{1}{5}R_3 \to R_3 \\ \frac{1}{15}R_4 \to R_4 \end{array} \begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 20 \\ 1 & 0 & -\frac{2}{3} & \frac{5}{3} & 0 & 0 & 50 \\ 0 & 0 & 1 & -3 & 1 & 0 & 30 \\ \hline 0 & 0 & \frac{1}{6} & \frac{19}{12} & 0 & 1 & \frac{335}{2} \end{array}$$

The optimal solution occurs when $x_1 = 50$ and $x_2 = 20$; that is, when 50 loaves of raisin bread and 20 raisin cakes are baked.

(b) $\frac{335}{2} = 167.5$; the maximum gross income is \$167.50.

(c) When $x_1 = 50$ and $x_2 = 20$, the number of units used are as follows.

Flour: $50 + 5(20) = 150$

This is the total amount of available flour.

Sugar: $50 + 2(20) = 90$

This is the total amount of available sugar.

Raisins: $2(50) + 20 = 120$

This leaves $150 - 120$, or 30 units, of raisins.

Since

$$\begin{aligned} 2x_1 + x_2 + s_3 &= 150, \\ 2(50) + 20 + s_3 &= 150 \\ s_3 &= 30. \end{aligned}$$

27. (a) Let x_1 represent the number of racing bicycles, x_2 the number of touring bicycles, and x_3 the number of mountain bicycles.

From Exercise 28 in Section 4.1, the initial simplex tableau is as follows.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrrr|r} 17 & 27 & \boxed{34} & 1 & 0 & 0 & 91{,}800 \\ 12 & 21 & 15 & 0 & 1 & 0 & 42{,}000 \\ \hline -8 & -12 & -22 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 34 in row 1, column 3.

$$\begin{array}{r} \\ -15R_1 + 34R_2 \to R_2 \\ 11R_1 + 17R_3 \to R_3 \end{array} \left[\begin{array}{rrrrrr|r} 17 & 27 & 34 & 1 & 0 & 0 & 91{,}800 \\ 153 & 309 & 0 & -15 & 34 & 0 & 51{,}000 \\ \hline 51 & 93 & 0 & 11 & 0 & 17 & 1{,}009{,}800 \end{array}\right]$$

(columns: x_1, x_2, x_3, s_1, s_2, z)

This is a final tableau, since all of the indicators are nonnegative. Create a 1 in the columns corresponding to x_3, s_2, and z.

$$\begin{array}{r} \frac{1}{34}R_1 \to R_1 \\ \frac{1}{34}R_2 \to R_2 \\ \frac{1}{17}R_3 \to R_3 \end{array} \left[\begin{array}{rrrrrr|r} \frac{1}{2} & \frac{27}{34} & 1 & \frac{1}{34} & 0 & 0 & 2700 \\ \frac{9}{2} & \frac{309}{34} & 0 & -\frac{15}{34} & 1 & 0 & 1500 \\ \hline 3 & \frac{93}{17} & 0 & \frac{11}{17} & 0 & 1 & 59{,}400 \end{array}\right]$$

(columns: x_1, x_2, x_3, s_1, s_2, z)

From the tableau, $x_1 = 0$, $x_2 = 0$, and $x_3 = 2700$. The company should make no racing or touring bicycles and 2700 mountain bicycles.

(b) From the third row of the final tableau, the maximum profit is \$59,400.

(c) When $x_1 = 0$, $x_2 = 0$, and $x_3 = 2700$, the number of units of steel used is

$$17(0) + 27(0) + 34(2700) = 91{,}800$$

which is all the steel available. The number of units of aluminum used is

$$12(0) + 21(0) + 15(2700) = 40{,}500$$

which leaves $42{,}000 - 40{,}500 = 1500$ units of aluminum unused.

Checking the second constraint:

$$\begin{aligned} 12x_1 + 21x_2 + 15x_3 + s_2 &= 42{,}000 \\ 12(0) + 21(0) + 15(2700) + s_2 &= 42{,}000 \\ s_2 &= 1500. \end{aligned}$$

28. (a) The tableau and set up were explained in Exercise 30 of Section 4.1.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{matrix} \\ \left[\begin{array}{rrrrrrr|r} 2 & 2 & 3 & 1 & 0 & 0 & 0 & 800 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 400 \\ 0 & 1 & \boxed{1} & 0 & 0 & 1 & 0 & 200 \\ \hline -30 & -40 & -60 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 1 in row 3, column 3.

$$\begin{array}{r} -3R_3 + R_1 \to R_1 \\ -R_3 + R_2 \to R_2 \\ \\ 60R_3 + R_4 \to R_4 \end{array} \left[\begin{array}{rrrrrrr|r} \boxed{2} & -1 & 0 & 1 & 0 & -3 & 0 & 200 \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 & 200 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 200 \\ \hline -30 & 20 & 0 & 0 & 0 & 60 & 1 & 12{,}000 \end{array}\right]$$

(columns: x_1, x_2, x_3, s_1, s_2, s_3, z)

Pivot on the 2 in row 1, column 1.

$$\begin{array}{r} \\ -R_1 + 2R_2 \to R_2 \\ \\ 15R_1 + R_4 \to R_4 \end{array} \left[\begin{array}{rrrrrrr|r} 2 & -1 & 0 & 1 & 0 & -3 & 0 & 200 \\ 0 & 1 & 0 & -1 & 2 & 1 & 0 & 200 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 200 \\ \hline 0 & 5 & 0 & 15 & 0 & 15 & 1 & 15{,}000 \end{array}\right]$$

(columns: x_1, x_2, x_3, s_1, s_2, s_3, z)

Create a 1 in the column corresponding to x_1.

$$\frac{1}{2}R_1 \to R_1 \left[\begin{array}{rrrrrrr|r} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{3}{2} & 0 & 100 \\ 0 & 1 & 0 & -1 & 2 & 1 & 0 & 200 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 200 \\ \hline 0 & 5 & 0 & 15 & 0 & 15 & 1 & 15{,}000 \end{array}\right]$$

(columns: x_1, x_2, x_3, s_1, s_2, s_3, z)

The maximum profit is \$15,000 when $x_1 = 100$, $x_2 = 0$, and $x_3 = 200$, that is, when 100 basic sets, no regular sets, and 200 deluxe sets are made.

(b) Even though regular sets make a larger profit, there are only 200 slicers available. Since slicers are used in regular and deluxe sets, and deluxe sets account for \$20 more profit, slicers should be used in deluxe sets (as many as possible) with any leftovers used in regular sets.

29. **(a)** Let x_1 be the number of newspaper ads, x_2 be the number of Internet banner ads, and x_3 be the number of TV ads. Here is the initial tableau:

$$\begin{array}{rrrrrrrr|r}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z & \\
\hline
400 & 20 & \boxed{2000} & 1 & 0 & 0 & 0 & 0 & 8000 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 30 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 60 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 10 \\
\hline
-4000 & -3000 & -10,000 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}$$

Pivot on the 2000 in row 1, column 3.

$$\begin{array}{r|rrrrrrrr|r}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z & \\
\hline
 & 400 & 20 & 2000 & 1 & 0 & 0 & 0 & 0 & 8000 \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 30 \\
 & 0 & \boxed{1} & 0 & 0 & 0 & 1 & 0 & 0 & 60 \\
-R_1 + 2000R_4 \to R_4 & -400 & -20 & 0 & -1 & 0 & 0 & 2000 & 0 & 12,000 \\
\hline
5R_1 + R_5 \to R_5 & -2000 & -2900 & 0 & 5 & 0 & 0 & 0 & 1 & 40,000
\end{array}$$

Pivot on the 1 in row 3, column 2.

$$\begin{array}{r|rrrrrrrr|r}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z & \\
\hline
-20R_3 + R_1 \to R_1 & \boxed{400} & 20 & 2000 & 1 & 0 & -20 & 0 & 0 & 6800 \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 30 \\
 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 60 \\
20R_3 + R_4 \to R_4 & -400 & 0 & 0 & -1 & 0 & 20 & 2000 & 0 & 13,200 \\
\hline
2900R_3 + R_5 \to R_5 & -2000 & 0 & 0 & 5 & 0 & 2900 & 0 & 1 & 214,000
\end{array}$$

Pivot on the 400 in row 1, column 1.

$$\begin{array}{r|rrrrrrrr|r}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z & \\
\hline
 & 400 & 0 & 2000 & 1 & 0 & -20 & 0 & 0 & 6800 \\
-R_1 + 400R_2 \to R_2 & 0 & 0 & -2000 & -1 & 400 & 20 & 0 & 0 & 5200 \\
 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 60 \\
R_1 + R_4 \to R_4 & 0 & 0 & 2000 & 0 & 0 & 0 & 2000 & 0 & 20,000 \\
\hline
5R_1 + R_5 \to R_5 & 0 & 0 & 10,000 & 10 & 0 & 2800 & 0 & 1 & 248,000
\end{array}$$

Create a 1 in the columns corresponding to $x_1, s_2,$ and s_4.

$$\begin{array}{r|rrrrrrrr|r}
 & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z & \\
\hline
\frac{1}{400}R_1 \to R_1 & 1 & 0 & 5 & \frac{1}{400} & 0 & -\frac{1}{20} & 0 & 0 & 17 \\
\frac{1}{400}R_2 \to R_2 & 0 & 0 & -5 & -\frac{1}{400} & 1 & \frac{1}{20} & 0 & 0 & 13 \\
 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 60 \\
\frac{1}{2000}R_4 \to R_4 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 10 \\
\hline
 & 0 & 0 & 10,000 & 10 & 0 & 2800 & 0 & 1 & 248,000
\end{array}$$

This is the final tableau. The maximum exposure is 248,000 women when 17 newspaper ads, 60 Internet banner ads, and no TV ads are used.

30. **(a)** Let x_1 represent the number of toy trucks and x_2 the number of toy fire engines.

Maximize $\quad z = 8.50x_1 + 12.10x_2$

subject to:

$$\begin{aligned} 2x_1 + 3x_2 &\le 24{,}000 \\ x_1 &\le 6600 \\ x_2 &\le 5500 \\ x_1 + x_2 &\le 10{,}000 \end{aligned}$$

The initial tableau:

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\left[\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & z & \\ 2 & 3 & 1 & 0 & 0 & 0 & 0 & 24{,}000 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 10{,}000 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 6600 \\ 0 & \boxed{1} & 0 & 0 & 0 & 1 & 0 & 5500 \\ \hline -850 & -1210 & 0 & 0 & 0 & 0 & 100 & 0 \end{array}\right]$$

We pivot on the indicated 1 in column 2.

$$\begin{array}{r} \\ -3R_4 + R_1 \to R_1 \\ -R_4 + R_2 \to R_2 \\ \\ \\ 1210R_4 + R_5 \to R_5 \end{array}\left[\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & z & \\ \boxed{2} & 0 & 1 & 0 & 0 & -3 & 0 & 7500 \\ 1 & 0 & 0 & 1 & 0 & -1 & 0 & 4500 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 6600 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 5500 \\ \hline -850 & 0 & 0 & 0 & 0 & 1210 & 100 & 6{,}655{,}000 \end{array}\right]$$

Next pivot on the 2 in column 1, row 1.

$$\begin{array}{r} \\ \\ -R_1 + 2R_2 \to R_2 \\ -R_1 + 2R_3 \to R_3 \\ \\ 425R_1 + R_5 \to R_5 \end{array}\left[\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & z & \\ 2 & 0 & 1 & 0 & 0 & -3 & 0 & 7500 \\ 0 & 0 & -1 & 2 & 0 & \boxed{1} & 0 & 1500 \\ 0 & 0 & -1 & 0 & 2 & 3 & 0 & 5700 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 5500 \\ \hline 0 & 0 & 425 & 0 & 0 & -65 & 100 & 9{,}842{,}500 \end{array}\right]$$

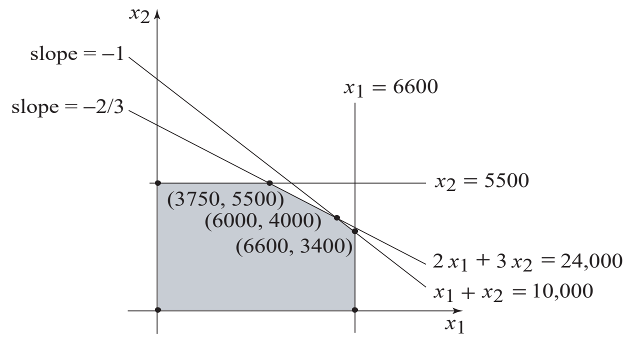
Finally, pivot on the 1 in column 6, row 2.

$$\begin{array}{r} \\ 3R_2 + R_1 \to R_1 \\ \\ -3R_2 + R_3 \to R_3 \\ -R_2 + 1R_4 \to R_4 \\ 65R_2 + R_5 \to R_5 \end{array}\left[\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & z & \\ 2 & 0 & -2 & 6 & 0 & 0 & 0 & 12{,}000 \\ 0 & 0 & -1 & 2 & 0 & 1 & 0 & 1500 \\ 0 & 0 & 2 & -6 & 2 & 0 & 0 & 1200 \\ 0 & 1 & 1 & -2 & 0 & 0 & 0 & 4000 \\ \hline 0 & 0 & 360 & 130 & 0 & 0 & 100 & 9{,}940{,}000 \end{array}\right]$$

From this tableau we can read the solution:

$$x_1 = \frac{12{,}000}{2} = 6000, \; x_2 = 4000, \; z = \frac{9{,}940{,}000}{100} = 99{,}400$$

Produce 6000 trucks and 4000 fire engines for a maximum profit of \$99,400.



The slopes of the two diagonal constraint lines are -1 and $-\frac{2}{3}$.

(b) As we decrease the profit p for five engines from \$12.10, the slope of the objective function line $z = 8.50x_1 + px_2$ moves from $-\frac{8.50}{12.10}$ toward -1. When $p = 8.50$, the slope reaches -1 and now the corner point at $(6600, 3400)$ yields a maximum profit. So, for a fire-engine profit of \$8.50 produce 6600 trucks and 3400 fire engines.

(c) As we increase the profit p for fire engines from \$12.10, the slope of the objective function line $z = 8.50x_1 + px_2$ moves from $-\frac{8.50}{12.10}$ toward $-\frac{2}{3}$. When $p = 12.75$, the slope is $-\frac{8.50}{12.75} = -\frac{2}{3}$ and now the corner point at $(3750, 5500)$ yields a maximum profit. So, for a fire-engine profit of \$12.75 produce 3750 trucks and 5500 fire engines.

31. (a) The coefficients of the objective function are the profit coefficients from the table: 5, 4, and 3; choice (3) is correct.

(b) The constraints are the available man-hours for the 2 departments, 400 and 600; choice (4) is correct.

(c) $2X_1 + 3X_2 + 1X_3 \leq 400$ is the constraint on department 1; choice (3) is correct.

32. (a) Look at the first table, which has to do with the profits. The profit-maximization formula is

$$\$2\text{A} + \$5\text{B} + \$4\text{C} = X,$$

so the answer is choice (1).

(b) Look at the "Painting" row of the second chart. The "Painting" constraint is

$$1\text{A} + 2\text{B} + 2\text{C} \leq 38{,}000,$$

so the answer is choice (3).

33. Maximize $z = 100x + 200y$

subject to: $2x + 2y \leq 16$
$x + 3y \leq 12$

with $x \geq 0, y \geq 0.$

Using Excel, we enter the variables x and y in cells Al and Bl, respectively. Enter the x- and y-coordinates of the initial corner point of the feasible region, (0, 0), in cells A2 and B2, respectively, and NAME these cells x and y, respectively. In cells C2, C4, C5, C6, and C7, enter the formula for the function to maximize and each of the constraints: $100x + 200y$, $2x + 2y$, $x + 3y$, x, and y. Since x and y have been set to 0, all the cells containing formulas should also show the value 0, as below.

	A	B	C
1	x	y	
2	0	0	0
3			
4			0
5			0
6			0
7			0

Using the SOLVER, ask Excel to maximize the value in cell C2 subject to the constraints $C4 \le 16$, $C5 \le 12$, $C6 \ge 0$, $C7 \ge 0$. Make sure you have checked off the box *Assume Linear Model* in SOLVER OPTIONS.

Excel returns the following values and allows you to choose a report.

	A	B	C
1	x	y	
2	6	2	1000
3			
4			16
5			12
6			6
7			2

Select the sensitivity report. The report will appear on a new sheet of the spread sheet.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
A2	*x*	6	0	100	100	33.33333333
B2	*y*	2	0	200	100	100

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C4		16	25	16	8	8
C5		12	50	12	12	4
C6		6	0	0	6	1E + 30
C7		2	0	0	2	1E + 30

The church group's allowable increase is \$100 and the allowable decrease is \$33.33. So their contribution can be as high as $\$100 + \$100 = \$200$ or as low as $\$100 - \$33.33 = \$66.67$ and the original solution is still optimal. The unions' allowable increase is \$100 and the allowable decrease is \$100. So their contribution can be as high as $\$200 + \$100 = \$300$ or as low as $\$200 - \$100 = \$100$ and the original solution is still optimal.

34. Maximize $z = 350x + 500y$

subject to:
$$\begin{aligned} 5x + 7y &\le 3600 \\ x + 2y &\le 900 \\ 4x + 4y &\le 2600 \end{aligned}$$

with $x \ge 0, y \ge 0.$

Using Excel, we enter the variables *x* and *y* in cells A1 and B1, respectively. Enter the *x*- and *y*-coordinates of the initial corner point of the feasible region, (0, 0), in cells A2 and B2, respectively, and NAME these cells *x* and *y,* respectively. In cells C2, C4, C5, C6, C7, and C8, enter the formula for the function to maximize and each of the constraints:

$350x + 500y,\ \ 5x + 7y,\ \ x + 2y, 4x + 4y, x,$ and y. Since x and y have been set to 0, all the cells containing formulas should also show the value 0, as below.

	A	B	C
1	x	y	
2	0	0	0
3			
4			0
5			0
6			0
7			0
8			0

Using the SOLVER, ask Excel to maximize the value in cell C2 subject to the constraints $C4 \le 3600,\ \ C5 \le 900,\ \ C6 \le 3600,\ \ C7 \ge 0,\ \ C8 \ge 0$. Make sure you have checked off the box *Assume Linear Model* in SOLVER OPTIONS.

Excel returns the following values and allows you to choose a report.

	A	B	C
1	x	y	
2	300	300	255000
3			
4			3600
5			900
6			2400
7			300
8			300

Select the sensitivity report. The report will appear on a new sheet of the spread sheet.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$A\$2	x	300	0	350	7.142857143	100
\$B\$2	y	300	0	500	200	10

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$C\$8		3600	66.66666667	3600	150	450
\$C\$4		900	16.66666667	900	128.5714286	75
\$C\$7		2400	0	2600	1E + 30	200
\$C\$5		300	0	0	300	1E + 30
\$C\$6		300	0	0	300	1E + 30

For the Flexscan sets, the allowable increase is \$7.14 and the allowable decrease is \$100. So the profit from the bargain sets can be as high as $\$350 + \$7.14 = \$357.14$ or as low as $\$350 - \$100 = \$250$ and the original solution is still optimal. For the Panoramic I sets, the allowable increase is \$200 and the allowable decrease is \$10. So the profit from the Panoramic I sets can be as high as $\$500 + \$200 = \$700$ or as low as $\$500 - \$10 = \$490$ and the original solution is still optimal.

35. Let $x_1 =$ number of hours running, x_2 be the number of hours biking, and x_3 be the number hours walking. The problem can be stated as follows.

Maximize $\quad z = 531x_1 + 472x_2 + 354x_3$

subject to:
$$\begin{aligned} x_1 + x_2 + x_3 &\le 15 \\ x_1 \qquad\qquad &\le 3 \\ 2x_2 - x_3 &\le 0 \end{aligned}$$

with $\quad x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

We need three slack variables, s_1, s_2, and s_3. The initial simplex tableau as follows.

$$\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 15 \\ \boxed{1} & 0 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 2 & -1 & 0 & 0 & 1 & 0 & 0 \\ \hline -531 & -472 & -354 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Pivot on the 1 in row 2, column 1.

$$\begin{array}{r} \\ -R_2 + R_1 \to R_1 \\ \\ \\ 531R_2 + R_4 \to R_4 \end{array} \left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 & 12 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & \boxed{2} & -1 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & -472 & -354 & 0 & 351 & 0 & 1 & 1593 \end{array}\right]$$

Pivot on the 2 in row 3, column 2.

$$\begin{array}{r} \\ -R_3 + 2R_1 \to R_1 \\ \\ \\ 236R_3 + R_4 \to R_4 \end{array} \left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & \boxed{3} & 2 & -2 & -1 & 0 & 24 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 2 & -1 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -590 & 0 & 531 & 236 & 1 & 1593 \end{array}\right]$$

Finally pivot on the 3 in row 1, column 3.

$$\begin{array}{r} \\ \\ \\ R_1 + 3R_3 \to R_3 \\ 590R_1 + 3R_4 \to R_4 \end{array} \left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 0 & 1 & 3 & 2 & -2 & -1 & 0 & 24 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 6 & 0 & 2 & -2 & 2 & 0 & 24 \\ \hline 0 & 0 & 0 & 1180 & 413 & 118 & 3 & 18{,}939 \end{array}\right]$$

Create a 1 in the columns corresponding to x_2, x_3, and z.

$$\begin{array}{r} \\ \frac{1}{3}R_1 \to R_1 \\ \\ \frac{1}{6}R_3 \to R_3 \\ \frac{1}{3}R_4 \to R_4 \end{array} \left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} & 0 & 8 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 4 \\ \hline 0 & 0 & 0 & \frac{1180}{3} & \frac{413}{3} & \frac{118}{3} & 1 & 6313 \end{array}\right]$$

Lauren should run 3 hours, bike 4 hours, and walk 8 hours for a maximum calorie expenditure of 6313 calories.

36. **(a)** Let x_1 represent the number of hours doing calisthenics, x_2 be the number of hours swimming, and x_3 be the number of hours playing the drums. The problem can be stated as follows.

Maximize $z = 388x_1 + 518x_2 + 345x_3$

subject to:
$$\begin{aligned} x_1 + x_2 + x_3 &\le 10 \\ -x_1 + 2x_2 - x_3 &\le 0 \\ x_3 &\le 4 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

We need three slack variables. The initial simplex tableau is as follows.

$$\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ -1 & \boxed{2} & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 4 \\ \hline -388 & -518 & -345 & 0 & 0 & 0 & 1 & 0 \end{array} \end{array}$$

Pivot on the 2 in row 2, column 2.

$$\begin{array}{r} -R_2 + 2R_1 \to R_1 \\ \\ \\ 259R_2 + R_4 \to R_4 \end{array} \left[\begin{array}{ccccccc|c} \boxed{3} & 0 & 3 & 2 & -1 & 0 & 0 & 20 \\ -1 & 2 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 4 \\ \hline -647 & 0 & -604 & 0 & 259 & 0 & 1 & 0 \end{array}\right]$$

(columns: x_1, x_2, x_3, s_1, s_2, s_3, z)

Pivot on the 3 in row 1, column 1.

$$\begin{array}{r} \\ R_1 + 3R_2 \to R_2 \\ \\ 647R_1 + 3R_4 \to R_4 \end{array} \left[\begin{array}{ccccccc|c} 3 & 0 & 3 & 2 & -1 & 0 & 0 & 20 \\ 0 & 6 & 0 & 2 & 2 & 0 & 0 & 20 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 4 \\ \hline 0 & 0 & 129 & 1294 & 130 & 0 & 3 & 12{,}940 \end{array}\right]$$

(columns: x_1, x_2, x_3, s_1, s_2, s_3, z)

Create a 1 in the columns corresponding to $x_1, x_2,$ and z.

$$\begin{array}{r} \frac{1}{3}R_1 \to R_1 \\ \frac{1}{6}R_2 \to R_2 \\ \\ \frac{1}{3}R_4 \to R_4 \end{array} \left[\begin{array}{ccccccc|c} 1 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & \frac{20}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{10}{3} \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 4 \\ \hline 0 & 0 & 43 & \frac{1294}{3} & \frac{130}{3} & 0 & 1 & \frac{12{,}940}{3} \end{array}\right]$$

(columns: x_1, x_2, x_3, s_1, s_2, s_3, z)

Joe should do $\frac{20}{3}$ hours of calisthenics, $\frac{10}{3}$ hours of swimming, and 0 hours of playing the drums for a maximum calorie expenditure of $\frac{12{,}940}{3}$ or $4313\frac{1}{3}$ calories.

37. **(a)** Let x_1 represent the number of species A, x_2 represent the number of species B, and x_3 represent the number of species C.

Maximize $z = 1.62x_1 + 2.14x_2 + 3.01x_3$

subject to:
$$\begin{aligned} 1.32x_1 + 2.1x_2 + 0.86x_3 &\le 490 \\ 2.9x_1 + 0.95x_2 + 1.52x_3 &\le 897 \\ 1.75x_1 + 0.6x_2 + 2.01x_3 &\le 653 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

Use a graphing calculator or computer to solve this problem and find that the answer is to stock none of species A, 114 of species B, and 291 of species C for a maximum combined weight of 1119.72 kg.

(b) When $x_1 = 0,\ x_2 = 114,$ and $x_3 = 291,$ the number of units used are as follows.

$$\text{Food I: } 1.32(0) + 2.1(114) + 0.86(291) = 489.66$$

or 490 units, which is the total amount available of Food I.

$$\text{Food II: } 2.9(0) + 0.95(114) + 1.52(291) = 550.62$$

or 551 units, which leaves $897 - 551,$ or 346 units of Food II available.

$$\text{Food III: } 1.75(0) + 0.6(114) + 2.01(291) = 653.31$$

or 653 units, which is the total amount available of Food III.

(c) Many answers are possible. The idea is to choose average weights for species B and C that are considerably smaller than the average weight chosen for species A, so that species A dominates the objective function.

(d) Many answers are possible. The idea is to choose average weights for species A and B that are considerably smaller than the average weight chosen for species C.

38. (a) Let $x_1 =$ amount of P, $x_2 =$ amount of Q, $x_3 =$ amount of R, and $x_4 =$ amount of S (all in kilograms).

We desire to maximize

$$z = 90x_1 + 70x_2 + 60x_3 + 50x_4$$

subject to:

$$\begin{aligned} 0.375x_3 + 0.625x_4 &\le 500 \\ 0.75x_2 + 0.5x_3 + 0.375x_4 &\le 600 \\ x_1 + 0.25x_2 + 0.125x_3 \qquad &\le 300 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0.$

If we rewrite the constraints as

$$\begin{aligned} \frac{3}{8}x_3 + \frac{5}{8}x_4 &\le 500 \\ \frac{3}{4}x_2 + \frac{1}{2}x_3 + \frac{3}{8}x_4 &\le 600 \\ x_1 + \frac{1}{4}x_2 + \frac{1}{8}x_3 \qquad &\le 300, \end{aligned}$$

and then multiply each inequality by the least common denominator, 8, we get a set of constraints without fractions.

$$\begin{aligned} 3x_3 + 5x_4 &\le 4000 \\ 6x_2 + 4x_3 + 3x_4 &\le 4800 \\ 8x_1 + 2x_2 + x_3 \qquad &\le 2400 \end{aligned}$$

We need three slack variables. The initial simplex tableau is as follows.

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 3 & 5 & 1 & 0 & 0 & 0 & 4000 \\ 0 & 6 & 4 & 3 & 0 & 1 & 0 & 0 & 4800 \\ \boxed{8} & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 2400 \\ \hline -90 & -70 & -60 & -50 & 0 & 0 & 0 & 1 & 0 \end{array}$$

The first pivot is the 8 in row 3, column 1.

$$\begin{array}{r} \\ \\ \\ \\ 45R_3 + 4R_4 \to R_4 \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 3 & \boxed{5} & 1 & 0 & 0 & 0 & 4000 \\ 0 & 6 & 4 & 3 & 0 & 1 & 0 & 0 & 4800 \\ 8 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 2400 \\ \hline 0 & -190 & -195 & -200 & 0 & 0 & 45 & 4 & 108{,}000 \end{array}$$

Pivot on the 5 in row 1, column 4.

$$\begin{array}{r} \\ \\ -3R_1 + 5R_2 \to R_2 \\ \\ 40R_1 + R_4 \to R_4 \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 3 & 5 & 1 & 0 & 0 & 0 & 4000 \\ 0 & \boxed{30} & 11 & 0 & -3 & 5 & 0 & 0 & 12{,}000 \\ 8 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 2400 \\ \hline 0 & -190 & -75 & 0 & 40 & 0 & 45 & 4 & 268{,}000 \end{array}$$

Pivot on the 30 in row 2, column 2.

$$\begin{array}{r} \\ \\ \\ -R_2 + 15R_3 \to R_3 \\ 19R_2 + 3R_4 \to R_4 \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 3 & 5 & 1 & 0 & 0 & 0 & 4000 \\ 0 & 30 & \boxed{11} & 0 & -3 & 5 & 0 & 0 & 12{,}000 \\ 120 & 0 & 4 & 0 & 3 & -5 & 15 & 0 & 24{,}000 \\ \hline 0 & 0 & -16 & 0 & 63 & 95 & 135 & 12 & 1{,}032{,}000 \end{array}$$

Pivot on the 11 in row 2, column 3.

$$\begin{array}{r} \\ -3R_2 + 11R_1 \to R_1 \\ \\ -4R_2 + 11R_3 \to R_3 \\ 16R_2 + 11R_4 \to R_4 \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ 0 & -90 & 0 & 55 & 20 & -15 & 0 & 0 & 8000 \\ 0 & 30 & 11 & 0 & -3 & 5 & 0 & 0 & 12{,}000 \\ 1320 & -120 & 0 & 0 & 45 & -75 & 165 & 0 & 216{,}000 \\ \hline 0 & 480 & 0 & 0 & 645 & 1125 & 1485 & 132 & 11{,}544{,}000 \end{array}$$

$$\begin{array}{r} \\ \frac{1}{55}R_1 \to R_1 \\ \frac{1}{11}R_2 \to R_2 \\ \frac{1}{1320}R_3 \to R_3 \\ \frac{1}{132}R_4 \to R_4 \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ 0 & -\frac{18}{11} & 0 & 1 & \frac{4}{11} & -\frac{3}{11} & 0 & 0 & \frac{1600}{11} \\ 0 & \frac{30}{11} & 1 & 0 & -\frac{3}{11} & \frac{5}{11} & 0 & 0 & \frac{12{,}000}{11} \\ 1 & -\frac{1}{11} & 0 & 0 & \frac{3}{88} & -\frac{15}{264} & \frac{1}{8} & 0 & \frac{1800}{11} \\ \hline 0 & \frac{40}{11} & 0 & 0 & \frac{215}{44} & \frac{1125}{132} & \frac{45}{4} & 1 & \frac{962{,}000}{11} \end{array}$$

This final tableau gives the solution

$$x_1 = \frac{1800}{11} \approx 163.6,\ x_2 = 0,$$
$$x_3 = \frac{12{,}000}{11} \approx 1090.9,$$
$$x_4 = \frac{1600}{11} \approx 145.5,$$

and $z = \dfrac{962{,}000}{11} \approx 87{,}454.5.$

Produce 163.6 kg of food P, none of food Q, 1090.9 kg of R, and 145.5 kg of S.

(b) The maximum total growth value is read from the bottom row of the final tableau: $\frac{962{,}000}{11} \approx 87{,}454.5.$

(c) When $x_1 = \frac{1800}{11}$, $x_2 = 0$, $x_3 = \frac{12{,}000}{11}$, and $x_4 = \frac{1600}{11}$, the number of units of nutrient A used is

$$0.375\left(\frac{12{,}000}{11}\right) + 0.625\left(\frac{1600}{11}\right) = 500$$

which is the total amount of nutrient A available. The number of units of nutrient B used is

$$0.75(0) + 0.5\left(\frac{12{,}000}{11}\right) + 0.375\left(\frac{1600}{11}\right) = 600$$

which is all the units of nutrient B. The amount of nutrient C used is

$$\left(\frac{1800}{11}\right) + 0.25(0) + 0.125\left(\frac{12{,}000}{11}\right) = 300$$

which is all of the nutrient C. So none of the nutrients are left over.

39. Let x_1 represent the number of minutes for the senator, x_2 the number of minutes for the congresswoman, and x_3 the number of minutes for the governor.

Of the half-hour show's time, at most only $30 - 3 = 27$ min are available to be allotted to the politicians. The given information leads to the inequality

$$x_1 + x_2 + x_3 \le 27$$

and the inequalities

$$x_1 \ge 2x_3 \quad \text{and} \quad x_1 + x_3 \ge 2x_2,$$

and we are to maximize the objective function

$$z = 35x_1 + 40x_2 + 45x_3.$$

Rewrite the equation as

$$x_3 \le 27 - x_1 - x_2$$

and the inequalities as

$$-x_1 + 2x_3 \le 0 \quad \text{and} \quad -x_1 + 2x_2 - x_3 \le 0.$$

Substitute $27 - x_1 - x_2$ for x_3 in the objective function and the inequalities, and the problem is as follows.

Maximize $z = 35x_1 + 40x_2 + 45x_3$

subject to:
$$\begin{aligned} -x_1 \quad\quad\ + 2x_3 &\le 0 \\ -x_1 + 2x_2 - x_3 &\le 0 \\ x_1 + x_2 + x_3 &\le 27 \end{aligned}$$

with $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$

We need three slack variables. The initial simplex tableau is as follows.

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ -1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & \boxed{2} & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 27 \\ \hline -35 & -40 & -45 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 2 in row 2, column 2.

$$\begin{array}{r} \\ \\ \\ -R_2 + 2R_3 \to R_3 \\ 20R_2 + R_4 \to R_4 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ -1 & 0 & \boxed{2} & 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & -1 & 2 & 0 & 54 \\ \hline -55 & 0 & -65 & 0 & 20 & 0 & 1 & 0 \end{array}$$

Pivot on the 2 in row 1, column 3.

$$\begin{array}{r} \\ \\ R_1 + 2R_2 \to R_2 \\ -3R_1 + 2R_3 \to R_3 \\ 65R_1 + 2R_4 \to R_4 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ -1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ -3 & 4 & 0 & 1 & 2 & 0 & 0 & 0 \\ \boxed{9} & 0 & 0 & -3 & -2 & 4 & 0 & 108 \\ \hline -175 & 0 & 0 & 65 & 40 & 0 & 2 & 0 \end{array}$$

Pivot on the 9 in row 3, column 1.

$$\begin{array}{r} \\ R_3 + 9R_1 \to R_1 \\ R_3 + 3R_2 \to R_2 \\ \\ 175R_3 + 9R_4 \to R_4 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 18 & 6 & -2 & 4 & 0 & 108 \\ 0 & 12 & 0 & 0 & 4 & 4 & 0 & 108 \\ 9 & 0 & 0 & -3 & -2 & 4 & 0 & 108 \\ \hline 0 & 0 & 0 & 60 & 10 & 700 & 18 & 18{,}900 \end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, x_3, and z.

$$\begin{array}{r} \\ \frac{1}{18}R_1 \to R_1 \\ \frac{1}{12}R_2 \to R_2 \\ \frac{1}{9}R_3 \to R_3 \\ \frac{1}{18}R_4 \to R_4 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{9} & \frac{2}{9} & 0 & 6 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 9 \\ 1 & 0 & 0 & -\frac{1}{3} & -\frac{2}{9} & \frac{4}{9} & 0 & 12 \\ \hline 0 & 0 & 0 & \frac{10}{3} & \frac{5}{9} & \frac{350}{9} & 1 & 1050 \end{array}$$

The maximum value of z is 1050 when $x_1 = 12$, $x_2 = 9$, and $x_3 = 6$. That is, for a maximum of 1,050,000 viewers, the time allotments should be 12 minutes for the senator, 9 minutes for the congresswoman, and 6 minutes for the governor.

40. **(a)** Let x_1 = number of large fund-raising parties,
x_2 = number of letters requesting funds,
and x_3 = number of dinner parties.

Maximize $z = 200{,}000x_1 + 100{,}000x_2 + 600{,}000x_3$

subject to:
$$\begin{aligned} x_1 + x_2 + x_3 &\le 25 \\ 3000x_1 + 1000x_2 + 12{,}000x_3 &\le 102{,}000 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

We need two slack variables. The initial simplex tableau is as follows.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 25 \\ 3000 & 1000 & \boxed{12{,}000} & 0 & 1 & 0 & 102{,}000 \\ \hline -200{,}000 & -100{,}000 & -600{,}000 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 12,000 in row 2, column 3.

$$\begin{array}{r} -R_2 + 12{,}000R_1 \to R_1 \\ \\ 50R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{cccccc|c} \boxed{9000} & 11{,}000 & 0 & 12{,}000 & -1 & 0 & 198{,}000 \\ 3000 & 1000 & 12{,}000 & 0 & 1 & 0 & 102{,}000 \\ \hline -50{,}000 & -50{,}000 & 0 & 0 & 50 & 1 & 5{,}100{,}000 \end{array}\right]$$

Pivot on 9000 in row 1 column 1.

$$\begin{array}{r} \\ -R_1 + 3R_2 \to R_2 \\ 50R_1 + 9R_3 \to R_3 \end{array} \left[\begin{array}{cccccc|c} 9000 & 11{,}000 & 0 & 12{,}000 & -1 & 0 & 198{,}000 \\ 0 & -8000 & 36{,}000 & -12{,}000 & 4 & 0 & 108{,}000 \\ \hline 0 & 100{,}000 & 0 & 600{,}000 & 400 & 9 & 55{,}800{,}000 \end{array}\right]$$

Create a 1 in the columns corresponding to x_1, x_3, and z.

$$\begin{array}{r} \frac{1}{9000}R_1 \to R_1 \\ \frac{1}{36{,}000}R_2 \to R_2 \\ \frac{1}{9}R_3 \to R_3 \end{array} \left[\begin{array}{cccccc|c} 1 & \frac{11}{9} & 0 & \frac{4}{3} & -\frac{1}{9000} & 0 & 22 \\ 0 & -\frac{2}{9} & 1 & -\frac{1}{3} & \frac{1}{9000} & 0 & 3 \\ \hline 0 & \frac{100{,}000}{9} & 0 & \frac{200{,}000}{3} & \frac{400}{9} & 1 & 6{,}200{,}000 \end{array}\right]$$

The maximum amount of money is \$6,200,000 when $x_1 = 22$, $x_2 = 0$, and $x_3 = 3$, that is, when 22 fund-raising parties, no mailings, and 3 dinner parties are planned.

4.3 Minimization Problems; Duality

Your Turn 1

Write the augmented matrix.

$$\left[\begin{array}{ccc|c} 3 & 3 & 4 & 24 \\ 5 & 1 & 3 & 27 \\ \hline 25 & 12 & 27 & 0 \end{array}\right]$$

Transpose to get the matrix for the dual problem.

$$\left[\begin{array}{cc|c} 3 & 5 & 25 \\ 3 & 1 & 12 \\ 4 & 3 & 27 \\ \hline 24 & 27 & 0 \end{array}\right]$$

Write the dual problem.

Maximize $z = 24x_1 + 27x_2$

subject to: $3x_1 + 5x_2 \le 25$

$3x_1 + x_2 \le 12$

$4x_1 + 3x_2 \le 27$

with $x_1 \ge 0,\ x_2 \ge 0$

Your Turn 2

Write the augmented matrix.

$$\left[\begin{array}{cc|c} 3 & 5 & 20 \\ 3 & 1 & 18 \\ \hline 15 & 12 & 0 \end{array}\right]$$

Transpose to get the matrix for the dual problem.

$$\left[\begin{array}{cc|c} 3 & 3 & 15 \\ 5 & 1 & 12 \\ \hline 20 & 18 & 0 \end{array}\right]$$

Write the dual problem.

Maximize $z = 20x_1 + 18x_2$

subject to: $3x_1 + 3x_2 \le 15$

$5x_1 + x_2 \le 12$

with $x_1 \ge 0,\ x_2 \ge 0.$

The initial tableau for this problem is

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{l} \quad x_1 \quad x_2 \quad s_1 \quad s_2 \quad z \end{array}$$
$$\left[\begin{array}{ccccc|c} 3 & 3 & 1 & 0 & 0 & 15 \\ \boxed{5} & 1 & 0 & 1 & 0 & 12 \\ \hline -20 & -18 & 0 & 0 & 1 & 0 \end{array}\right]$$

Pivot around the indicated 5.

$$\begin{array}{r} -3R_2 + 5R_1 \to R_1 \\ \\ 4R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 0 & \boxed{12} & 5 & -3 & 0 & 39 \\ 5 & 1 & 0 & 1 & 0 & 12 \\ \hline 0 & -14 & 0 & 4 & 1 & 48 \end{array}\right]$$

Now pivot around the indicated 12:

$$\begin{array}{r} \\ -R_1 + 12R_2 \to R_2 \\ 7R_1 + 6R_3 \to R_3 \end{array} \left[\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 0 & 12 & 5 & -3 & 0 & 39 \\ 60 & 0 & -5 & 15 & 0 & 105 \\ \hline 0 & 0 & 35 & 3 & 6 & 561 \end{array}\right]$$

Finally divide the last row by 6 to produce a 1 in the z column:

$$\begin{array}{r} \\ \\ R_3/6 \to R_3 \end{array} \left[\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 0 & 12 & 5 & -3 & 0 & 39 \\ 60 & 0 & -5 & 15 & 0 & 105 \\ \hline 0 & 0 & \frac{35}{6} & \frac{1}{2} & 1 & \frac{187}{2} \end{array}\right]$$

In the original problem, w has a minimum of $\frac{187}{2}$ when $y_1 = \frac{35}{6}$ and $y_2 = \frac{1}{2}$.

4.3 Warmup Exercises

W1. Maximize $z = 5x_1 + 3x_2$

subject to: $5x_1 + 2x_2 \le 20$

$3x_1 + 6x_2 \le 18$

$x_1 \ge 0$

$x_2 \ge 0$

Introduce slack variables to change the constraints into equalities.

$5x_1 + 2x_2 + s_1 = 20$

$3x_1 + 6x_2 + s_2 = 18$

Write the objective function with all variables on the left side of the equals sign.

$z - 5x_1 - 3x_2 = 0$

The intial simplex tableau is

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 5 & 2 & 1 & 0 & 0 & 20 \\ 3 & 6 & 0 & 1 & 0 & 18 \\ \hline -5 & -3 & 0 & 0 & 1 & 0 \end{array}$$

The most negative element in the last rwo is in column 1, and the corresponding quotients are

$\frac{20}{5} = 4$ and $\frac{18}{3} = 6$

Since $4 < 6$, we pivot on the 5 in the upper left corner.

	x_1	x_2	s_1	s_2	z	
	5	2	1	0	0	20
$-3R_1 + 5R_2 \to R_2$	0	24	−3	5	0	30
$R_1 + R_3 \to R_3$	0	−1	1	0	1	20

A negative element remains in column 2 of the last row. The corresponding quotients are

$\frac{20}{2} = 10$ and $\frac{30}{24} = 1.25$

Since $1.25 < 10$, we now pivot on the 24 in column 2.

	x_1	x_2	s_1	s_2	z	
$(-1)R_2 + 12R_1 \to R_1$	60	0	15	−5	0	210
	0	24	−3	5	0	30
$R_2 + 24R_3 \to R_3$	0	0	21	5	24	510

The last row has no negative elements, so we can now read the solution.

$x_1 = \frac{210}{60} = 3.5 \quad x_2 = \frac{30}{24} = 1.25 \quad z = \frac{510}{24} = 21.25$

A maximum of 21.25 is obtained at $x_1 = 3.5$ and $x_2 = 1.25$.

W2. Maximize $z = 3x_1 + 2x_2$

subject to:

$$2x_1 + 3x_2 \le 18$$
$$4x_1 + 3x_2 \le 24$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$

Introduce slack variables to change the constraints into equalities.

$$2x_1 + 3x_2 + s_1 = 18$$
$$4x_1 + 3x_2 + s_2 = 24$$

Write the objective function with all variables on the left side of the equals sign.

$$z - 3x_1 - 2x_2 = 0$$

The initial simplex tableau is

x_1	x_2	s_1	s_2	z	
2	3	1	0	0	18
4	3	0	1	0	24
−3	−2	0	0	1	0

The most negative element in the last row is in column 1, and the corresponding quotients are

$\frac{18}{2} = 9$ and $\frac{24}{4} = 6$

Since $6 < 9$, we pivot on the 4 in column 1.

	x_1	x_2	s_1	s_2	z	
$(-1)R_2 + 2R_1 \to R_1$	0	3	2	−1	0	12
	4	3	0	1	0	24
$3R_2 + 4R_3 \to R_3$	0	1	0	3	4	72

The last row has no negative elements, do we can now read the solution.

$x_1 = \frac{24}{4} = 6 \quad x_2 = 0 \quad z = \frac{72}{4} = 18$

A maximum of 18 is obtained at $x_1 = 6$ and $x_2 = 0$.

4.3 Exercises

1. To form the transpose of a matrix, the rows of the original matrix are written as the columns of the transpose. The transpose of

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 10 & 0 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 10 \\ 3 & 1 & 0 \end{bmatrix}.$$

2. The transpose of a matrix is found by exchanging the rows and columns. The transpose of

$$\begin{bmatrix} 3 & 4 & -2 & 0 & 1 \\ 2 & 0 & 11 & 5 & 7 \end{bmatrix}$$

is

$$\begin{bmatrix} 3 & 2 \\ 4 & 0 \\ -2 & 11 \\ 0 & 5 \\ 1 & 7 \end{bmatrix}.$$

3. The transpose of

$$\begin{bmatrix} 4 & 5 & -3 & 15 \\ 7 & 14 & 20 & -8 \\ 5 & 0 & -2 & 23 \end{bmatrix}$$

is

$$\begin{bmatrix} 4 & 7 & 5 \\ 5 & 14 & 0 \\ -3 & 20 & -2 \\ 15 & -8 & 23 \end{bmatrix}.$$

4. The transpose of

$$\begin{bmatrix} 1 & 11 & 15 \\ 0 & 10 & -6 \\ 4 & 12 & -2 \\ 1 & -1 & 13 \\ 2 & 25 & -1 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 4 & 1 & 2 \\ 11 & 10 & 12 & -1 & 25 \\ 15 & -6 & -2 & 13 & -1 \end{bmatrix}.$$

5. Maximize $z = 4x_1 + 3x_2 + 2x_3$

subject to:
$$\begin{aligned} x_1 + x_2 + x_3 &\le 5 \\ x_1 + x_2 &\le 4 \\ 2x_1 + x_2 + 3x_3 &\le 15 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

To form the dual, first write the augmented matrix for the given problem.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 1 & 0 & 4 \\ 2 & 1 & 3 & 15 \\ \hline 4 & 3 & 2 & 0 \end{array}\right]$$

Then form the transpose of this matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 3 \\ 1 & 0 & 3 & 2 \\ \hline 5 & 4 & 15 & 0 \end{array}\right]$$

The dual problem is stated from this second matrix (using y instead of x).

Minimize $w = 5y_1 + 4y_2 + 15y_3$

subject to:
$$\begin{aligned} y_1 + y_2 + 2y_3 &\ge 4 \\ y_1 + y_2 + y_3 &\ge 3 \\ y_1 + 3y_3 &\ge 2 \end{aligned}$$

with $y_1 \ge 0, y_2 \ge 0, y_3 \ge 0.$

6. Maximize $z = 2x_1 + 7x_2 + 4x_3$

subject to:
$$\begin{aligned} 4x_1 + 2x_2 + x_3 &\le 26 \\ x_1 + 7x_2 + 8x_3 &\le 33 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

To find the dual, first write the augmented matrix for the problem.

$$\left[\begin{array}{ccc|c} 4 & 2 & 1 & 26 \\ 1 & 7 & 8 & 33 \\ \hline 2 & 7 & 4 & 0 \end{array}\right]$$

Then form the transpose of this matrix.

$$\left[\begin{array}{cc|c} 4 & 1 & 2 \\ 2 & 7 & 7 \\ 1 & 8 & 4 \\ \hline 26 & 33 & 0 \end{array}\right]$$

The dual problem is:

Minimize $w = 26y_1 + 33y_2$

subject to:
$$\begin{aligned} 4y_1 + y_2 &\ge 2 \\ 2y_1 + 7y_2 &\ge 7 \\ y_1 + 8y_2 &\ge 4 \end{aligned}$$

with $y_1 \ge 0,\ y_2 \ge 0.$

7. Minimize $w = 3y_1 + 6y_2 + 4y_3 + y_4$

subject to:
$$\begin{aligned} y_1 + y_2 + y_3 + y_4 &\ge 150 \\ 2y_1 + 2y_2 + 3y_3 + 4y_4 &\ge 275 \end{aligned}$$

with $y_1 \ge 0, y_2 \ge 0, y_3 \ge 0, y_4 \ge 0.$

To find the dual problem, first write the augmented matrix for the problem.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 150 \\ 2 & 2 & 3 & 4 & 275 \\ \hline 3 & 6 & 4 & 1 & 0 \end{array}\right]$$

Then form the transpose of this matrix.

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 2 & 6 \\ 1 & 3 & 4 \\ 1 & 4 & 1 \\ \hline 150 & 275 & 0 \end{array}\right]$$

The dual problem is

Maximize $z = 150x_1 + 275x_2$

subject to: $x_1 + 2x_2 \le 3$
$x_1 + 2x_2 \le 6$
$x_1 + 3x_2 \le 4$
$x_1 + 4x_2 \le 1$

with $x_1 \ge 0,\ x_2 \ge 0.$

8. Minimize $w = y_1 + y_2 + 4y_3$

subject to:
$$\begin{aligned} y_1 + 2y_2 + 3y_3 &\ge 115 \\ 2y_1 + y_2 + 8y_3 &\ge 200 \\ y_1 \qquad + y_3 &\ge 50 \end{aligned}$$

with $y_1 \ge 0, y_2 \ge 0, y_3 \ge 0.$

Write the augmented matrix for the problem.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 115 \\ 2 & 1 & 8 & 200 \\ 1 & 0 & 1 & 50 \\ \hline 1 & 1 & 4 & 0 \end{array}\right]$$

Form the transpose of this matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 8 & 1 & 4 \\ \hline 115 & 200 & 50 & 0 \end{array}\right]$$

The dual problem is:

Maximize $z = 115x_1 + 200x_2 + 50x_3$

subject to:
$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 1 \\ 2x_1 + x_2 \qquad &\le 1 \\ 3x_1 + 8x_2 + x_3 &\le 4 \end{aligned}$$

with $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$

9. Find $y_1 \ge 0$ and $y_2 \ge 0$ such that

$$\begin{aligned} 2y_1 + 3y_2 &\ge 6 \\ 2y_1 + y_2 &\ge 7 \end{aligned}$$

and $w = 5y_1 + 2y_2$ is minimized.

Write the augmented matrix for this problem.

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 2 & 1 & 7 \\ \hline 5 & 2 & 0 \end{array}\right]$$

Form the transpose of this matrix.

$$\left[\begin{array}{cc|c} 2 & 2 & 5 \\ 3 & 1 & 2 \\ \hline 6 & 7 & 0 \end{array}\right]$$

Use this matrix to write the dual problem.

Find $x_1 \ge 0$ and $x_2 \ge 0$ such that

$$\begin{aligned} 2x_1 + 2x_2 &\le 5 \\ 3x_1 + x_2 &\le 2 \end{aligned}$$

and $z = 6x_1 + 7x_2$ is maximized.

Introduce slack variables s_1 and s_2. The initial tableau is as follows.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 2 & 1 & 0 & 0 & 5 \\ 3 & \boxed{1} & 0 & 1 & 0 & 2 \\ \hline -6 & -7 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 1 in row 2, column 2, since that column has the most negative indicator and that row has the smallest nonnegative quotient.

$$\begin{array}{r} \\ -2R_2 + R_1 \to R_1 \\ \\ 7R_2 + R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} -4 & 0 & 1 & -2 & 0 & 1 \\ 3 & 1 & 0 & 1 & 0 & 2 \\ \hline 15 & 0 & 0 & 7 & 1 & 14 \end{array}\right] \end{array}$$

The minimum value of w is the same as the maximum value of z. The minimum value of w is 14 when $y_1 = 0$ and $y_2 = 7$. (Note that the values of y_1 and y_2 are given by the entries in the bottom row of the columns corresponding to the slack variables in the final tableau.)

10. Find $y_1 \ge 0$ and $y_2 \ge 0$ such that

$$\begin{aligned} 2y_1 + 3y_2 &\ge 15 \\ 5y_1 + 6y_2 &\ge 35 \end{aligned}$$

and $w = 2y_1 + 3y_2$ is minimized.

Write the augmented matrix for this problem.

$$\left[\begin{array}{cc|c} 2 & 3 & 15 \\ 5 & 6 & 35 \\ \hline 2 & 3 & 0 \end{array}\right]$$

Form the transpose to get the matrix for the dual problem.

$$\left[\begin{array}{cc|c} 2 & 5 & 2 \\ 3 & 6 & 3 \\ \hline 15 & 35 & 0 \end{array}\right]$$

Use this matrix to write the dual problem:

Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$2x_1 + 5x_2 \leq 2$$
$$3x_1 + 6x_2 \leq 3$$

and $z = 15x_1 + 35x_2$ is maximized.

Introduce slack variables and write the initial tableau.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 2 & \boxed{5} & 1 & 0 & 0 & 2 \\ 3 & 6 & 0 & 1 & 0 & 3 \\ \hline -15 & -35 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 5 in row 1, column 2.

$$\begin{array}{r} \\ \\ -6R_1 + 5R_2 \to R_2 \\ 7R_1 + R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 2 & 5 & 1 & 0 & 0 & 2 \\ \boxed{3} & 0 & -6 & 5 & 0 & 3 \\ \hline -1 & 0 & 7 & 0 & 1 & 14 \end{array}\right] \end{array}$$

Because the quotients in the pivot column are the same, we have a choice for the second pivot. Choose the 3 in row 2, column 1, as the second pivot.

$$\begin{array}{r} \\ -2R_2 + 3R_1 \to R_1 \\ \\ R_2 + 3R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 15 & 15 & -10 & 0 & 0 \\ 3 & 0 & -6 & 5 & 0 & 3 \\ \hline 0 & 0 & 15 & 5 & 3 & 45 \end{array}\right] \end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, and z.

$$\begin{array}{r} \\ \frac{1}{15}R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \\ \frac{1}{3}R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & 1 & -\frac{2}{3} & 0 & 0 \\ 1 & 0 & -2 & \frac{5}{3} & 0 & 1 \\ \hline 0 & 0 & 5 & \frac{5}{3} & 1 & 15 \end{array}\right] \end{array}$$

The minimum is 15 when $y_1 = 5$ and $y_2 = \frac{5}{3}$.

For the second pivot, if the 2 in row 1, column 1, was chosen instead, the minimum would still be 15 but would occur when $y_1 = \frac{15}{2}$ and $y_2 = 0$. So, any point on the line segment between $\left(5, \frac{5}{3}\right)$ and $\left(\frac{15}{2}, 0\right)$ is a solution.

11. Find $y_1 \geq 0$ and $y_2 \geq 0$ such that

$$10y_1 + 5y_2 \geq 100$$
$$20y_1 + 10y_2 \geq 150$$

and $w = 4y_1 + 5y_2$ is minimized.

Write the augmented matrix for this problem.

$$\left[\begin{array}{cc|c} 10 & 5 & 100 \\ 20 & 10 & 150 \\ \hline 4 & 5 & 0 \end{array}\right]$$

Form the transpose of this matrix.

$$\left[\begin{array}{cc|c} 10 & 20 & 4 \\ 5 & 10 & 5 \\ \hline 100 & 150 & 0 \end{array}\right]$$

Write the dual problem from this matrix.

Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$10x_1 + 20x_2 \leq 4$$
$$5x_1 + 10x_2 \leq 5$$

and $z = 100x_1 + 150x_2$ is maximized.

The initial simplex tableau is as follows.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 10 & \boxed{20} & 1 & 0 & 0 & 4 \\ 5 & 10 & 0 & 1 & 0 & 5 \\ \hline -100 & -150 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 20 in row 1, column 2.

$$\begin{array}{r} \\ \\ -R_1 + 2R_1 \to R_2 \\ 15R_1 + 2R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} \boxed{10} & 20 & 1 & 0 & 0 & 4 \\ 0 & 0 & -1 & 2 & 0 & 6 \\ \hline -50 & 0 & 15 & 0 & 2 & 60 \end{array}\right] \end{array}$$

Pivot on the 10 in row 1, column 1.

$$\begin{array}{r} \\ \\ \\ 5R_1 + R_3 \to R_3 \end{array}\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 10 & 20 & 1 & 0 & 0 & 4 \\ 0 & 0 & -1 & 2 & 0 & 6 \\ \hline 0 & 100 & 20 & 0 & 2 & 80 \end{array}\right] \end{array}$$

Create a 1 in the columns corresponding to x_1, s_2, and z.

$$\begin{array}{r} \\ \frac{1}{10}R_1 \to R_1 \\ \frac{1}{2}R_2 \to R_2 \\ \frac{1}{2}R_3 \to R_3 \end{array}\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & \frac{1}{10} & 0 & 0 & \frac{2}{5} \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & 3 \\ \hline 0 & 50 & 10 & 0 & 1 & 40 \end{array}\right] \end{array}$$

The minimum value of w is 40 when $y_1 = 10$ and $y_2 = 0$. (These values of y_1 and y_2 are read from the last row of the columns corresponding to s_1 and s_2 in the final tableau.)

12. Minimize $w = 29y_1 + 10y_2$

subject to: $3y_1 + 2y_2 \geq 2$
$5y_1 + y_2 \geq 3$

with $y_1 \geq 0, y_2 \geq 0.$

Write the augmented matrix for this problem.

$$\left[\begin{array}{cc|c} 3 & 2 & 2 \\ 5 & 1 & 3 \\ \hline 29 & 10 & 0 \end{array}\right]$$

From the transpose to get the matrix for the dual problem.

$$\left[\begin{array}{cc|c} 3 & 5 & 29 \\ 2 & 1 & 10 \\ \hline 2 & 3 & 0 \end{array}\right]$$

Write the dual problem from this matrix:

Maximize $z = 2x_1 + 3x_2$

subject to: $3x_1 + 5x_2 \leq 29$
$2x_1 + x_2 \leq 10$

with $x_1 \geq 0, x_2 \geq 0.$

Write the initial tableau.

$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 3 & \boxed{5} & 1 & 0 & 0 & 29 \\ 2 & 1 & 0 & 1 & 0 & 10 \\ \hline -2 & -3 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 5 in row 1, column 2.

$$\begin{array}{r} \\ \\ -R_1 + 5R_2 \to R_2 \\ 3R_1 + 5R_3 \to R_3 \end{array}\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 3 & 5 & 1 & 0 & 0 & 29 \\ \boxed{7} & 0 & -1 & 5 & 0 & 21 \\ \hline -1 & 0 & 3 & 0 & 5 & 87 \end{array}\right] \end{array}$$

Pivot on the 7 in row 2, column 1.

$$\begin{array}{r} \\ -3R_2 + 7R_1 \to R_1 \\ \\ R_2 + 7R_3 \to R_3 \end{array}\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 35 & 10 & -15 & 0 & 140 \\ 7 & 0 & -1 & 5 & 0 & 21 \\ \hline 0 & 0 & 20 & 5 & 35 & 630 \end{array}\right] \end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, and z.

$$\begin{array}{r} \\ \frac{1}{35}R_1 \to R_1 \\ \frac{1}{7}R_2 \to R_2 \\ \frac{1}{35}R_3 \to R_3 \end{array}\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & \frac{2}{7} & -\frac{3}{7} & 0 & 4 \\ 1 & 0 & -\frac{1}{7} & \frac{5}{7} & 0 & 3 \\ \hline 0 & 0 & \frac{4}{7} & \frac{1}{7} & 1 & 18 \end{array}\right] \end{array}$$

The minimum is 18 when $y_1 = \frac{4}{7}$ and $y_2 = \frac{1}{7}$.

13. Minimize $w = 6y_1 + 10y_2$

subject to: $3y_1 + 5y_2 \geq 15$
$4y_1 + 7y_2 \geq 20$

with $y_1 \geq 0,\ y_2 \geq 0.$

Write the augmented matrix.

$$\left[\begin{array}{cc|c} 3 & 5 & 15 \\ 4 & 7 & 20 \\ \hline 6 & 10 & 0 \end{array}\right]$$

Transpose to get the matrix for the dual problem.

$$\left[\begin{array}{cc|c} 3 & 4 & 6 \\ 5 & 7 & 10 \\ \hline 15 & 20 & 0 \end{array}\right]$$

Write the dual problem.

Maximize $z = 15x_1 + 20x_2$

subject to: $3x_1 + 4x_2 \le 6$

$5x_1 + 7x_2 \le 10$

with $x_1 \ge 0,\ x_2 \ge 0.$

Write the initial tableau for this problem.

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \boxed{3} & 4 & 1 & 0 & 0 & 6 \\ 5 & 7 & 0 & 1 & 0 & 10 \\ \hline -15 & -20 & 0 & 0 & 1 & 0 \end{array}$$

Pivot around the indicated 3 to obtain this final tableau:

$$\begin{array}{r} \\ \\ -5R_1 + 3R_2 \to R_2 \\ 5R_1 + R_3 \to R_3 \end{array}\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 3 & 4 & 1 & 0 & 0 & 6 \\ 0 & 1 & -5 & 3 & 0 & 0 \\ \hline 0 & 0 & 5 & 0 & 1 & 30 \end{array}$$

Instead we could pivot around the 5 in the first row, second column. This produces the following final tableau:

$$\begin{array}{r} \\ -3R_2 + 5R_1 \to R_1 \\ \\ 3R_2 + R_3 \to R_3 \end{array}\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 0 & -1 & 5 & -3 & 0 & 6 \\ 5 & 7 & 0 & 1 & 0 & 10 \\ \hline 0 & 1 & 0 & 3 & 1 & 30 \end{array}$$

In the original problem, w has a minimum of 30 when $y_1 = 5$ and $y_2 = 0$ (reading from the first final tableau) or when $y_1 = 0$ and $y_2 = 3$ (reading from the second final tableau). Any point on the line segment between $(5, 0)$ and $(0, 3)$ also gives the minimum of 30.

14. Minimize $w = 3y_1 + 2y_2$

subject to: $y_1 + 2y_2 \ge 10$

$y_1 + y_2 \ge 8$

$2y_1 + y_2 \ge 12$

with $y_1 \ge 0, y_2 \ge 0.$

Write the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 2 & 10 \\ 1 & 1 & 8 \\ 2 & 1 & 12 \\ \hline 3 & 2 & 0 \end{array}\right]$$

Transpose to get the matrix for the dual problem.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 2 & 1 & 1 & 2 \\ \hline 10 & 8 & 12 & 0 \end{array}\right]$$

Write the dual problem:

Maximize $z = 10x_1 + 8x_2 + 12x_3$

subject to: $x_1 + x_2 + 2x_3 \le 3$

$2x_1 + x_2 + x_3 \le 2$

with $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$

Write the initial tableau.

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 1 & 1 & \boxed{2} & 1 & 0 & 0 & 3 \\ 2 & 1 & 1 & 0 & 1 & 0 & 2 \\ \hline -10 & -8 & -12 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 2 in row 1, column 3.

$$\begin{array}{r} \\ \\ -R_1 + 2R_2 \to R_2 \\ R_3 + 6R_1 \to R_3 \end{array}\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 1 & 1 & 2 & 1 & 0 & 0 & 3 \\ \boxed{3} & 1 & 0 & -1 & 2 & 0 & 1 \\ \hline -4 & -2 & 0 & 6 & 0 & 1 & 18 \end{array}$$

Pivot on the 3 in row 2, column 1.

$$\begin{array}{r} \\ -R_2 + 3R_1 \to R_1 \\ \\ 4R_2 + 3R_3 \to R_3 \end{array}\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 0 & 2 & 6 & 4 & -2 & 0 & 8 \\ 3 & 1 & 0 & -1 & 2 & 0 & 1 \\ \hline 0 & -2 & 0 & 14 & 8 & 3 & 58 \end{array}$$

Pivot on the 1 in row 2, column 2.

$$\begin{array}{r} \\ -2R_2 + R_1 \to R_1 \\ \\ R_3 + 2R_2 \to R_3 \end{array}\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ -6 & 0 & 6 & 6 & -6 & 0 & 6 \\ 3 & 1 & 0 & -1 & 2 & 0 & 1 \\ \hline 6 & 0 & 0 & 12 & 12 & 3 & 60 \end{array}$$

Create a 1 in the columns corresponding to x_2, x_3, and z.

$$\begin{array}{r} \\ \frac{1}{6}R_1 \to R_1 \\ \\ \frac{1}{3}R_3 \to R_3 \end{array}\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ -1 & 0 & 1 & 1 & -1 & 0 & 1 \\ 3 & 1 & 0 & -1 & 2 & 0 & 1 \\ \hline 2 & 0 & 0 & 4 & 4 & 1 & 20 \end{array}$$

This solution is optimal. The minimum is 20 when $y_1 = 4$ and $y_2 = 4$.

15. Minimize $w = 2y_1 + y_2 + 3y_3$

subject to: $y_1 + y_2 + y_3 \geq 100$
$2y_1 + y_2 \geq 50$

with $y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0.$

Write the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 2 & 1 & 0 & 50 \\ \hline 2 & 1 & 3 & 0 \end{array}\right]$$

Form the transpose of this matrix.

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 3 \\ \hline 100 & 50 & 0 \end{array}\right]$$

The dual problem is as follows.

Maximize $z = 100x_1 + 50x_2$

subject to: $x_1 + 2x_2 \leq 2$
$x_1 + x_2 \leq 1$
$x_1 \leq 3$

with $x_1 \geq 0, x_2 \geq 0.$

The initial simplex tableau is as follows.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 2 & 1 & 0 & 0 & 0 & 2 \\ \boxed{1} & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 3 \\ \hline -100 & -50 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 1 in row 2, column 1.

$$\begin{array}{r} \\ -R_2 + R_1 \to R_1 \\ \\ -R_2 + R_3 \to R_3 \\ 100R_2 + R_4 \to R_4 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 & 2 \\ \hline 0 & 50 & 0 & 100 & 0 & 1 & 100 \end{array}\right] \end{array}$$

The minimum value of w is 100 when $y_1 = 0$, $y_2 = 100$, and $y_3 = 0$.

16. Minimize $w = 4y_1 + 7y_2 + 9y_3$

subject to: $2y_1 + 3y_2 + 4y_3 \geq 45$
$y_1 + 5y_2 + 2y_3 \geq 40$

with $y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0.$

Write the augmented matrix for this problem.

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 45 \\ 1 & 5 & 2 & 40 \\ \hline 4 & 7 & 9 & 0 \end{array}\right]$$

Form the transpose of this matrix for the dual problem.

$$\left[\begin{array}{cc|c} 2 & 1 & 4 \\ 3 & 5 & 7 \\ 4 & 2 & 9 \\ \hline 45 & 40 & 0 \end{array}\right]$$

This corresponds to the following dual problem.

Maximize $z = 45x_1 + 40x_2$

subject to: $2x_1 + x_2 \leq 4$
$3x_1 + 5x_2 \leq 7$
$4x_1 + 2x_2 \leq 9$

with $x_1 \geq 0,\ x_2 \geq 0.$

Write the initial tableau.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{cccccc|c} \boxed{2} & 1 & 1 & 0 & 0 & 0 & 4 \\ 3 & 5 & 0 & 1 & 0 & 0 & 7 \\ 4 & 2 & 0 & 0 & 1 & 0 & 9 \\ \hline -45 & -40 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot around the indicated 2.

$$\begin{array}{r} \\ -3R_1 + 2R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \\ 45R_1 + 2R_4 \to R_4 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{cccccc|c} 2 & 1 & 1 & 0 & 0 & 0 & 4 \\ 0 & \boxed{7} & -3 & 2 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 & 1 & 0 & 1 \\ \hline 0 & -35 & 45 & 0 & 0 & 2 & 180 \end{array}\right] \end{array}$$

Now pivot around the indicated 7.

$$\begin{array}{r} -R_2 + 7R_1 \to R_1 \\ \\ \\ 5R_2 + R_4 \to R_4 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{cccccc|c} 14 & 0 & 10 & -2 & 0 & 0 & 26 \\ 0 & 7 & -3 & 2 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 30 & 10 & 0 & 2 & 190 \end{array}\right] \end{array}$$

Now divide the last row by 2.

$$\begin{array}{r} \\ \\ \\ R_4/2 \rightarrow R_4 \end{array} \left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 14 & 0 & 10 & -2 & 0 & 0 & 26 \\ 0 & 7 & -3 & 2 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 15 & 5 & 0 & 1 & 95 \end{array}\right]$$

The minimum is 95 when $y_1 = 15$, $y_2 = 5$, and $y_3 = 0$.

17. Minimize $z = x_1 + 2x_2$

subject to: $-2x_1 + x_2 \geq 1$

$x_1 - 2x_2 \geq 1$

with $x_1 \geq 0$, $x_2 \geq 0$.

A quick sketch of the constraints $-2x_1 + x_2 \geq 1$ and $x_1 - 2x_2 \geq 1$ will verify that the two corresponding half planes do not overlap in the first quadrant of the x_1x_2-plane. Therefore, this problem (P) has no feasible solution. The dual of the given problem is as follows:

Maximize $w = y_1 + y_2$

subject to: $-2y_1 + y_2 \leq 1$

$y_1 - 2y_2 \leq 2$

with $y_1 \geq 0$, $y_2 \geq 0$.

A quick sketch here will verify that there is a feasible region in the y_1y_2-plane, and it is unbounded. Therefore, there is no maximum value of w in this problem (D).

(P) has no feasible solution and the objective function of (D) is unbounded; this is choice (a).

18. Since the constraints in Example 4 allow arbitrarily large y_1 and y_2, the objective function can be made negative and as large in absolute value as desired by choosing a large enough positive value of y_1.

19. **(a)** Let $y_1 =$ the number of units of regular beer and $y_2 =$ the number of units of light beer.

Minimize $w = 32,000y_1 + 50,000y_2$

subject to:

$$\begin{aligned} y_1 \quad\quad\quad\quad\quad\quad\quad &\geq 10 \\ y_2 &\geq 15 \\ y_1 + \quad\quad\quad y_2 &\geq 45 \\ 120,000y_1 + 300,000y_2 &\geq 9,000,000 \\ y_1 + \quad\quad\quad y_2 &\geq 20 \end{aligned}$$

with $y_1 \geq 0$, $y_2 \geq 0$.

Write the augmented matrix for this problem.

$$\left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 15 \\ 1 & 1 & 45 \\ 120,000 & 300,000 & 9,000,000 \\ 1 & 1 & 20 \\ \hline 32,000 & 50,000 & 0 \end{array}\right]$$

Form the transpose of this matrix for the dual problem.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 120,000 & 1 & 32,000 \\ 0 & 1 & 1 & 300,000 & 1 & 50,000 \\ \hline 10 & 15 & 45 & 9,000,000 & 20 & 0 \end{array}\right]$$

The dual problem is

Maximize $z = 10x_1 + 15x_2 + 45x_3 + 9{,}000{,}000x_4 + 20x_5$

subject to: $x_1 + x_3 + 120{,}000x_4 + x_5 \leq 32{,}000$

$x_2 + x_3 + 300{,}000x_4 + x_5 \leq 50{,}000$

with $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0.$

Write the initial simplex tableau.

$$\begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & s_1 & s_2 & z & \\ 1 & 0 & 1 & 120{,}000 & 1 & 1 & 0 & 0 & 32{,}000 \\ 0 & 1 & 1 & \boxed{300{,}000} & 1 & 0 & 1 & 0 & 50{,}000 \\ \hline -10 & -15 & -45 & -9{,}000{,}000 & -20 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 300,000 in row 2, column 3.

$$\begin{array}{r} \\ -2R_2 + 5R_1 \to R_1 \\ \\ 30R_2 + R_3 \to R_3 \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & s_1 & s_2 & z & \\ 5 & -2 & \boxed{3} & 0 & 3 & 5 & -2 & 0 & 60{,}000 \\ 0 & 1 & 1 & 300{,}000 & 1 & 0 & 1 & 0 & 50{,}000 \\ \hline -10 & 15 & -15 & 0 & 10 & 0 & 30 & 1 & 1{,}500{,}000 \end{array}$$

Pivot on the 3 in row 1, column 3.

$$\begin{array}{r} \\ \\ -R_1 + 3R_2 \to R_2 \\ 5R_1 + R_3 \to R_3 \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & s_1 & s_2 & z & \\ 5 & -2 & 3 & 0 & 3 & 5 & -2 & 0 & 60{,}000 \\ -5 & 5 & 0 & 900{,}000 & 0 & -5 & 5 & 0 & 90{,}000 \\ \hline 15 & 5 & 0 & 0 & 25 & 25 & 20 & 1 & 1{,}800{,}000 \end{array}$$

Create a 1 in the columns corresponding to x_3 and x_4.

$$\begin{array}{r} \\ \frac{1}{3}R_1 \to R_1 \\ \frac{1}{900{,}000}R_3 \to R_3 \\ \\ \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & s_1 & s_2 & z & \\ \frac{5}{3} & -\frac{2}{3} & 1 & 0 & 1 & \frac{5}{3} & -\frac{2}{3} & 0 & 20{,}000 \\ -\frac{1}{180{,}000} & \frac{1}{180{,}000} & 0 & 1 & 0 & -\frac{1}{180{,}000} & \frac{1}{180{,}000} & 0 & \frac{1}{10} \\ \hline 15 & 5 & 0 & 0 & 25 & 25 & 20 & 1 & 1{,}800{,}000 \end{array}$$

The minimum value of w is 1,800,000 when $y_1 = 25$ and $y_2 = 20$.

Therefore, 25 units of regular beer and 20 units of light beer should be made for a minimum cost of $1,800,000.

(b) The shadow cost for revenue is $\frac{1}{10}$ dollar or $0.10. An increase in $500,000 in revenue will increase costs to

$$\$1{,}800{,}000 + \$0.10(500{,}000) = \$1{,}850{,}000.$$

20. **(a)** Let $y_1 =$ the number of small test tubes and $y_2 =$ the number of large test tubes.

Minimize $w = 18y_1 + 15y_2$

Subject to:
$$\begin{aligned} y_1 \quad &\geq 900 \\ y_2 &\geq 600 \\ y_1 + y_2 &\geq 2700 \\ y_1 \quad &\geq 2y_2 \end{aligned}$$

with $y_1 \geq 0, y_2 \geq 0.$

The last constraint can be written as

$$y_1 - 2y_2 \geq 0.$$

Write the augmented matrix for this problem.

$$\left[\begin{array}{cc|c} 1 & 0 & 900 \\ 0 & 1 & 600 \\ 1 & 1 & 2700 \\ 1 & -2 & 0 \\ \hline 18 & 15 & 0 \end{array}\right]$$

Transpose to get the matrix for the dual problem.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 18 \\ 0 & 1 & 1 & -2 & 15 \\ \hline 900 & 600 & 2700 & 0 & 0 \end{array}\right]$$

Write the dual problem.

Maximize $z = 900x_1 + 600x_2 + 2700x_3$

Subject to:
$$\begin{aligned} x_1 \quad + x_3 + x_4 &\leq 18 \\ x_2 + x_3 - 2x_4 &\leq 15 \end{aligned}$$

with $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.$

Write the initial simplex tableau.

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 18 \\ 0 & 1 & \boxed{1} & -2 & 0 & 1 & 0 & 15 \\ \hline -900 & -600 & -2700 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 1 in row 2, column 3.

$$\begin{array}{r} \\ -R_2 + R_1 \to R_1 \\ \\ 2700R_2 + R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & -1 & 0 & \boxed{3} & 1 & -1 & 0 & 3 \\ 0 & 1 & 1 & -2 & 0 & 1 & 0 & 15 \\ \hline -900 & 2100 & 0 & -5400 & 0 & 2700 & 1 & 40,500 \end{array}\right] \end{array}$$

Pivot on the 3 in row 1, column 4.

$$\begin{array}{r} \\ \\ 2R_1 + 3R_2 \to R_2 \\ 1800R_1 + R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & -1 & 0 & 3 & 1 & -1 & 0 & 3 \\ 2 & 1 & 3 & 0 & 2 & 1 & 0 & 51 \\ \hline 900 & 300 & 0 & 0 & 1800 & 900 & 1 & 45,900 \end{array}\right] \end{array}$$

Create a 1 in the columns corresponding to x_3 and x_4.

$$\begin{array}{r} \\ \frac{1}{3}R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccccc|c} \frac{1}{3} & -\frac{1}{3} & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & 1 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 17 \\ \hline 900 & 300 & 0 & 0 & 1800 & 900 & 1 & 45,900 \end{array}\right] \end{array}$$

The minimum cost is 45,900¢, or \$459, when 1800 small test tubes and 900 test tubes are ordered.

(b) The shadow cost for the test tubes is \$0.17. An increase in the minimum number of test tubes by $(3000 - 2700) = 300$ will increase the cost to

$$\$459 + \$0.17(300) = \$510.$$

21. **(a)** The initial matrix for the original problem is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 400 & 160 & 280 & 20,000 \\ \hline 120 & 40 & 60 & 0 \end{array}\right].$$

The transposed matrix, for the dual problem, is

$$\left[\begin{array}{cc|c} 1 & 400 & 120 \\ 1 & 160 & 40 \\ 1 & 280 & 60 \\ \hline 100 & 20,000 & 0 \end{array}\right].$$

Minimize $w = 100y_1 + 20,000y_2$

subject to:
$$\begin{aligned} y_1 + 400y_2 &\geq 120 \\ y_1 + 160y_2 &\geq 40 \\ y_1 + 280y_2 &\geq 60 \end{aligned}$$

with $y_1 \geq 0, y_2 \geq 0.$

(b) We apply the simplex algorithm to the original maximization problem. The initial tableau is

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 100 \\ \boxed{400} & 160 & 280 & 0 & 1 & 0 & 20,000 \\ \hline -120 & -40 & -60 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 400 in row 2, column 1.

$$\begin{array}{r} -R_2 + 400R_1 \to R_1 \\ \\ \frac{3}{10}R_2 + R_3 \to R_3 \end{array}
\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\
\left[\begin{array}{cccccc|c} 0 & 240 & 120 & 400 & -1 & 0 & 20{,}000 \\ 400 & 160 & 280 & 0 & 1 & 0 & 20{,}000 \\ \hline 0 & 8 & 24 & 0 & 0.3 & 1 & 6000 \end{array}\right] \end{array}$$

Create a 1 in the columns corresponding to x_1 and s_1.

$$\begin{array}{r} \frac{1}{400}R_1 \to R_1 \\ \frac{1}{400}R_2 \to R_2 \\ \\ \end{array}
\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\
\left[\begin{array}{cccccc|c} 0 & 0.6 & 0.3 & 1 & -\frac{1}{400} & 0 & 50 \\ 1 & 0.4 & 0.7 & 0 & \frac{1}{400} & 0 & 50 \\ \hline 0 & 8 & 24 & 0 & 0.3 & 1 & 6000 \end{array}\right] \end{array}$$

This solution is optimal. A maximum profit of \$6000 is achieved by planting 50 acres of potatoes, 0 acres of corn, and 0 acres of cabbage.

From the dual solution, the shadow cost of acreage is 0 and of capital is $\frac{3}{10}$.

$$\begin{aligned} \text{New profit} &= 6000 + 0(-10) + \left(\frac{3}{10}\right)1000 \\ &= \$6300 \end{aligned}$$

Now calculate the number of acres of each:

$$\begin{aligned} \text{Profit} &= 120P + 40C + 60B \\ 6300 &= 120P + 40(0) + 60(0) \\ P &= 52.5. \end{aligned}$$

The farmer will make a profit of \$6300 by planting 52.5 acres of potatoes and no corn or cabbage.

(c) $$\begin{aligned} \text{New profit} &= 6000 + 0(10) + \left(\frac{3}{10}\right)(-1000) \\ &= \$5700 \end{aligned}$$

Calculate the number of acres of each:

$$\begin{aligned} \text{Profit} &= 120P + 40C + 60B \\ 5700 &= 120P + 40(0) + 60(0) \\ P &= 47.5. \end{aligned}$$

The farmer will make a profit of \$5700 by planting 47.5 acres of potatoes and no corn or cabbage.

22. **(a)** Maximize $x_1 + 1.5x_2 = z$

subject to:
$$\begin{aligned} x_1 + 2x_2 &\le 200 \\ 4x_1 + 3x_2 &\le 600 \\ 0 \le x_2 &\le 90 \end{aligned}$$

with $x_1 \ge 0$.

(b) Write the initial tableau.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\
\left[\begin{array}{cccccc|c} 1 & 2 & 1 & 0 & 0 & 0 & 200 \\ 4 & 3 & 0 & 1 & 0 & 0 & 600 \\ 0 & \boxed{1} & 0 & 0 & 1 & 0 & 90 \\ \hline -1 & -1.5 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 1 in row 3, column 2.

$$\begin{array}{r} -2R_3 + R_1 \to R_1 \\ -3R_3 + R_2 \to R_2 \\ \\ 1.5R_3 + R_4 \to R_4 \end{array}
\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\
\left[\begin{array}{cccccc|c} \boxed{1} & 0 & 1 & 0 & -2 & 0 & 20 \\ 4 & 0 & 0 & 1 & -3 & 0 & 330 \\ 0 & 1 & 0 & 0 & 1 & 0 & 90 \\ \hline -1 & 0 & 0 & 0 & 1.5 & 1 & 135 \end{array}\right] \end{array}$$

Pivot on the 1 in row 1, column 1.

$$\begin{array}{r} \\ -4R_1 + R_2 \to R_2 \\ \\ R_1 + R_4 \to R_4 \end{array}
\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\
\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & -2 & 0 & 20 \\ 0 & 0 & -4 & 1 & \boxed{5} & 0 & 250 \\ 0 & 1 & 0 & 0 & 1 & 0 & 90 \\ \hline 0 & 0 & 1 & 0 & -0.5 & 1 & 155 \end{array}\right] \end{array}$$

Pivot on the 5 in row 2, column 5.

$$\begin{array}{r} \frac{2}{5}R_2 + R_1 \to R_1 \\ \frac{1}{5}R_2 \to R_2 \\ -\frac{1}{5}R_2 + R_3 \to R_3 \\ \frac{1}{10}R_2 + R_4 \to R_4 \end{array}
\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\
\left[\begin{array}{cccccc|c} 1 & 0 & -\frac{3}{5} & \frac{2}{5} & 0 & 0 & 120 \\ 0 & 0 & -\frac{4}{5} & \frac{1}{5} & 1 & 0 & 50 \\ 0 & 1 & \frac{4}{5} & -\frac{1}{5} & 0 & 0 & 40 \\ \hline 0 & 0 & 0.6 & 0.1 & 0 & 1 & 180 \end{array}\right] \end{array}$$

The maximum profit is \$180 when $x_1 = 120$ and $x_2 = 40$, that is, when 120 bears and 40 monkeys are produced.

(c) The corresponding dual problem is as follows:

$$\text{Minimize} \quad w = 200y_1 + 600y_2 + 90y_3$$

$$\begin{aligned}\text{subject to:} \quad y_1 + 4y_2 \quad &\geq 1\\ 2y_1 + 3y_2 + y_3 &\geq 1.5\end{aligned}$$

$$\text{with} \quad y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$$

(d) From the given final tableau, the optimal solution to the dual problem is $y_1 = 0.6$, $y_2 = 0.1$, $y_3 = 0$, and $w = 180$.

(e) The shadow value for felt is 0.6; an increase in supply of 10 units of felt will increase profit to

$$\$180 + 0.6(10) = \$186.$$

(f) The shadow values are 0.1 for stuffing and 0 for trim. If stuffing and trim are each decreased by 10 units, the profit will be

$$\$180 - 0.1(10) - 0(10) = \$179.$$

23. Let $y_1 =$ the number of political interviews conducted

and $y_2 =$ the number of market interviews conducted.

The problem is:

$$\text{Minimize} \quad w = 45y_1 + 55y_2$$

$$\begin{aligned}\text{subject to:} \quad y_1 + \quad y_2 &\geq 8\\ 8y_1 + 10y_2 &\geq 60\\ 6y_1 + 5y_2 &\geq 40\end{aligned}$$

$$\text{with} \quad y_1 \geq 0, y_2 \geq 0.$$

Write the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 1 & 8\\ 8 & 10 & 60\\ 6 & 5 & 40\\ \hline 45 & 55 & 0\end{array}\right]$$

Transpose to get the matrix for the dual problem.

$$\left[\begin{array}{ccc|c} 1 & 8 & 6 & 45\\ 1 & 10 & 5 & 55\\ \hline 8 & 60 & 40 & 0\end{array}\right]$$

Write the dual problem:

$$\text{Maximize} \quad z = 8x_1 + 60x_2 + 40x_3$$

$$\begin{aligned}\text{subject to:} \quad x_1 + 8x_2 + 6x_3 &\leq 45\\ x_1 + 10x_2 + 5x_3 &\leq 55\end{aligned}$$

$$\text{with} \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Write the initial tableau.

$$\begin{array}{c} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \end{array}\\ \left[\begin{array}{cccccc|c} 1 & 8 & 6 & 1 & 0 & 0 & 45\\ 1 & \boxed{10} & 5 & 0 & 1 & 0 & 55\\ \hline -8 & -60 & -40 & 0 & 0 & 1 & 0\end{array}\right]\end{array}$$

Pivot on the 10 in row 2, column 2.

$$\begin{array}{r} -4R_2 + 5R_1 \to R_1\\ \\ 6R_2 + R_3 \to R_3\end{array} \left[\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 1 & 0 & \boxed{10} & 5 & -4 & 0 & 5\\ 1 & 10 & 5 & 0 & 1 & 0 & 55\\ \hline -2 & 0 & -10 & 0 & 6 & 1 & 330\end{array}\right]$$

Pivot on the 10 in row 1, column 3.

$$\begin{array}{r} \\ -R_1 + 2R_2 \to R_2\\ R_1 + R_3 \to R_3\end{array} \left[\begin{array}{cccccc|c} \boxed{1} & 0 & 10 & 5 & -4 & 0 & 5\\ 1 & 20 & 0 & -5 & 6 & 0 & 105\\ \hline -1 & 0 & 0 & 5 & 2 & 1 & 335\end{array}\right]$$

Pivot on the 1 in row 1, column 1.

$$\begin{array}{r} \\ -R_1 + R_2 \to R_2\\ R_1 + R_3 \to R_3\end{array} \left[\begin{array}{cccccc|c} 1 & 0 & 10 & 5 & -4 & 0 & 5\\ 0 & 20 & -10 & -10 & \boxed{10} & 0 & 100\\ \hline 0 & 0 & 10 & 10 & -2 & 1 & 340\end{array}\right]$$

Pivot on the 10 in row 2, column 5.

$$\begin{array}{r} 2R_2 + 5R_1 \to R_1\\ \\ R_2 + 5R_3 \to R_3\end{array} \left[\begin{array}{cccccc|c} 5 & 40 & 30 & 5 & 0 & 0 & 225\\ 0 & 20 & -10 & -10 & 10 & 0 & 100\\ \hline 0 & 20 & 40 & 40 & 0 & 5 & 1800\end{array}\right]$$

Create a 1 in the columns corresponding to x_1, s_2, and z.

$$\begin{array}{r} \frac{1}{5}R_1 \to R_1\\ \frac{1}{10}R_2 \to R_2\\ \frac{1}{5}R_3 \to R_3\end{array} \left[\begin{array}{cccccc|c} 1 & 8 & 6 & 1 & 0 & 0 & 45\\ 0 & 2 & -1 & -1 & 1 & 0 & 10\\ \hline 0 & 4 & 8 & 8 & 0 & 1 & 360\end{array}\right]$$

(columns: x_1, x_2, x_3, s_1, s_2, z)

The minimum time spent is 360 min when $y_1 = 8$ and $y_2 = 0$, that is, when 8 political interviews and no market interviews are done.

24. **(a)** Let $y_1 =$ the number of grams of soybean meal,

$y_2 =$ the number of grams of meat byproducts

and $y_3 =$ the number of grams of grain.

Minimize $\quad w = 8y_1 + 9y_2 + 10y_3$

subject to:
$$\begin{aligned} 2.5y_1 + 4.5y_2 + 5y_3 &\geq 54 \\ 5y_1 + 3y_2 + 10y_3 &\geq 60 \end{aligned}$$

with $\quad y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0.$

Write the augmented matrix for this problem.

$$\left[\begin{array}{ccc|c} 2.5 & 4.5 & 5 & 54 \\ 5 & 3 & 10 & 60 \\ \hline 8 & 9 & 10 & 0 \end{array}\right]$$

Transpose to get the matrix for the dual problem.

$$\left[\begin{array}{cc|c} 2.5 & 5 & 8 \\ 4.5 & 3 & 9 \\ 5 & 10 & 10 \\ \hline 54 & 60 & 0 \end{array}\right]$$

Write the dual problem.

Maximize $\quad z = 54x_1 + 60x_2$

subject to:
$$\begin{aligned} 2.5x_1 + 5x_2 &\leq 8 \\ 4.5x_1 + 3x_2 &\leq 9 \\ 5x_1 + 10x_2 &\leq 10 \end{aligned}$$

with $\quad x_1 \geq 0,\ x_2 \geq 0.$

Write the initial tableau.

$$\begin{array}{c} \\ \\ \\ \\ \end{array}\left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 2.5 & 5 & 1 & 0 & 0 & 0 & 8 \\ 4.5 & 3 & 0 & 1 & 0 & 0 & 9 \\ 5 & \boxed{10} & 0 & 0 & 1 & 0 & 10 \\ \hline -54 & -60 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

To eliminate the decimal entries, multiply rows 1 and 2 by 2.

$$\left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 5 & 10 & 2 & 0 & 0 & 0 & 16 \\ 9 & 6 & 0 & 2 & 0 & 0 & 18 \\ 5 & \boxed{10} & 0 & 0 & 1 & 0 & 10 \\ \hline -54 & -60 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Pivot on the 10 in row 3, column 2.

$$\begin{array}{r} \\ -R_3 + R_1 \to R_1 \\ -3R_3 + 5R_2 \to R_2 \\ \\ 6R_3 + R_4 \to R_4 \end{array}\left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 2 & 0 & -1 & 0 & 6 \\ \boxed{30} & 0 & 0 & 10 & -3 & 0 & 60 \\ 5 & 10 & 0 & 0 & 1 & 0 & 10 \\ \hline -24 & 0 & 0 & 0 & 6 & 1 & 60 \end{array}\right]$$

Pivot on the 30 in row 2, column 1.

$$\begin{array}{r} \\ \\ \\ -R_2 + 6R_3 \to R_3 \\ 4R_2 + 5R_4 \to R_4 \end{array}\left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 2 & 0 & -1 & 0 & 6 \\ 30 & 0 & 0 & 10 & -3 & 0 & 60 \\ 0 & 60 & 0 & -10 & 9 & 0 & 0 \\ \hline 0 & 0 & 0 & 40 & 18 & 5 & 540 \end{array}\right]$$

Create a 1 in the columns representing x_1, x_2, s_1, and z.

$$\begin{array}{r} \\ \frac{1}{2}R_1 \to R_1 \\ \frac{1}{30}R_2 \to R_2 \\ \frac{1}{60}R_3 \to R_3 \\ \frac{1}{5}R_4 \to R_4 \end{array}\left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & 3 \\ 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{10} & 0 & 2 \\ 0 & 1 & 0 & -\frac{1}{6} & \frac{3}{20} & 0 & 0 \\ \hline 0 & 0 & 0 & 8 & 3.6 & 1 & 108 \end{array}\right]$$

The minimum cost is obtained when 0 g of soybean meal, 8g of meat by products, and 3.6 g of grain are used, or 0 g of soybean meal, 0 g of meat by products and 10.8 g of grain are used.

(b) The minimum cost is $1.08.

(c) After the initial pivot, the tableau is

$$\left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 2 & 0 & -1 & 0 & 6 \\ 30 & 0 & 0 & 10 & -3 & 0 & 60 \\ \boxed{5} & 10 & 0 & 0 & 1 & 0 & 10 \\ \hline -24 & 0 & 0 & 0 & 6 & 1 & 60 \end{array}\right]$$

Now pivot on the 5 in row 3, column 1.

$$\begin{array}{r} \\ \\ -6R_3 + R_2 \to R_2 \\ \\ 24R_3 + 5R_4 \to R_4 \end{array}\left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 2 & 0 & -1 & 0 & 6 \\ 0 & -60 & 0 & 10 & -9 & 0 & 0 \\ 5 & 10 & 0 & 0 & 1 & 0 & 10 \\ \hline 0 & 240 & 0 & 0 & 54 & 5 & 540 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ \\ \frac{1}{5}R_4 \to R_4 \end{array} \begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{matrix} \\ \left[\begin{array}{cccccc|c} 0 & 0 & 2 & 0 & -1 & 0 & 6 \\ 0 & -60 & 0 & 10 & -9 & 0 & 0 \\ 5 & 10 & 0 & 0 & 1 & 0 & 10 \\ \hline 0 & 48 & 0 & 0 & 10.8 & 1 & 108 \end{array}\right] \end{array}$$

The minimum cost is \$108 when $y_1 = 0$, $y_2 = 8$ and $y_3 = 10.8$, that is, when 0 grams of soybean meal, 0 grams of meat by-products, and 10.8 grams of grain are mixed.

25. Organize the information in a table.

	Units of Nutrient A (per bag)	Units of Nutrient B (per bag)	Cost (per bag)
Feed 1	1	2	\$3
Feed 2	3	1	\$2
Minimum	7	4	

Let $y_1 =$ the number of bags of feed 1

and $y_2 =$ the number of bags of feed 2.

(a) We want the cost to equal \$7 for 7 units of A and 4 units of B exactly. Therefore, use a system of equations rather than a system of inequalities.

$$\begin{aligned} 3y_1 + 2y_2 &= 7 \\ y_1 + 3y_2 &= 7 \\ 2y_1 + y_2 &= 4 \end{aligned}$$

Use Gauss-Jordan elimination to solve this system of equations.

$$\left[\begin{array}{cc|c} 3 & 2 & 7 \\ 1 & 3 & 7 \\ 2 & 1 & 4 \end{array}\right]$$

$$\begin{array}{r} \\ -R_1 + 3R_2 \to R_2 \\ -2R_1 + 3R_3 \to R_3 \end{array} \left[\begin{array}{cc|c} 3 & 2 & 7 \\ 0 & 7 & 14 \\ 0 & -1 & -2 \end{array}\right]$$

$$\begin{array}{r} -2R_2 + 7R_1 \to R_1 \\ \\ R_2 + 7R_3 \to R_3 \end{array} \left[\begin{array}{cc|c} 21 & 0 & 21 \\ 0 & 7 & 14 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{21}R_1 \to R_1 \\ \frac{1}{7}R_2 \to R_2 \\ \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $y_1 = 1$ and $y_2 = 2$, so use 1 bag of feed 1 and 2 bags of feed 2. The cost will be $3(1) + 2\ (2) = \$7$ as desired. The number of units of A is $1(1) + 3(2) = 7$, and the number of units of B is $2(1) + 1(2) = 4$.

(b)

	Units of Nutrient A (per bag)	Units of Nutrient B (per bag)	Cost (per bag)
Feed 1	1	2	\$3
Feed 2	3	1	\$2
Minimum	5	4	

The problem is:

Minimize $w = 3y_1 + 2y_2$

subject to:
$$\begin{aligned} y_1 + 3y_2 &\ge 5 \\ 2y_1 + y_2 &\ge 4 \end{aligned}$$

with $y_1 \ge 0, y_2 \ge 0$.

The dual problem is as follows.

Maximize $z = 5x_1 + 4x_2$

subject to:
$$\begin{aligned} x_1 + 2x_2 &\le 3 \\ 3x_1 + x_2 &\le 2 \end{aligned}$$

with $x_1 \ge 0,\ x_2 \ge 0$.

The initial tableau is as follows.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 3 \\ \boxed{3} & 1 & 0 & 1 & 0 & 2 \\ \hline -5 & -4 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot as indicated.

$$\begin{array}{r} \\ -R_2 + 3R_1 \to R_1 \\ \\ 5R_2 + 3R_3 \to R_3 \end{array} \begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{ccccc|c} 0 & \boxed{5} & 3 & -1 & 0 & 7 \\ 3 & 1 & 0 & 1 & 0 & 2 \\ \hline 0 & -7 & 0 & 5 & 3 & 10 \end{array}\right] \end{array}$$

$$\begin{array}{r} \\ \\ -R_1 + 5R_2 \to R_2 \\ 7R_1 + 5R_3 \to R_3 \end{array} \begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{ccccc|c} 0 & 5 & 3 & -1 & 0 & 7 \\ 15 & 0 & -3 & 6 & 0 & 3 \\ \hline 0 & 0 & 21 & 18 & 15 & 99 \end{array}\right] \end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, and z.

$$\begin{array}{r} \frac{1}{5}R_1 \to R_1 \\ \frac{1}{15}R_2 \to R_2 \\ \frac{1}{15}R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{7}{5} \\ 1 & 0 & -\frac{1}{5} & \frac{2}{5} & 0 & \frac{1}{5} \\ \hline 0 & 0 & \frac{7}{5} & \frac{6}{5} & 1 & \frac{33}{5} \end{array}\right] \end{array}$$

Reading from the final column of the final tableau, $x_2 = \$1.40$ is the cost of nutrient B and $x_1 = \$0.20$ is the cost of nutrient A. With 5 units of A and 4 units of B, this gives a minimum cost of

$$5(\$0.20) + 4(\$1.40) = \$6.60$$

as given in the lower right corner. 1.4 $\left(\text{or } \frac{7}{5}\right)$ bags of feed 1 and 1.2 $\left(\text{or } \frac{6}{5}\right)$ bags of feed 2 should be used.

26. Let $y_1 =$ the number of large bowls.

$y_2 =$ the number of small bowls.

$y_3 =$ the number of pots for plants.

Minimize $w = 5y_1 + 6y_2 + 4y_3$

subject to: $3y_1 + 2y_2 + 4y_3 \geq 72$

$6y_1 + 6y_2 + 2y_3 \geq 108$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$

Write the augmented matrix for this problem.

$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & 72 \\ 6 & 6 & 2 & 108 \\ \hline 5 & 6 & 4 & 0 \end{array}\right]$$

Transpose to get the matrix for the dual problem.

$$\left[\begin{array}{cc|c} 3 & 6 & 5 \\ 2 & 6 & 6 \\ 4 & 2 & 4 \\ \hline 72 & 108 & 0 \end{array}\right]$$

Write the dual problem.

Maximize $z = 72x_1 + 108x_2$

subject to: $3x_1 + 6x_2 \leq 5$

$2x_1 + 6x_2 \leq 6$

$4x_1 + 2x_2 \leq 4$

with $x_1 \geq 0, \ x_2 \geq 0.$

Write the initial tableau.

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z & \end{array} \\ \left[\begin{array}{cccccc|c} 3 & \boxed{6} & 1 & 0 & 0 & 0 & 5 \\ 2 & 6 & 0 & 1 & 0 & 0 & 6 \\ 4 & 2 & 0 & 0 & 1 & 0 & 4 \\ \hline -72 & -108 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 6 in row 1, column 2.

$$\begin{array}{r} \frac{1}{6}R_1 \to R_1 \\ -R_1 + R_2 \to R_2 \\ -\frac{1}{3}R_1 + R_3 \to R_3 \\ 18R_1 + R_4 \to R_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z & \end{array} \\ \left[\begin{array}{cccccc|c} \frac{1}{2} & 1 & \frac{1}{6} & 0 & 0 & 0 & \frac{5}{6} \\ -1 & 0 & -1 & 1 & 0 & 0 & 1 \\ \boxed{3} & 0 & -\frac{1}{3} & 0 & 1 & 0 & \frac{7}{3} \\ \hline -18 & 0 & 18 & 0 & 0 & 1 & 90 \end{array}\right] \end{array}$$

Pivot on the 3 in row 3, column 1.

$$\begin{array}{r} -\frac{1}{6}R_3 + R_1 \to R_1 \\ \frac{1}{3}R_3 + R_2 \to R_2 \\ \frac{1}{3}R_3 \to R_3 \\ 6R_3 + R_4 \to R_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 1 & \frac{2}{9} & 0 & -\frac{1}{6} & 0 & \frac{4}{9} \\ 0 & 0 & -\frac{10}{9} & 1 & \frac{1}{3} & 0 & \frac{16}{9} \\ 1 & 0 & -\frac{1}{9} & 0 & \frac{1}{3} & 0 & \frac{7}{9} \\ \hline 0 & 0 & 16 & 0 & 6 & 1 & 104 \end{array}\right] \end{array}$$

The minimum time is 104 hours when $y_1 = 16$, $y_2 = 0$, and $y_3 = 6$, that is, when 16 large bowls, 0 small bowls, and 6 pots for flowers are made.

27. Let $y_1 =$ the number of minutes spent walking,

$y_2 =$ the number of minutes spent cycling,

and $y_3 =$ the number of minutes spent swimming.

Minimize $w = y_1 + y_2 + y_3$

subject to: $3.5y_1 + 4y_2 + 8y_3 \geq 1500$

$y_1 + y_2 \geq 3y_3$

$y_1 \geq 30$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$

The second constraint can be written as

$$y_1 + y_2 - 3y_3 \geq 0.$$

Write the augmented matrix for this problem.

$$\left[\begin{array}{ccc|c} 3.5 & 4 & 8 & 1500 \\ 1 & 1 & -3 & 0 \\ 1 & 0 & 0 & 30 \\ \hline 1 & 1 & 1 & 0 \end{array}\right]$$

Transpose to get the matrix for the dual problem.

$$\left[\begin{array}{ccc|c} 3.5 & 1 & 1 & 1 \\ 4 & 1 & 0 & 1 \\ 8 & -3 & 0 & 1 \\ \hline 1500 & 0 & 30 & 0 \end{array}\right]$$

Write the dual problem.

Maximize $z = 1500x_1 + 30x_3$

subject to:
$$\begin{aligned} 3.5x_1 + x_2 + x_3 &\le 1 \\ 4x_1 + x_2 &\le 1 \\ 8x_1 - 3x_2 &\le 1 \end{aligned}$$

with $x_1 \ge 0, s_2 \ge 0, x_3 \ge 0.$

Write the initial simplex tableau.

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 3.5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 4 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 8 & -3 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline -1500 & 0 & -30 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Using a graphing calculator or computer program, such as Solver in Microsoft Excel, we obtain the optimal answer: 30 minutes walking, 197.25 minutes cycling, and 75.75 minutes swimming for a total minimum time of 303 minutes per week.

28. Let $y_1 =$ the number of #1 pills

and $y_2 =$ the number of #2 pills.

Organize the given information in a table.

	Vitamin A	Vitamin B_1	Vitamin C	Cost
#1	1600	1	20	\$0.10
#2	400	1	70	\$0.20
Total Needed	3200	5	200	

The problem is:

Minimize $w = 0.1y_1 + 0.2y_2$

subject to:
$$\begin{aligned} 1600y_1 + 400y_2 &\ge 3200 \\ y_1 + y_2 &\ge 5 \\ 20y_1 + 70y_2 &\ge 200 \end{aligned}$$

with $y_1 \ge 0, y_2 \ge 0.$

Write the augmented matrix for the given problem.

$$\left[\begin{array}{cc|c} 1600 & 400 & 3200 \\ 1 & 1 & 5 \\ 20 & 70 & 200 \\ \hline 0.1 & 0.2 & 0 \end{array}\right]$$

Form the transpose of this matrix for the dual problem.

$$\left[\begin{array}{ccc|c} 1600 & 1 & 20 & 0.1 \\ 400 & 1 & 70 & 0.2 \\ \hline 3200 & 5 & 200 & 0 \end{array}\right]$$

This corresponds to the following dual problem.

Maximize $z = 3200x_1 + 5x_2 + 200x_3$

subject to:
$$\begin{aligned} 1600x_1 + x_2 + 20x_3 &\le 0.1 \\ 400x_1 + x_2 + 70x_3 &\le 0.2 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

Write the initial tableau.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} \boxed{1600} & 1 & 20 & 1 & 0 & 0 & 0.1 \\ 400 & 1 & 70 & 0 & 1 & 0 & 0.2 \\ \hline -3200 & -5 & -200 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on 1600.

$$\begin{array}{r} \\ -R_1 + 4R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{cccccc|c} 1600 & 1 & 20 & 1 & 0 & 0 & 0.1 \\ 0 & 3 & \boxed{260} & -1 & 4 & 0 & 0.7 \\ \hline 0 & -3 & -160 & 2 & 0 & 1 & 0.2 \end{array}\right]$$

Pivot on 260.

$$\begin{array}{r} -R_2 + 13R_1 \to R_1 \\ \\ 8R_2 + 13R_3 \to R_3 \end{array} \left[\begin{array}{cccccc|c} 20{,}800 & \boxed{10} & 0 & 14 & -4 & 0 & 0.6 \\ 0 & 3 & 260 & -1 & 4 & 0 & 0.7 \\ \hline 0 & -15 & 0 & 18 & 32 & 13 & 8.2 \end{array}\right]$$

Pivot on 10.

$$\begin{array}{r} \\ -3R_1 + 10R_2 \to R_2 \\ 3R_1 + 2R_3 \to R_3 \end{array} \left[\begin{array}{cccccc|c} 20{,}800 & 10 & 0 & 14 & -4 & 0 & 0.6 \\ -62{,}400 & 0 & 2600 & -52 & 52 & 0 & 5.2 \\ \hline 62{,}400 & 0 & 0 & 78 & 52 & 26 & 18.2 \end{array}\right]$$

Now divide the last row by 26.

$$\begin{array}{r} \\ \\ \\ R_2/26 \to R_3 \end{array} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrrr|r} 20{,}800 & 10 & 0 & 14 & -4 & 0 & 0.6 \\ -62{,}400 & 0 & 2600 & -52 & 52 & 0 & 5.2 \\ \hline 2400 & 0 & 0 & 3 & 2 & 1 & 0.7 \end{array}\right] \end{array}$$

Greg should buy 3 of pill #1 and 2 of pill #2 for a minimum cost of \$0.70.

29. Let $y_1 =$ the number of units of ingredient I;

$y_2 =$ the number of units of ingredient II;

and $y_3 =$ the number of units of ingredient III.

The problem is:

Minimize $w = 4y_1 + 7y_2 + 5y_3$

subject to:
$$\begin{aligned} 4y_1 + y_2 + 10y_3 &\geq 10 \\ 3y_1 + 2y_2 + y_3 &\geq 12 \\ 4y_2 + 5y_3 &\geq 20 \end{aligned}$$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$

The dual problem is as follows.

Maximize $z = 10x_1 + 12x_2 + 20x_3$

subject to:
$$\begin{aligned} 4x_1 + 3x_2 &\leq 4 \\ x_1 + 2x_2 + 4x_3 &\leq 7 \\ 10x_1 + x_2 + 5x_3 &\leq 5 \end{aligned}$$

with $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

The initial tableau is as follows.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{matrix} \\ \left[\begin{array}{rrrrrrr|r} 4 & 3 & 0 & 1 & 0 & 0 & 0 & 4 \\ 1 & 2 & 4 & 0 & 1 & 0 & 0 & 7 \\ 10 & 1 & \boxed{5} & 0 & 0 & 1 & 0 & 5 \\ \hline -10 & -12 & -20 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot as indicated.

$$\begin{array}{r} \\ \\ -4R_3 + 5R_2 \to R_2 \\ \\ 4R_3 + R_4 \to R_4 \end{array} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{matrix} \\ \left[\begin{array}{rrrrrrr|r} 4 & \boxed{3} & 0 & 1 & 0 & 0 & 0 & 4 \\ -35 & 6 & 0 & 0 & 5 & -4 & 0 & 15 \\ 10 & 1 & 5 & 0 & 0 & 1 & 0 & 5 \\ \hline 30 & -8 & 0 & 0 & 0 & 4 & 1 & 20 \end{array}\right] \end{array}$$

$$\begin{array}{r} \\ \\ -2R_1 + R_2 \to R_2 \\ -R_1 + 3R_3 \to R_3 \\ 8R_1 + 3R_4 \to R_4 \end{array} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{matrix} \\ \left[\begin{array}{rrrrrrr|r} 4 & 3 & 0 & 1 & 0 & 0 & 0 & 4 \\ -43 & 0 & 0 & -2 & 5 & -4 & 0 & 7 \\ 26 & 0 & 15 & -1 & 0 & 3 & 0 & 11 \\ \hline 122 & 0 & 0 & 8 & 0 & 12 & 3 & 92 \end{array}\right] \end{array}$$

Create a 1 in the columns corresponding to x_2, x_3, and z.

$$\begin{array}{r} \frac{1}{3}R_1 \to R_1 \\ \\ \frac{1}{15}R_3 \to R_3 \\ \frac{1}{3}R_4 \to R_4 \end{array} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{matrix} \\ \left[\begin{array}{rrrrrrr|r} \frac{4}{3} & 1 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{4}{3} \\ -43 & 0 & 0 & -2 & 5 & -4 & 0 & 7 \\ \frac{26}{15} & 0 & 1 & -\frac{1}{15} & 0 & \frac{1}{5} & 0 & \frac{11}{5} \\ \hline \frac{122}{3} & 0 & 0 & \frac{8}{3} & 0 & 4 & 1 & \frac{92}{3} \end{array}\right] \end{array}$$

From the last row, the minimum value is $\frac{92}{3}$ when $y_1 = \frac{8}{3}$, $y_2 = 0$, and $y_3 = 4$. The biologist can meet his needs at a minimum cost of \$30.67 by using $\frac{8}{3}$ units of ingredient I and 4 units of ingredient III. (Ingredient II should not be used at all.)

4.4 Nonstandard Problems

Your Turn 1

Minimize $w = 6y_1 + 4y_2$

subject to:
$$\begin{aligned} 3y_1 + 4y_2 &\geq 10 \\ 9y_1 + 7y_2 &\leq 18 \end{aligned}$$

with $y_1 \geq 0, \ y_2 \geq 0.$

Instead we maximize $z = -w = -6y_1 - 4y_2$ subject to the same constraints. Inserting slack and surplus variables produces the following initial tableau.

$$\begin{array}{c} \begin{matrix} y_1 & y_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrr|r} 3 & 4 & -1 & 0 & 0 & 10 \\ \boxed{9} & 7 & 0 & 1 & 0 & 18 \\ \hline 6 & 4 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Because s_1 is negative, we choose the positive entry farthest to the left in row 1, which is the 3 in column 1. The entry 9 in this column gives the smallest quotient so we choose 9 as the pivot.

$$
\begin{array}{r}
-R_2+3R_1 \to R_1 \\
\\
-2R_2+3R_3 \to R_3
\end{array}
\overset{\begin{array}{ccccc} y_1 & y_2 & s_1 & s_2 & z \end{array}}{\left[\begin{array}{ccccc|r}
0 & \boxed{5} & -3 & -1 & 0 & 12 \\
9 & 7 & 0 & 1 & 0 & 18 \\
\hline
0 & -2 & 0 & -2 & 3 & -36
\end{array}\right]}
$$

s_2 is still negative, so we pivot on the 5 in column 2.

$$
\begin{array}{r}
\\
-7R_1+5R_2 \to R_2 \\
2R_1+5R_3 \to R_3
\end{array}
\overset{\begin{array}{ccccc} y_1 & y_2 & s_1 & s_2 & z \end{array}}{\left[\begin{array}{ccccc|r}
0 & 5 & -3 & -1 & 0 & 12 \\
45 & 0 & 21 & \boxed{12} & 0 & 6 \\
\hline
0 & 0 & -6 & -12 & 15 & -156
\end{array}\right]}
$$

Now we work on the largest negative indicator and pivot on the 12 in column 4.

$$
\begin{array}{r}
R_2+12R_1 \to R_1 \\
\\
R_2+R_3 \to R_3
\end{array}
\overset{\begin{array}{ccccc} y_1 & y_2 & s_1 & s_2 & z \end{array}}{\left[\begin{array}{ccccc|r}
45 & 60 & -15 & 0 & 0 & 150 \\
45 & 0 & 21 & 12 & 0 & 6 \\
\hline
45 & 0 & 15 & 0 & 15 & -150
\end{array}\right]}
$$

From this we can read the solution: The minimum is $-\left(\frac{-150}{15}\right) = 10$ when $y_1 = 0$ and $y_2 = \frac{150}{60} = \frac{5}{2}$.

Your Turn 2

We start with this tableau.

$$
\overset{\begin{array}{ccccccccc} y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & s_4 & z \end{array}}{\left[\begin{array}{ccccccccc|r}
0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 16 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 12 \\
0 & 0 & 1 & \boxed{1} & -1 & 0 & -1 & -1 & 0 & 8 \\
\hline
0 & 0 & 0 & -300 & 180 & 0 & 400 & 480 & 1 & -10,640
\end{array}\right]}
$$

We pivot on the 1 in row 4 of column 4.

$$
\begin{array}{r}
-R_4+R_1 \to R_1 \\
\\
R_4+R_3 \to R_3 \\
\\
300R_4+R_5 \to R_5
\end{array}
\overset{\begin{array}{ccccccccc} y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & s_4 & z \end{array}}{\left[\begin{array}{ccccccccc|r}
0 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 8 \\
0 & 0 & 0 & 0 & \boxed{1} & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 20 \\
0 & 0 & 1 & 1 & -1 & 0 & -1 & -1 & 0 & 8 \\
\hline
0 & 0 & 300 & 0 & -120 & 0 & 100 & 180 & 1 & -8240
\end{array}\right]}
$$

Finally we pivot on the 1 in row 2 of column 5.

$$
\begin{array}{r}
-R_2+R_1 \to R_1 \\
\\
\\
R_2+R_4 \to R_4 \\
120R_2+R_5 \to R_5
\end{array}
\overset{\begin{array}{ccccccccc} y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & s_4 & z \end{array}}{\left[\begin{array}{ccccccccc|r}
0 & 1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 8 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 20 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 8 \\
\hline
0 & 0 & 300 & 0 & 0 & 120 & 220 & 300 & 1 & -8240
\end{array}\right]}
$$

Since there are now no negative indicators this tableau gives the solution:
$y_1 = 20,\ y_2 = 8,\ y_3 = 0,\ y_4 = 8,$ with a minimum cost of \$8240.

4.4 Exercises

1. $2x_1 + 3x_2 \le 8$
$x_1 + 4x_2 \ge 7$

Introduce the slack variable s_1 and the surplus variable s_2 to obtain the following equations:

$$
\begin{aligned}
2x_1 + 3x_2 + s_1 \quad &= 8 \\
x_1 + 4x_2 \quad - s_2 &= 7.
\end{aligned}
$$

2. $3x_1 + 7x_2 \le 9$
$4x_1 + 5x_2 \ge 11$

We need one slack variable, s_1, and one surplus variable, s_2. The system becomes

$$
\begin{aligned}
3x_1 + 7x_2 + s_1 \quad &= 9 \\
4x_1 + 5x_2 \quad - s_2 &= 11.
\end{aligned}
$$

3. $2x_1 + x_2 + 2x_3 \le 50$
$x_1 + 3x_2 + x_3 \ge 35$
$x_1 + 2x_2 \ge 15$

Introduce the slack variable s_1 and the surplus variables s_2 and s_3 to obtain the following equations:

$$
\begin{aligned}
2x_1 + x_2 + 2x_3 + s_1 \quad &= 50 \\
x_1 + 3x_2 + x_3 \quad - s_2 &= 35 \\
x_1 + 2x_2 \quad - s_3 &= 15.
\end{aligned}
$$

4. $2x_1 + x_3 \le 40$
$x_1 + x_2 \ge 18$
$x_1 + x_3 \ge 20$

We need one slack variable, s_1, and two surplus variables, s_2 and s_3.

The system becomes

$$
\begin{aligned}
2x_1 + x_3 + s_1 \quad &= 40 \\
x_1 + x_2 \quad - s_2 &= 18 \\
x_1 + x_3 \quad - s_3 &= 20.
\end{aligned}
$$

5. Minimize $w = 3y_1 + 4y_2 + 5y_3$

subject to:
$$
\begin{aligned}
y_1 + 2y_2 + 3y_3 &\ge 9 \\
y_2 + 2y_3 &\ge 8 \\
2y_1 + y_2 + 2y_3 &\ge 6
\end{aligned}
$$

with $y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0.$

Change this to a maximization problem by letting $z = -w$. The problem can now be stated equivalently as follows:

Maximize $z = -3y_1 - 4y_2 - 5y_3$

subject to:
$$y_1 + 2y_2 + 3y_3 \geq 9$$
$$y_2 + 2y_3 \geq 8$$
$$2y_1 + y_2 + 2y_3 \geq 6$$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$

6. Minimize $w = 8y_1 + 3y_2 + y_3$

subject to:
$$7y_1 + 6y_2 + 8y_3 \geq 18$$
$$4y_1 + 5y_2 + 10y_3 \geq 20$$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$

To minimize $w = 8y_1 + 3y_2 + y_3$,

we maximize $z = -w = -8y_1 - 3y_2 - y_3$.

The constraints are not changed.

7. Minimize $w = y_1 + 2y_2 + y_3 + 5y_4$

subject to:
$$y_1 + y_2 + y_3 + y_4 \geq 50$$
$$3y_1 + y_2 + 2y_3 + y_4 \geq 100$$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0.$

Change this to a maximization problem by letting $z = -w$. The problem can now be stated equivalently as follows:

Maximize $z = -y_1 - 2y_2 - y_3 - 5y_4$

subject to:
$$y_1 + y_2 + y_3 + y_4 \geq 50$$
$$3y_1 + y_2 + 2y_3 + y_4 \geq 100$$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0.$

8. Minimize $w = y_1 + y_2 + 7y_3$

subject to:
$$5y_1 + 2y_2 + y_3 \geq 125$$
$$4y_1 + y_2 + 6y_3 \leq 75$$
$$6y_1 + 8y_2 \geq 84$$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$

To minimize $w = y_1 + y_2 + 7y_3$,

we maximize $z = -w = -y_1 - y_2 - 7y_3$.

The constraints are not changed.

9. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$x_1 + 2x_2 \geq 24$$
$$x_1 + x_2 \leq 40$$

and $z = 12x_1 + 10x_2$ is maximized.

Subtracting the surplus variable s_1 and adding the slack variable s_2 leads to the equations

$$x_1 + 2x_2 - s_1 = 24$$
$$x_1 + x_2 + s_2 = 40.$$

The initial simplex tableau is as follows.

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrr|r} \boxed{1} & 2 & -1 & 0 & 0 & 24 \\ 1 & 1 & 0 & 1 & 0 & 40 \\ \hline -12 & -10 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The initial basic solution is not feasible since $s_1 = -24$ is negative, so row transformations must be used. Pivot on the 1 in row 1, column 1, since it is the positive entry that is farthest to the left in the first row (the row containing the -1) and since, in the first column, $\frac{24}{1} = 24$ is a smaller quotient than $\frac{40}{1} = 40$. After row transformations, we obtain the following tableau.

$$\begin{array}{r} \\ \\ -R_1 + R_2 \rightarrow R_2 \\ 12R_1 + R_3 \rightarrow R_3 \end{array}\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrr|r} 1 & 2 & -1 & 0 & 0 & 24 \\ 0 & -1 & \boxed{1} & 1 & 0 & 16 \\ \hline 0 & 14 & -12 & 0 & 1 & 288 \end{array}\right] \end{array}$$

The basic solution is now feasible, but the problem is not yet finished since there is a negative indicator. Continue in the usual way. The 1 in column 3 is the next pivot. After row transformations, we get the following tableau.

$$\begin{array}{r} \\ R_1 + R_2 \rightarrow R_1 \\ \\ 12R_2 + R_3 \rightarrow R_3 \end{array}\begin{array}{c} \begin{matrix} x_1 & x_2 & s_1 & s_2 & z \end{matrix} \\ \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & 1 & 0 & 40 \\ 0 & -1 & 1 & 1 & 0 & 16 \\ \hline 0 & 2 & 0 & 12 & 1 & 480 \end{array}\right] \end{array}$$

This is a final tableau since the entries in the last row are all nonnegative. The maximum value is 480 when $x_1 = 40$ and $x_2 = 0$.

10. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$2x_1 + x_2 \geq 2$$
$$2x_1 + 5x_2 \leq 80$$

and $z = 6x_1 + 2x_2$ is maximized.

Introducing one surplus variable and one slack variable, the system becomes

$$\begin{aligned} 2x_1 + x_2 - s_1 \quad &= 2\\ 2x_1 + 5x_2 \quad + s_2 &= 80. \end{aligned}$$

The initial simplex tableau is

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline \boxed{2} & 1 & -1 & 0 & 0 & 2 \\ 2 & 5 & 0 & 1 & 0 & 80 \\ \hline -6 & -2 & 0 & 0 & 1 & 0 \end{array}.$$

The initial basic solution is not feasible since $s_1 = -2$. Pivot on the 2 in row 1, column 1.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ \hline & 2 & 1 & -1 & 0 & 0 & 2 \\ -R_1 + R_2 \to R_2 & 0 & 4 & \boxed{1} & 1 & 0 & 78 \\ \hline 3R_1 + R_3 \to R_3 & 0 & 1 & -3 & 0 & 1 & 6 \end{array}$$

Pivot on the 1 in row 2, column 3.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ \hline R_2 + R_1 \to R_1 & 2 & 5 & 0 & 1 & 0 & 80 \\ & 0 & 4 & 1 & 1 & 0 & 78 \\ \hline 3R_2 + R_3 \to R_3 & 0 & 13 & 0 & 3 & 1 & 240 \end{array}$$

Create a 1 in the column corresponding to x_1.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ \hline \frac{1}{2}R_1 \to R_1 & 1 & \frac{5}{2} & 0 & \frac{1}{2} & 0 & 40 \\ & 0 & 4 & 1 & 1 & 0 & 78 \\ \hline & 0 & 13 & 0 & 3 & 1 & 240 \end{array}$$

The maximum is 240 when $x_1 = 40$ and $x_2 = 0$.

11. Find $x_1 \geq 0,\ x_2 \geq 0,$ and $x_3 \geq 0$ such that

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 150\\ x_1 + x_2 + x_3 &\geq 100 \end{aligned}$$

and $z = 2x_1 + 5x_2 + 3x_3$ is maximized.

The initial tableau is as follows.

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 150 \\ \boxed{1} & 1 & 1 & 0 & -1 & 0 & 100 \\ \hline -2 & -5 & -3 & 0 & 0 & 1 & 0 \end{array}$$

Note that s_1 is a slack variable, while s_2 is a surplus variable. The initial basic solution is not feasible, since $s_2 = -100$ is negative. Pivot on the 1 in row 2, column 1.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline -R_2 + R_1 \to R_1 & 0 & 0 & 0 & 1 & 1 & 0 & 50 \\ & 1 & \boxed{1} & 1 & 0 & -1 & 0 & 100 \\ \hline 2R_2 + R_3 \to R_3 & 0 & -3 & -1 & 0 & -2 & 1 & 200 \end{array}$$

Pivot on the 1 in row 2, column 2.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline & 0 & 0 & 0 & 1 & \boxed{1} & 0 & 50 \\ & 1 & 1 & 1 & 0 & -1 & 0 & 100 \\ \hline 3R_2 + R_3 \to R_3 & 3 & 0 & 2 & 0 & -5 & 1 & 500 \end{array}$$

Pivot on the 1 in row 1, column 5.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline & 0 & 0 & 0 & 1 & 1 & 0 & 50 \\ R_1 + R_2 \to R_2 & 1 & 1 & 1 & 1 & 0 & 0 & 150 \\ \hline 5R_1 + R_3 \to R_3 & 3 & 0 & 2 & 5 & 0 & 1 & 750 \end{array}$$

This is a final tableau. The maximum value is 750 when $x_1 = 0,\ x_2 = 150,$ and $x_3 = 0.$

12. Find $x_1 \geq 0,\ x_2 \geq 0,$ and $x_3 \geq 0$ such that

$$\begin{aligned} x_1 + \ x_2 + \ x_3 &\leq 15\\ 4x_1 + 4x_2 + 2x_3 &\geq 48 \end{aligned}$$

and $z = 2x_1 + x_2 + 3x_3$ is maximized.

The initial simplex tableau is

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline 1 & 1 & \boxed{1} & 1 & 0 & 0 & 15 \\ 4 & 4 & 2 & 0 & -1 & 0 & 48 \\ \hline -2 & -1 & -3 & 0 & 0 & 1 & 0 \end{array}.$$

The initial basic solution is not feasible since $s_2 = -48$. Pivot on the 1 in row 1, column 3.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline & 0 & 1 & 1 & 1 & 0 & 0 & 15 \\ -2R_1 + R_2 \to R_2 & \boxed{2} & 2 & 0 & -2 & -1 & 0 & 18 \\ \hline 3R_1 + R_3 \to R_3 & 1 & 2 & 0 & 3 & 0 & 1 & 45 \end{array}$$

The initial basic solution is still not feasible since $s_2 = -18$. To choose a pivot, locate the positive entry farthest to the left in row 2. The 2 in row 2, column 1, determines the pivot column and is also the pivot element, since it forms the smaller quotient.

$$\begin{array}{r} \\ -R_2 + 2R_1 \to R_1 \\ \\ -R_2 + 2R_3 \to R_3 \end{array} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline 0 & 0 & 2 & 4 & 1 & 0 & 12 \\ 2 & 2 & 0 & -2 & -1 & 0 & 18 \\ \hline 0 & 2 & 0 & 8 & 1 & 2 & 72 \end{array}$$

The solution is feasible since all variables and indicators are nonnegative. Therefore, create a 1 in the columns corresponding to x_1, x_3, and z.

$$\begin{array}{r} \\ \frac{1}{2}R_1 \to R_1 \\ \frac{1}{2}R_2 \to R_2 \\ \frac{1}{2}R_3 \to R_3 \end{array} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline 0 & 0 & 1 & 2 & \frac{1}{2} & 0 & 6 \\ 1 & 1 & 0 & -1 & -\frac{1}{2} & 0 & 9 \\ \hline 0 & 1 & 0 & 4 & \frac{1}{2} & 1 & 36 \end{array}$$

The maximum is 36 when $x_1 = 9$, $x_2 = 0$, and $x_3 = 6$.

13. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$\begin{aligned} x_1 + x_2 &\leq 100 \\ 2x_1 + 3x_2 &\leq 75 \\ x_1 + 4x_2 &\geq 50 \end{aligned}$$

and $z = 5x_1 - 3x_2$ is maximized.

The initial simplex tableau is

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 100 \\ \boxed{2} & 3 & 0 & 1 & 0 & 0 & 75 \\ 1 & 4 & 0 & 0 & -1 & 0 & 50 \\ \hline -5 & 3 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

The initial basic solution is not feasible since $s_3 = -50$. Pivot on the 2 in row 2, column 1.

$$\begin{array}{r} \\ -R_2 + 2R_1 \to R_1 \\ \\ -R_2 + 2R_3 \to R_3 \\ 5R_2 + 2R_4 \to R_4 \end{array} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline 0 & -1 & 2 & -1 & 0 & 0 & 125 \\ 2 & 3 & 0 & 1 & 0 & 0 & 75 \\ 0 & \boxed{5} & 0 & -1 & -2 & 0 & 25 \\ \hline 0 & 21 & 0 & 5 & 0 & 2 & 375 \end{array}.$$

This solution is still not feasible since $s_3 = -\frac{25}{2}$.

Pivot on the 5 in row 3, column 2.

$$\begin{array}{r} \\ R_3 + 5R_1 \to R_1 \\ -3R_3 + 5R_2 \to R_2 \\ \\ -21R_3 + 5R_4 \to R_4 \end{array} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline 0 & 0 & 10 & -6 & -2 & 0 & 650 \\ 10 & 0 & 0 & 8 & 6 & 0 & 300 \\ 0 & \boxed{5} & 0 & -1 & -2 & 0 & 25 \\ \hline 0 & 0 & 0 & 46 & 42 & 10 & 1350 \end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, s_1, and z.

$$\begin{array}{r} \\ \frac{1}{10}R_1 \to R_1 \\ \frac{1}{10}R_2 \to R_2 \\ \frac{1}{5}R_3 \to R_3 \\ \frac{1}{10}R_4 \to R_4 \end{array} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline 0 & 0 & 1 & -\frac{3}{5} & -\frac{1}{5} & 0 & 65 \\ 1 & 0 & 0 & \frac{4}{5} & \frac{3}{5} & 0 & 30 \\ 0 & 1 & 0 & -\frac{1}{5} & -\frac{2}{5} & 0 & 5 \\ \hline 0 & 0 & 0 & \frac{23}{5} & \frac{21}{5} & 1 & 135 \end{array}.$$

This is a final tableau. The maximum is 135 when $x_1 = 30$, $x_2 = 5$.

14. Find $x_1 \geq 0$ and $x_2 \geq 0$ such that

$$\begin{aligned} x_1 + 2x_2 &\leq 18 \\ x_1 + 3x_2 &\geq 12 \\ 2x_1 + 2x_2 &\leq 24 \end{aligned}$$

and $z = 5x_1 - 10x_2$ is maximized.

Introduce slack and surplus variables to get the system

$$\begin{aligned} x_1 + 2x_2 + s_1 \qquad\qquad &= 18 \\ x_1 + 3x_2 \qquad - s_2 \qquad &= 12 \\ 2x_1 + 2x_2 \qquad\qquad + s_3 &= 24. \end{aligned}$$

The initial tableau is

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ \hline 1 & 2 & 1 & 0 & 0 & 0 & 18 \\ \boxed{1} & 3 & 0 & -1 & 0 & 0 & 12 \\ 2 & 2 & 0 & 0 & 1 & 0 & 24 \\ \hline -5 & 10 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

$s_2 = -12$ is not a feasible solution. Pivot on the 1 in row 1, column 1.

$$\begin{array}{r} \\ -R_2 + R_1 \to R_1 \\ \\ -2R_2 + R_3 \to R_3 \\ 5R_2 + R_4 \to R_4 \end{array} \begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ \hline 0 & -1 & 1 & 1 & 0 & 0 & 6 \\ 1 & 3 & 0 & -1 & 0 & 0 & 12 \\ 0 & -4 & 0 & \boxed{2} & 1 & 0 & 0 \\ \hline 0 & 25 & 0 & -5 & 0 & 1 & 60 \end{array}$$

Pivot on the 2 in row 3, column 4.

$$\begin{array}{r} \\ -R_3 + 2R_1 \to R_1 \\ R_3 + 2R_2 \to R_2 \\ \\ 5R_3 + 2R_4 \to R_4 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 6 & 2 & 0 & -1 & 0 & 12 \\ 2 & 2 & 0 & 0 & 1 & 0 & 24 \\ 0 & -4 & 0 & 2 & 1 & 0 & 0 \\ \hline 0 & 30 & 0 & 0 & 5 & 2 & 120 \end{array}\right] \end{array}$$

$$\begin{array}{r} \\ \frac{1}{2}R_1 \to R_1 \\ \frac{1}{2}R_2 \to R_2 \\ \frac{1}{2}R_3 \to R_3 \\ \frac{1}{2}R_4 \to R_4 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 3 & 1 & 0 & -\frac{1}{2} & 0 & 6 \\ 1 & 1 & 0 & 0 & \frac{1}{2} & 0 & 12 \\ 0 & -2 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ \hline 0 & 15 & 0 & 0 & \frac{5}{2} & 1 & 60 \end{array}\right] \end{array}$$

The maximum is 60 when $x_1 = 12$ and $x_2 = 0$.

15. Find $y_1 \geq 0$, $y_2 \geq 0$, and $y_3 \geq 0$ such that

$$5y_1 + 3y_2 + 2y_3 \leq 150$$
$$5y_1 + 10y_2 + 3y_3 \geq 90$$

and $w = 10y_1 + 12y_2 + 10y_3$ is minimized.

Let $z = -w = -10y - 12y_2 - 10y_3$.
Maximize z.

The initial simplex tableau is

$$\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 5 & 3 & 2 & 1 & 0 & 0 & 150 \\ \boxed{5} & 10 & 3 & 0 & -1 & 0 & 90 \\ \hline 10 & 12 & 10 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The initial basic solution is not feasible since $s_2 = -90$. Pivot on the 5 in row 2, column 1.

$$\begin{array}{r} \\ -R_2 + R_1 \to R_1 \\ \\ -2R_2 + R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 0 & -7 & -1 & 1 & 1 & 0 & 60 \\ 5 & \boxed{10} & 3 & 0 & -1 & 0 & 90 \\ \hline 0 & -8 & 4 & 0 & 2 & 1 & -180 \end{array}\right] \end{array}$$

Pivot on the 10 in row 2, column 2.

$$\begin{array}{r} \\ 7R_2 + 10R_1 \to R_1 \\ \\ 8R_2 + 10R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 35 & 0 & 11 & 10 & 3 & 0 & 1230 \\ 5 & 10 & 3 & 0 & -1 & 0 & 90 \\ \hline 40 & 0 & 64 & 0 & 12 & 10 & -1080 \end{array}\right] \end{array}$$

Create a 1 in the columns corresponding to y_2, s_1, and z.

$$\begin{array}{r} \\ \frac{1}{10}R_1 \to R_1 \\ \frac{1}{10}R_2 \to R_2 \\ \frac{1}{10}R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} \frac{7}{2} & 0 & \frac{11}{10} & 1 & \frac{3}{10} & 0 & 123 \\ \frac{1}{2} & 1 & \frac{3}{10} & 0 & -\frac{1}{10} & 0 & 9 \\ \hline 4 & 0 & \frac{32}{5} & 0 & \frac{6}{5} & 1 & -108 \end{array}\right] \end{array}$$

This is a final tableau. The minimum is 108 when $y_1 = 0$, $y_2 = 9$, and $y_3 = 0$.

16. Minimize $w = 3y_1 + 2y_2 + 3y_3$

subject to: $2y_1 + 3y_2 + 6y_3 \leq 60$
$y_1 + 4y_2 + 5y_3 \geq 40$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$

Let $z = -w = -3y_1 - 2y_2 - 3y_3$. Maximize z.

The initial simplex tableau is

$$\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} \boxed{2} & 3 & 6 & 1 & 0 & 0 & 60 \\ 1 & 4 & 5 & 0 & -1 & 0 & 40 \\ \hline 3 & 2 & 3 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The initial basic solution is not feasible since $s_2 = -40$. Pivot on the 2 in row 1, column 1.

$$\begin{array}{r} \\ \\ -R_1 + 2R_2 \to R_2 \\ -3R_1 + 2R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 2 & 3 & 6 & 1 & 0 & 0 & 60 \\ 0 & 5 & \boxed{4} & -1 & -2 & 0 & 20 \\ \hline 0 & -5 & -12 & -3 & 0 & 2 & -180 \end{array}\right] \end{array}$$

There are negative indicators, so now pivot on the 4 in row 2, column 3.

$$\begin{array}{r} \\ -3R_2 + 2R_1 \to R_1 \\ \\ 3R_2 + R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 4 & -9 & 0 & \boxed{5} & 6 & 0 & 60 \\ 0 & 5 & 4 & -1 & -2 & 0 & 20 \\ \hline 0 & 10 & 0 & -6 & -6 & 2 & -120 \end{array}\right] \end{array}$$

Pivot on the 5 in row 1, column 4.

$$\begin{array}{r} \\ \\ R_1 + 5R_2 \to R_2 \\ 6R_1 + 5R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 4 & -9 & 0 & 5 & 6 & 0 & 60 \\ 4 & \boxed{16} & 20 & 0 & -4 & 0 & 160 \\ \hline 24 & -4 & 0 & 0 & 6 & 10 & -240 \end{array}\right] \end{array}$$

Pivot on the 16 in row 2, column 2.

$$\begin{array}{r} \\ 9R_2 + 16R_1 \to R_1 \\ \\ R_2 + 4R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} \boxed{100} & 0 & 180 & 80 & 60 & 0 & 2400 \\ 4 & 16 & 20 & 0 & -4 & 0 & 160 \\ \hline 100 & 0 & 20 & 0 & 20 & 40 & -800 \end{array}\right] \end{array}$$

$$\begin{array}{r} \\ \frac{1}{80}R_1 \to R_1 \\ \frac{1}{16}R_2 \to R_2 \\ \frac{1}{40}R_3 \to R_3 \end{array}\begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & z & \\ \frac{5}{4} & 0 & \frac{9}{4} & 1 & \frac{3}{4} & 0 & 30 \\ \frac{1}{4} & 1 & \frac{5}{4} & 0 & -\frac{1}{4} & 0 & 10 \\ \hline \frac{5}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & -20 \end{array}$$

The minimum is $w = 20$ when $y_1 = 0$, $y_2 = 10$, and $y_3 = 0$.

17. Maximize $z = 3x_1 + 2x_1$

subject to:
$$\begin{aligned} x_1 + x_2 &= 50 \\ 4x_1 + 2x_2 &\ge 120 \\ 5x_1 + 2x_2 &\le 200 \end{aligned}$$
with $x_1 \ge 0, x_2 \ge 0.$

The artificial variable a_1 is used to rewrite $x_1 + x_2 = 50$ as $x_1 + x_2 + a_1 = 50$; note that a_1 must equal 0 for this equation to be a true statement. Also the surplus variable s_1 and the slack variable s_2 are needed. The initial tableau is as follows.

$$\begin{array}{cccccc|c} x_1 & x_2 & a_1 & s_1 & s_2 & z & \\ 1 & 1 & 1 & 0 & 0 & 0 & 50 \\ \boxed{4} & 2 & 0 & -1 & 0 & 0 & 120 \\ 5 & 2 & 0 & 0 & 1 & 0 & 200 \\ \hline -3 & -2 & 0 & 0 & 0 & 1 & 0 \end{array}$$

The initial basic solution is not feasible. Pivot on the 4 in row 2, column 1.

$$\begin{array}{r} \\ -R_2 + 4R_1 \to R_1 \\ \\ -5R_2 + 4R_3 \to R_3 \\ 3R_2 + 4R_4 \to R_4 \end{array}\begin{array}{cccccc|c} x_1 & x_2 & a_1 & s_1 & s_2 & z & \\ 0 & 2 & 4 & 1 & 0 & 0 & 80 \\ 4 & 2 & 0 & -1 & 0 & 0 & 120 \\ 0 & -2 & 0 & \boxed{5} & 4 & 0 & 200 \\ \hline 0 & -2 & 0 & -3 & 0 & 4 & 360 \end{array}$$

The basic solution is now feasible, but there are negative indicators. Pivot on the 5 in row 3, column 4 (which is the column with the most negative indicator and the row with the smallest nonnegative quotient).

$$\begin{array}{r} \\ -R_3 + 5R_1 \to R_1 \\ R_3 + 5R_2 \to R_2 \\ \\ 3R_3 + 5R_4 \to R_4 \end{array}\begin{array}{cccccc|c} x_1 & x_2 & a_1 & s_1 & s_2 & z & \\ 0 & \boxed{12} & 20 & 0 & -4 & 0 & 200 \\ 20 & 8 & 0 & 0 & 4 & 0 & 800 \\ 0 & -2 & 0 & 5 & 4 & 0 & 200 \\ \hline 0 & -16 & 0 & 0 & 12 & 20 & 2400 \end{array}$$

Pivot on the 12 in row 1, column 2.

$$\begin{array}{r} \\ \\ -2R_1 + 3R_2 \to R_2 \\ R_1 + 6R_3 \to R_3 \\ 4R_1 + 3R_4 \to R_4 \end{array}\begin{array}{cccccc|c} x_1 & x_2 & a_1 & s_1 & s_2 & z & \\ 0 & 12 & 20 & 0 & -4 & 0 & 200 \\ 60 & 0 & -40 & 0 & 20 & 0 & 2000 \\ 0 & 0 & 20 & 30 & 20 & 0 & 1400 \\ \hline 0 & 0 & 80 & 0 & 20 & 60 & 8000 \end{array}$$

We now have $a_1 = 0$, so drop the a_1 column.

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 0 & 12 & 0 & -4 & 0 & 200 \\ 60 & 0 & 0 & 20 & 0 & 2000 \\ 0 & 0 & 30 & 20 & 0 & 1400 \\ \hline 0 & 0 & 0 & 20 & 60 & 8000 \end{array}$$

We are finished pivoting. Create a 1 in the columns corresponding to x_1, x_2, s_1, and z.

$$\begin{array}{r} \\ \frac{1}{12}R_1 \to R_1 \\ \frac{1}{60}R_2 \to R_2 \\ \frac{1}{30}R_3 \to R_3 \\ \frac{1}{60}R_4 \to R_4 \end{array}\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 0 & 1 & 0 & -\frac{1}{3} & 0 & \frac{50}{3} \\ 1 & 0 & 0 & \frac{1}{3} & 0 & \frac{100}{3} \\ 0 & 0 & 1 & \frac{2}{3} & 0 & \frac{140}{3} \\ \hline 0 & 0 & 0 & \frac{1}{3} & 1 & \frac{400}{3} \end{array}$$

The maximum value is $\frac{400}{3}$ when $x_1 = \frac{100}{3}$ and $x_2 = \frac{50}{3}$.

18. Maximize $z = 5x_1 + 7x_2$

subject to:
$$\begin{aligned} x_1 + x_2 &= 15 \\ 2x_1 + 4x_2 &\ge 30 \\ 3x_1 + 5x_2 &\ge 10 \end{aligned}$$
with $x_1 \ge 0, x_2 \ge 0.$

With artificial, slack, and surplus variables, we have

$$\begin{aligned} x_1 + x_2 + a \quad\quad &= 15 \\ 2x_1 + 4x_2 - s_1 \quad &= 30 \\ 3x_1 + 5x_2 - s_2 &= 10. \end{aligned}$$

The initial tableau is

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & a & z & \\ \boxed{1} & 1 & 0 & 0 & 1 & 0 & 15 \\ 2 & 4 & -1 & 0 & 0 & 0 & 30 \\ 3 & 5 & 0 & -1 & 0 & 0 & 10 \\ \hline -5 & -7 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

First, eliminate the artificial variable a. Pivot on the 1 in row 1, column 1.

$$\begin{array}{r} \\ \\ -2R_1+R_2 \to R_2 \\ -3R_1+R_3 \to R_3 \\ 5R_1+R_4 \to R_4 \end{array} \begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & a & z & \\ \hline 1 & 1 & 0 & 0 & 1 & 0 & 15 \\ 0 & 2 & -1 & 0 & -2 & 0 & 0 \\ 0 & 2 & 0 & -1 & -3 & 0 & 35 \\ \hline 0 & -2 & 0 & 0 & 5 & 1 & 75 \end{array}$$

Now $a = 0$, so we can drop the a column.

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline 1 & 1 & 0 & 0 & 0 & 15 \\ 0 & \boxed{2} & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 35 \\ \hline 0 & -2 & 0 & 0 & 1 & 75 \end{array}$$

Because $s_2 = -35$, we choose the 2 in row 2, column 2, as the next pivot. (Note that $s_1 = 0$.)

$$\begin{array}{r} \\ -R_2+2R_1 \to R_1 \\ \\ R_2-R_3 \to R_3 \\ R_2+R_4 \to R_4 \end{array} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline 2 & 0 & \boxed{1} & 0 & 0 & 30 \\ 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 35 \\ \hline 0 & 0 & -1 & 0 & 1 & 75 \end{array}$$

Pivot on the 1 in row 1, column 3.

$$\begin{array}{r} \\ \\ R_1+R_2 \to R_2 \\ R_1+R_3 \to R_3 \\ R_1+R_4 \to R_4 \end{array} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline 2 & 0 & 1 & 0 & 0 & 30 \\ 2 & 2 & 0 & 0 & 0 & 30 \\ 2 & 0 & 0 & 1 & 0 & 65 \\ \hline 2 & 0 & 0 & 0 & 1 & 105 \end{array}$$

Create a 1 in the column for x_2.

$$\begin{array}{r} \\ \\ \frac{1}{2}R_2 \to R_2 \\ \\ \\ \end{array} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline 2 & 0 & 1 & 0 & 0 & 30 \\ 1 & 1 & 0 & 0 & 0 & 15 \\ 2 & 0 & 0 & 1 & 0 & 65 \\ \hline 2 & 0 & 0 & 0 & 1 & 105 \end{array}$$

The maximum is 105 when $x_1 = 0$ and $x_2 = 15$.

19. Minimize $w = 32y_1 + 40y_2 + 48y_3$

subject to:
$$\begin{aligned} 20y_1 + 10y_2 + 5y_3 &= 200 \\ 25y_1 + 40y_2 + 50y_3 &\le 500 \\ 18y_1 + 24y_2 + 12y_3 &\ge 300 \end{aligned}$$

with $y_1 \ge 0, y_2 \ge 0, y_3 \ge 0$

With artificial, slack, and surplus variables, this problem becomes

Maximize $z = -32y_1 - 40y_2 - 48y_3$

subject to:
$$\begin{aligned} 20y_1 + 10y_2 + 5y_3 + a_1 &= 200 \\ 25y_1 + 40y_2 + 50y_3 + s_1 &= 500 \\ 18y_1 + 24y_2 + 12y_3 - s_2 &= 300. \end{aligned}$$

The initial tableau is as follows.

$$\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & a_1 & s_1 & s_2 & z & \\ \hline \boxed{20} & 10 & 5 & 1 & 0 & 0 & 0 & 200 \\ 25 & 40 & 50 & 0 & 1 & 0 & 0 & 500 \\ 18 & 24 & 12 & 0 & 0 & -1 & 0 & 300 \\ \hline 32 & 40 & 48 & 0 & 0 & 0 & 1 & 0 \end{array}$$

The initial basic tableau is not feasible. Pivot on the 20 in row 1, column 1.

$$\begin{array}{r} \\ \\ -5R_1+4R_2 \to R_2 \\ -9R_1+10R_3 \to R_3 \\ -8R_1+5R_4 \to R_4 \end{array} \begin{array}{ccccccc|c} y_1 & y_2 & y_3 & a_1 & s_1 & s_2 & z & \\ \hline 20 & 10 & 5 & 1 & 0 & 0 & 0 & 200 \\ 0 & 110 & 175 & -5 & 4 & 0 & 0 & 1000 \\ 0 & 150 & 75 & -9 & 0 & -10 & 0 & 1200 \\ \hline 0 & 120 & 200 & -8 & 0 & 0 & 5 & -1600 \end{array}$$

Eliminate the a_1 column.

$$\begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & z & \\ \hline 20 & 10 & 5 & 0 & 0 & 0 & 200 \\ 0 & 110 & 175 & 4 & 0 & 0 & 1000 \\ 0 & \boxed{150} & 75 & 0 & -10 & 0 & 1200 \\ \hline 0 & 120 & 200 & 0 & 0 & 5 & -1600 \end{array}$$

Pivot on the 150 in row 3, column 2.

$$\begin{array}{r} \\ -R_3+15R_1 \to R_1 \\ -11R_3+15R_2 \to R_2 \\ \\ -4R_3+5R_4 \to R_4 \end{array} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & z & \\ \hline 300 & 0 & 0 & 0 & 10 & 0 & 1800 \\ 0 & 0 & 1800 & 60 & 110 & 0 & 1800 \\ 0 & 150 & 75 & 0 & -10 & 0 & 1200 \\ \hline 0 & 0 & 700 & 0 & 40 & 25 & -12{,}800 \end{array}$$

Create ones in the columns corresponding to y_1, y_2, s_1, and z.

$$\begin{array}{r} \\ \frac{1}{300}R_1 \to R_1 \\ \frac{1}{60}R_2 \to R_2 \\ \frac{1}{150}R_3 \to R_3 \\ \frac{1}{25}R_4 \to R_4 \end{array} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & z & \\ \hline 1 & 0 & 0 & 0 & \frac{1}{30} & 0 & 6 \\ 0 & 0 & 30 & 1 & \frac{11}{6} & 0 & 30 \\ 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{15} & 0 & 8 \\ \hline 0 & 0 & 28 & 0 & \frac{8}{5} & 1 & -512 \end{array}$$

This is a final tableau. The minimum value is 512 when $y_1 = 6$, $y_2 = 8$, and $y_3 = 0$.

20. Minimize $w = 15y_1 + 12y_2 + 18y_3$

subject to:
$$y_1 + 2y_2 + 3y_3 \le 12$$
$$3y_1 + y_2 + 3y_3 \ge 18$$
$$y_1 + y_2 + y_3 = 10$$

with $y_1 \ge 0, y_2 \ge 0, y_3 \ge 0$

Let $z = -w = -15y_1 - 12y_2 - 18y_3$ and maximize z. Introduce the slack variable s_1, the surplus variable s_2, and the artificial variable a_1. The initial tableau is as follows.

$$\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & a_1 & z & \\ \hline 1 & 2 & 3 & 1 & 0 & 0 & 0 & 12 \\ 3 & 1 & 3 & 0 & -1 & 0 & 0 & 18 \\ \boxed{1} & 1 & 1 & 0 & 0 & 1 & 0 & 10 \\ \hline 15 & 12 & 18 & 0 & 0 & 0 & 1 & 0 \end{array}$$

First, eliminate the artificial variable a_1. Pivot on the 1 in row 3, column 1.

$$\begin{array}{r} \\ -R_3 + R_1 \to R_1 \\ -3R_3 + R_2 \to R_2 \\ \\ -15R_3 + R_4 \to R_4 \end{array} \begin{array}{ccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & a_1 & z & \\ 0 & 1 & 2 & 1 & 0 & -1 & 0 & 2 \\ 0 & -2 & 0 & 0 & -1 & -3 & 0 & -12 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 10 \\ \hline 0 & -3 & 3 & 0 & 0 & -15 & 1 & -150 \end{array}$$

Now $a_1 = 0$, so we can drop the a_1 column.

$$\begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & z & \\ 0 & \boxed{1} & 2 & 1 & 0 & 0 & 2 \\ 0 & -2 & 0 & 0 & -1 & 0 & -12 \\ 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ \hline 0 & -3 & 3 & 0 & 0 & 1 & -150 \end{array}$$

Pivot on the 1 in row 1, column 2.

$$\begin{array}{r} \\ \\ 2R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \\ 3R_1 + R_4 \to R_4 \end{array} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & z & \\ 0 & 1 & 2 & 1 & 0 & 0 & 2 \\ 0 & 0 & 4 & 2 & -1 & 0 & -8 \\ 1 & 0 & -1 & -1 & 0 & 0 & 8 \\ \hline 0 & 0 & 9 & 3 & 0 & 1 & -144 \end{array}$$

The maximum value of $z = -w$ is –144.

Therefore, the minimum value of w is 144 when $y_1 = 8$, $y_2 = 2$, and $y_3 = 0$.

23. **(a)** Let $y_1 =$ amount shipped from S_1 to D_1,

$y_2 =$ amount shipped from S_1 to D_2,

$y_3 =$ amount shipped from S_2 to D_1,

and $y_4 =$ amount shipped from S_2 to D_2.

Minimize $w = 30y_1 + 20y_2 + 25y_3 + 22y_4$

subject to:

$$\begin{aligned} y_1 + y_3 &\ge 3000 \\ y_2 + y_4 &\ge 5000 \\ y_1 + y_2 &\le 5000 \\ y_3 + y_4 &\le 5000 \\ 2y_1 + 6y_2 + 5y_3 + 4y_4 &\le 40{,}000 \end{aligned}$$

with $y_1 \ge 0, y_2 \ge 0, y_3 \ge 0, y_4 \ge 0.$

Maximize $z = -w = -30y_1 - 20y_2 - 25y_3 - 22y_4.$

$$\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccccccc|c} y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & s_4 & s_5 & z & \\ \boxed{1} & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 3000 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 5000 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 5000 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 5000 \\ 2 & 6 & 5 & 4 & 0 & 0 & 0 & 0 & 1 & 0 & 40{,}000 \\ \hline 30 & 20 & 25 & 22 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 1 in row 1, column 1 since the feasible solution has a negative value, $s_1 = -3000.$

$$\begin{array}{r} \\ \\ \\ -R_1 + R_3 \to R_3 \\ \\ -2R_1 + R_5 \to R_5 \\ -30R_1 + R_6 \to R_6 \end{array}\begin{array}{cccccccccc|c} y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & s_4 & s_5 & z & \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 3000 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 5000 \\ 0 & \boxed{1} & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2000 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 5000 \\ 0 & 6 & 3 & 4 & 2 & 0 & 0 & 0 & 1 & 0 & 34{,}000 \\ \hline 0 & 20 & -5 & 22 & 30 & 0 & 0 & 0 & 0 & 1 & -90{,}000 \end{array}$$

Since the feasible solution has a negative value $(s_2 = -5000),$ pivot on the 1 in row 3, column 2.

$$\begin{array}{r} \\ \\ -R_3 + R_2 \to R_2 \\ \\ \\ -6R_3 + R_5 \to R_5 \\ -20R_3 + R_6 \to R_6 \end{array}\begin{array}{cccccccccc|c} y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & s_4 & s_5 & z & \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 3000 \\ 0 & 0 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 3000 \\ 0 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2000 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 5000 \\ 0 & 0 & \boxed{9} & 4 & -4 & 0 & -6 & 0 & 1 & 0 & 22{,}000 \\ \hline 0 & 0 & 15 & 22 & 10 & 0 & -20 & 0 & 0 & 1 & -130{,}000 \end{array}$$

Since the feasible solution has a negative value ($s_2 = -3000$), pivot on the 9 in row 5, column 3.

$$\begin{array}{r} \\ -R_5 + 9R_1 \to R_1 \\ -R_5 + 9R_2 \to R_2 \\ R_5 + 9R_3 \to R_3 \\ -R_5 + 9R_4 \to R_4 \\ \\ -5R_5 + 3R_6 \to R_6 \end{array}\begin{array}{cccccccccc|c} y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & s_4 & s_5 & z & \\ 9 & 0 & 0 & -4 & -5 & 0 & 6 & 0 & -1 & 0 & 5000 \\ 0 & 0 & 0 & \boxed{5} & -5 & -9 & -3 & 0 & -1 & 0 & 5000 \\ 0 & 9 & 0 & 4 & 5 & 0 & 3 & 0 & 1 & 0 & 40{,}000 \\ 0 & 0 & 0 & 5 & 4 & 0 & 6 & 9 & -1 & 0 & 23{,}000 \\ 0 & 0 & 9 & 4 & -4 & 0 & -6 & 0 & 1 & 0 & 22{,}000 \\ \hline 0 & 0 & 0 & 46 & 50 & 0 & -30 & 0 & -5 & 3 & -500{,}000 \end{array}$$

Pivot on the 5 in row 2, column 4.

$$
\begin{array}{r|cccccccccc|r}
 & y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & s_4 & s_5 & z & \\
4R_2 + 5R_1 \to R_1 & 45 & 0 & 0 & 0 & -45 & -36 & 18 & 0 & -9 & 0 & 45{,}000 \\
 & 0 & 0 & 0 & 5 & -5 & -9 & -3 & 0 & -1 & 0 & 5000 \\
-4R_2 + 5R_3 \to R_3 & 0 & 45 & 0 & 0 & 45 & 36 & 27 & 0 & 9 & 0 & 180{,}000 \\
-R_2 + R_4 \to R_4 & 0 & 0 & 0 & 0 & 9 & 9 & \boxed{9} & 9 & 0 & 0 & 18{,}000 \\
-4R_2 + 5R_5 \to R_5 & 0 & 0 & 45 & 0 & 0 & 36 & -18 & 0 & 9 & 0 & 90{,}000 \\
\hline
-46R_2 + 5R_6 \to R_6 & 0 & 0 & 0 & 0 & 480 & 414 & -12 & 0 & 21 & 15 & -2{,}730{,}000
\end{array}
$$

Pivot on the 9 in row 4, column 7.

$$
\begin{array}{r|cccccccccc|r}
 & y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & s_4 & s_5 & z & \\
-2R_4 + R_1 \to R_1 & 45 & 0 & 0 & 0 & -63 & -54 & 0 & -18 & -9 & 0 & 9000 \\
R_4 + 3R_2 \to R_2 & 0 & 0 & 0 & 15 & -6 & -18 & 0 & 9 & -3 & 0 & 33{,}000 \\
-3R_4 + R_3 \to R_3 & 0 & 45 & 0 & 0 & 18 & 9 & 0 & -27 & 9 & 0 & 126{,}000 \\
 & 0 & 0 & 0 & 0 & 9 & 9 & 9 & 9 & 0 & 0 & 18{,}000 \\
2R_4 + R_5 \to R_5 & 0 & 0 & 45 & 0 & 18 & 54 & 0 & 18 & 9 & 0 & 126{,}000 \\
\hline
4R_4 + 3R_6 \to R_6 & 0 & 0 & 0 & 0 & 1476 & 450 & 0 & 36 & 63 & 45 & -8{,}118{,}000
\end{array}
$$

Create a 1 in the columns corresponding to y_1, y_2, y_3, y_4, and z.

$$
\begin{array}{r|cccccccccc|r}
 & y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & s_4 & s_5 & z & \\
\frac{1}{45}R_1 \to R_1 & 1 & 0 & 0 & 0 & -\frac{7}{5} & -\frac{6}{5} & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & 200 \\
\frac{1}{15}R_2 \to R_2 & 0 & 0 & 0 & 1 & -\frac{2}{5} & -\frac{6}{5} & 0 & \frac{3}{5} & -\frac{1}{5} & 0 & 2200 \\
\frac{1}{45}R_3 \to R_3 & 0 & 1 & 0 & 0 & \frac{2}{5} & \frac{1}{5} & 0 & -\frac{3}{5} & \frac{1}{5} & 0 & 2800 \\
 & 0 & 0 & 0 & 0 & 9 & 9 & 9 & 9 & 0 & 0 & 18{,}000 \\
\frac{1}{45}R_5 \to R_5 & 0 & 0 & 1 & 0 & \frac{2}{5} & \frac{6}{5} & 0 & \frac{2}{5} & \frac{1}{5} & 0 & 2800 \\
\hline
\frac{1}{45}R_6 \to R_6 & 0 & 0 & 0 & 0 & \frac{164}{5} & 10 & 0 & \frac{4}{5} & \frac{7}{5} & 1 & -180{,}400
\end{array}
$$

Here, $y_1 = 200$, $y_2 = 2800$, $y_3 = 2800$, $y_4 = 2200$, and $-z = w = 180{,}400$. So, ship 200 barrels of oil from supplier S_1 to distributor D_1. Ship 2800 barrels of oil from supplier S_1 to distributor D_2. Ship 2800 barrels of oil from supplier S_2 to distributor D_1. Ship 2200 barrels of oil from supplier S_2 to distributor D_2. The minimum cost is \$180,400.

(b) From the final tableau, $9s_3 = 18{,}000$, so $s_3 = 2000$. Therefore, S_1 could furnish 2000 more barrels of oil.

24. Let $y_1 =$ amount shipped from S_1 to D_1,
$y_2 =$ amount shipped from S_1 to D_2,
$y_3 =$ amount shipped from S_2 to D_1,
and $y_4 =$ amount shipped from S_2 to D_2.

Maximize $w = 30y_1 + 20y_2 + 25y_3 + 22y_4$

subject to:

$$
\begin{aligned}
y_1 + y_3 &\ge 3000 \\
y_2 + y_4 &\ge 5000 \\
y_1 + y_2 &= 5000 \\
y_3 + y_4 &= 5000 \\
2y_1 + 6y_2 + 5y_3 + 4y_4 &\le 40{,}000
\end{aligned}
$$

with $y_1 \ge 0$, $y_2 \ge 0$, $y_3 \ge 0$, $y_4 \ge 0$.

Maximize $z = -w = -30y_1 - 20y_2 - 25y_3 - 22y_4$.

y_1	y_2	y_3	y_4	s_1	s_2	s_3	a_1	a_2	z	
1	0	1	0	−1	0	0	0	0	0	3000
0	1	0	1	0	−1	0	0	0	0	5000
[1]	1	0	0	0	0	0	1	0	0	5000
0	0	1	1	0	0	0	0	1	0	5000
2	6	5	4	0	0	1	0	0	0	40,000
30	20	25	22	0	0	0	0	0	1	0

Pivot on the 1 in row 3, column 1 to remove the *a* columns.

	y_1	y_2	y_3	y_4	s_1	s_2	s_3	a_1	a_2	z	
$-R_1 + R_3 \to R_1$	0	1	−1	0	1	0	0	1	0	0	2000
	0	1	0	1	0	−1	0	0	0	0	5000
	1	1	0	0	0	0	0	1	0	0	5000
	0	0	1	1	0	0	0	0	1	0	5000
$-2R_3 + R_5 \to R_5$	0	4	5	4	0	0	1	−2	0	0	30,000
$-30R_3 + R_6 \to R_6$	0	−10	25	22	0	0	0	−30	0	1	−150,000

Column a_1 can be removed since $a_1 = 0$.

y_1	y_2	y_3	y_4	s_1	s_2	s_3	a_2	z	
0	1	−1	0	1	0	0	0	0	2000
0	1	0	1	0	−1	0	0	0	5000
1	1	0	0	0	0	0	0	0	5000
0	0	[1]	1	0	0	0	1	0	5000
0	4	5	4	0	0	1	0	0	30,000
0	−10	25	22	0	0	0	0	1	−150,000

Pivot on the 1 in row 4, column 3.

	y_1	y_2	y_3	y_4	s_1	s_2	s_3	a_2	z	
$R_4 + R_1 \to R_1$	0	1	0	1	1	0	0	1	0	7000
	0	[1]	0	1	0	−1	0	0	0	5000
	1	1	0	0	0	0	0	0	0	5000
	0	0	1	1	0	0	0	1	0	5000
$-5R_4 + R_5 \to R_5$	0	4	0	−1	0	0	1	−5	0	5000
$-25R_4 + R_6 \to R_6$	0	−10	0	−3	0	0	0	−25	1	−275,000

Column a_2 can be removed since $a_2 = 0$. Pivot on the 1 in row 2, column 2 since the basic solution is not yet feasible ($s_2 = -5000$).

$$
\begin{array}{r}
\\
-R_2 + R_1 \to R_1 \\
\\
-R_2 + R_3 \to R_3 \\
\\
-4R_2 + R_5 \to R_5 \\
10R_2 + R_6 \to R_6
\end{array}
\begin{array}{cccccccc|c}
y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & z & \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2000 \\
0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 5000 \\
1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 5000 \\
0 & 0 & 0 & \boxed{-5} & 0 & 4 & 1 & 0 & -15,000 \\
\hline
0 & 0 & 0 & 7 & 0 & -10 & 0 & 1 & -225,000
\end{array}
$$

Pivot on the –5 in row 5, column 4.

$$
\begin{array}{r}
\\
\\
R_5 + 5R_2 \to R_2 \\
R_5 - 5R_3 \to R_3 \\
R_5 + 5R_4 \to R_4 \\
\\
7R_5 + 5R_6 \to R_6
\end{array}
\begin{array}{cccccccc|c}
y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & z & \\
0 & 0 & 0 & 0 & 1 & \boxed{1} & 0 & 0 & 2000 \\
0 & 5 & 0 & 0 & 0 & -1 & 1 & 0 & 10,000 \\
-5 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -15,000 \\
0 & 0 & 5 & 0 & 0 & 4 & 1 & 0 & 10,000 \\
0 & 0 & 0 & -5 & 0 & 4 & 1 & 0 & -15,000 \\
\hline
0 & 0 & 0 & 0 & 0 & -22 & 7 & 5 & -1,230,000
\end{array}
$$

Pivot on the 1 in row 1, column 6.

$$
\begin{array}{r}
\\
\\
R_1 + R_2 \to R_2 \\
R_1 + R_3 \to R_3 \\
-4R_1 + R_4 \to R_4 \\
-4R_1 + R_5 \to R_5 \\
22R_1 + R_6 \to R_6
\end{array}
\begin{array}{cccccccc|c}
y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & z & \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2000 \\
0 & 5 & 0 & 0 & 1 & 0 & 1 & 0 & 12,000 \\
-5 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -13,000 \\
0 & 0 & 5 & 0 & -4 & 0 & 1 & 0 & 2000 \\
0 & 0 & 0 & -5 & -4 & 0 & 1 & 0 & -23,000 \\
\hline
0 & 0 & 0 & 0 & 22 & 0 & 7 & 5 & -1,186,000
\end{array}
$$

$$
\begin{array}{r}
\\
\\
\frac{1}{5}R_2 \to R_2 \\
-\frac{1}{5}R_3 \to R_3 \\
\frac{1}{5}R_4 \to R_4 \\
-\frac{1}{5}R_5 \to R_5 \\
\frac{1}{5}R_6 \to R_6
\end{array}
\begin{array}{cccccccc|c}
y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & s_3 & z & \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2000 \\
0 & 1 & 0 & 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & 2400 \\
1 & 0 & 0 & 0 & -\frac{1}{5} & 0 & -\frac{1}{5} & 0 & 2600 \\
0 & 0 & 1 & 0 & -\frac{4}{5} & 0 & \frac{1}{5} & 0 & 400 \\
0 & 0 & 0 & 1 & \frac{4}{5} & 0 & -\frac{1}{5} & 0 & 4600 \\
\hline
0 & 0 & 0 & 0 & \frac{22}{5} & 0 & \frac{7}{5} & 1 & -237,200
\end{array}
$$

Here, $y_1 = 2600$, $y_2 = 2400$, $y_3 = 400$, $y_4 = 4600$, and $z = -w = 237,200$ Therefore, ship 2600 barrels from S_1 to D_1, 2400 barrels from S_1 to D_2, 400 barrels from S_2 to D_1, and 4600 barrels from S_2 to D_2 for a minimum cost of \$237,200.

25. **(a)** Let x_1 = the number of computers shipped from W_1 to D_1,
x_2 = the number of computers shipped from W_1 to D_2,
x_3 = the number of computers shipped from W_2 to D_1,
and x_4 = the number of computers shipped from W_2 to D_2.

Minimize $w = 14x_1 + 12x_2 + 12x_3 + 10x_4$

subject to:

$$\begin{aligned} x_1 + x_3 &\geq 32 \\ x_2 + x_4 &\geq 20 \\ x_1 + x_2 &\leq 25 \\ x_3 + x_4 &\leq 30 \end{aligned}$$

with $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0.$

Maximize
$z = -w = -14x_1 - 12x_2 - 12x_3 - 10x_4.$

The initial tableau looks like the following.

x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	z	
1	0	1	0	−1	0	0	0	0	32
0	1	0	1	0	−1	0	0	0	20
[1]	1	0	0	0	0	1	0	0	25
0	0	1	1	0	0	0	1	0	30
14	12	12	10	0	0	0	0	1	0

The variable s_1 is negative; we pivot on the 1 in row 3 of column 1.

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	z	
$-R_3 + R_1 \to R_1$	0	−1	[1]	0	−1	0	−1	0	0	7
	0	1	0	1	0	−1	0	0	0	20
	1	1	0	0	0	0	1	0	0	25
	0	0	1	1	0	0	0	1	0	30
$-14R_3 + R_5 \to R_5$	0	−2	12	10	0	0	−14	0	1	−350

The variable s_1 is still negative; we pivot on the 1 in row 1 of column 3.

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	z	
	0	−1	1	0	−1	0	−1	0	0	7
	0	[1]	0	1	0	−1	0	0	0	20
	1	1	0	0	0	0	1	0	0	25
$-R_1 + R_4 \to R_4$	0	1	0	1	1	0	1	1	0	23
$-12R_1 + R_5 \to R_5$	0	10	0	10	12	0	−2	0	1	−434

The variable s_2 is still negative; we pivot on the 1 in row 2 of column 2.

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	z	
$R_2 + R_1 \to R_1$	0	0	1	1	−1	−1	−1	0	0	27
	0	1	0	1	0	−1	0	0	0	20
$-R_2 + R_3 \to R_3$	1	0	0	−1	0	1	1	0	0	5
$-R_2 + R_4 \to R_4$	0	0	0	0	1	1	[1]	1	0	3
$-10R_2 + R_5 \to R_5$	0	0	0	0	12	10	−2	0	1	−634

Now we eliminate the only negative indicator by pivoting on the 1 in row 4 of column 7.

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	z	
$R_4 + R_1 \to R_1$	0	0	1	1	0	0	0	1	0	30
	0	1	0	1	0	0	0	0	0	20
$-R_4 + R_3 \to R_3$	1	0	0	−1	−1	0	0	−1	0	2
	0	0	0	0	1	1	1	1	0	3
$2R_4 + R_5 \to R_5$	0	0	0	0	14	12	0	2	1	−628

From this we can read the solution: Ship 2 computers from W_1 to D_1, ship 20 computers from W_1 to D_2, ship 30 computers from W_2 to D_1, and 0 computers from W_2 to D_2. The resulting minimum cost is \$628.

(b) From the final tableau, $s_3 = 3$. Therefore, warehouse W_1 has three more computers that it could ship.

26. Let x_1 = the amount invested in government securities,
x_2 = the amount invested in municipal bonds,
and x_3 = the amount invested in mutual funds.

Maximize $z = 0.07x_1 + 0.06x_2 + 0.10x_3$

subject to:

$$\begin{aligned} x_1 + x_2 + x_3 &= 100{,}000 \\ x_1 &\geq 40{,}000 \\ x_2 + x_3 &\geq 50{,}000 \\ 0.02x_1 + 0.01x_2 + 0.03x_3 &\leq 2400 \end{aligned}$$

or $\quad 2x_1 + x_2 + 3x_3 \leq 240{,}000$

with $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

x_1	x_2	x_3	s_1	s_2	s_3	s_4	z	
1	1	1	1	0	0	0	0	100,000
[1]	0	0	0	−1	0	0	0	40,000
0	1	1	0	0	−1	0	0	50,000
2	1	3	0	0	0	1	0	240,000
−0.07	−0.06	−0.10	0	0	0	0	1	0

Since s_2 is negative we pivot on the 1 in row 2, column 1.

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	z	
$-R_2 + R_1 \to R_1$	0	1	1	1	1	0	0	0	60,000
	1	0	0	0	−1	0	0	0	40,000
	0	[1]	1	0	0	−1	0	0	50,000
$-2R_2 + R_4 \to R_4$	0	1	3	0	2	0	1	0	160,000
$7R_2 + 100R_5 \to R_5$	0	−6	−10	0	−7	0	0	100	280,000

s_2 is no longer basic, but s_3 is negative, so we pivot on the 1 in row 3 of column 2.

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	z	
$-R_3 + R_1 \to R_1$	0	0	0	1	[1]	1	0	0	10,000
	1	0	0	0	−1	0	0	0	40,000
	0	1	1	0	0	−1	0	0	50,000
$-R_3 + R_4 \to R_4$	0	0	2	0	2	1	1	0	110,000
$6R_3 + R_5 \to R_5$	0	0	−4	0	−7	−6	0	100	580,000

Now all the basic variables are nonnegative, so we work on the most negative indicator in the last row, which is −7. We pivot on the 1 in the first row of column 5.

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	z	
	0	0	0	1	1	1	0	0	10,000
$R_1 + R_2 \to R_2$	1	0	0	1	0	1	0	0	50,000
	0	1	1	0	0	−1	0	0	50,000
$-2R_1 + R_4 \to R_4$	0	0	[2]	−2	0	−1	1	0	90,000
$7R_1 + R_5 \to R_5$	0	0	−4	7	0	1	0	100	650,000

Now we have just a single negative indicator, so we pivot on the 2 in row 4 of column 3.

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	z	
	0	0	0	1	1	[1]	0	0	10,000
	1	0	0	1	0	1	0	0	50,000
$-R_4 + 2R_3 \to R_3$	0	2	0	2	0	−1	−1	0	10,000
	0	0	2	−2	0	−1	1	0	90,000
$2R_4 + R_5 \to R_5$	0	0	0	3	0	−1	2	100	830,000

There is still a negative indicator, so we pivot on the 1 in row 1 of column 6.

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	z	
	0	0	0	1	1	1	0	0	10,000
$-R_1 + R_2 \to R_2$	1	0	0	0	−1	0	0	0	40,000
$R_1 + R_3 \to R_3$	0	2	0	3	1	0	−1	0	20,000
$R_1 + R_4 \to R_4$	0	0	2	−1	1	0	1	0	100,000
$R_1 + R_5 \to R_5$	0	0	0	4	1	0	2	100	840,000

We now read the solution.

$$x_1 = 40{,}000,$$
$$x_2 = \frac{20{,}000}{2} = 10{,}000,$$
$$x_3 = \frac{100{,}000}{2} = 50{,}000,$$

Invest $40,000 in government securities, $10,000 in municipal bonds, and $50,000 in mutual funds; the maximum interest is $8400.

27. Let x_1 = the number of million dollars for home loans

and x_2 = the number of million dollars for commercial loans.

Maximize $z = 0.12x_1 + 0.10x_2$

subject to:
$$x_1 \ge 4x_2 \text{ or } x_1 - 4x_2 \ge 0$$
$$x_1 + x_2 \ge 10$$
$$3x_1 + 2x_2 \le 72$$
$$x_1 + x_2 \le 25$$

with $x_1 \ge 0, x_2 \ge 0.$

x_1	x_2	s_1	s_2	s_3	s_4	z	
1	−4	−1	0	0	0	0	0
1	1	0	−1	0	0	0	10
3	2	0	0	1	0	0	72
1	1	0	0	0	1	0	25
−0.12	−0.10	0	0	0	0	1	0

Eliminate the decimals in the last row by multiplying by 100

x_1	x_2	s_1	s_2	s_3	s_4	z	
1	−4	−1	0	0	0	0	0
[1]	1	0	−1	0	0	0	10
3	2	0	0	1	0	0	72
1	1	0	0	0	1	0	25
−12	−10	0	0	0	0	100	0

Pivot on the 1 in row 2, column 1.

	x_1	x_2	s_1	s_2	s_3	s_4	z	
$-R_2 + R_1 \to R_1$	0	−5	−1	1	0	0	0	−10
	1	1	0	−1	0	0	0	10
$-3R_2 + R_3 \to R_3$	0	−1	0	[3]	1	0	0	42
$-R_2 + R_4 \to R_4$	0	0	0	1	0	1	0	15
$12R_2 + R_5 \to R_5$	0	2	0	−12	0	0	100	120

Pivot on the 3 in row 3, column 4.

$$\begin{array}{r} \\ -R_3+3R_1\to R_1 \\ R_3+3R_2\to R_2 \\ \\ -R_3+3R_4\to R_4 \\ 4R_3+R_5\to R_5 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & z & \\ \hline 0 & -14 & -3 & 0 & -1 & 0 & 0 & -72 \\ 3 & 2 & 0 & 0 & -2 & 0 & 0 & 72 \\ 0 & -1 & 0 & 3 & 1 & 0 & 0 & 42 \\ 0 & \boxed{1} & 0 & 0 & -1 & 3 & 0 & 3 \\ \hline 0 & -2 & 0 & 0 & 4 & 0 & 100 & 288 \end{array}$$

Pivot on the 1 in row 4, column 2.

$$\begin{array}{r} \\ 14R_4+R_1\to R_1 \\ -2R_4+R_2\to R_2 \\ R_4+R_3\to R_3 \\ \\ 2R_4+R_5\to R_5 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & z & \\ \hline 0 & 0 & -3 & 0 & -15 & 42 & 0 & -30 \\ 3 & 0 & 0 & 0 & 0 & -6 & 0 & 66 \\ 0 & 0 & 0 & 3 & 0 & 3 & 0 & 45 \\ 0 & 1 & 0 & 0 & -1 & 3 & 0 & 3 \\ \hline 0 & 0 & 0 & 0 & 2 & 6 & 100 & 294 \end{array}$$

Create a 1 in the columns corresponding to x_1 and z.

$$\begin{array}{r} \\ \\ \frac{1}{3}R_2\to R_2 \\ \\ \\ \frac{1}{100}R_5\to R_5 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & z & \\ \hline 0 & 0 & -3 & 0 & -15 & 42 & 0 & -30 \\ 1 & 0 & 0 & 0 & 0 & -2 & 0 & 22 \\ 0 & 0 & 0 & 3 & 0 & 3 & 0 & 45 \\ 0 & 1 & 0 & 0 & -1 & 3 & 0 & 3 \\ \hline 0 & 0 & 0 & 0 & 0.02 & 0.06 & 1 & 2.94 \end{array}$$

Here, $x_1 = 22$, $x_2 = 3$, and $z = 2.94$. Make \$22 million (\$22,000,000) in home loans and \$3 million (\$3,000,000) in commercial loans for a maximum return of \$2.94 million, or \$2,940,000.

28. Let $x_1 =$ the number of pounds of bluegrass seed,

$x_2 =$ the number of pounds of rye seed,

and $x_3 =$ the number of pounds of Bermuda seed.

If each batch must contain at least 25% bluegrass seed, then

$$y_1 \geq 0.25(y_1 + y_2 + y_3)$$
$$0.75y_1 - 0.25y_2 - 0.25y_3 \geq 0.$$

And if the amount of Bermuda must be no more than $\frac{2}{3}$ the amount of rye, then

$$y_3 \leq \frac{2}{3}y_2$$
$$-2y_2 + 3y_3 = 0.$$

Using these forms for our constraints, we can now state the problem as follows.

Minimize $w = 16y_1 + 14y_2 + 12y_3$

subject to:
$$\begin{aligned} 0.75y_1 - 0.25y_2 - 0.25y_3 &\geq 0 \\ -2y_2 + 3y_3 &\leq 0 \\ y_1 + y_2 + y_3 &\geq 6000 \end{aligned}$$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$

The initial simplex tableau is

$$\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & z & \\ \hline 0.75 & -0.25 & -0.25 & -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 3 & 0 & 1 & 0 & 0 & 0 \\ \boxed{1} & 1 & 1 & 0 & 0 & -1 & 0 & 6000 \\ \hline 16 & 14 & 12 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Since $s_3 = -6000$, the basic solution is not feasible. So pivot on the 1 in row 3, column 1.

$$\begin{array}{r} \\ 0.75R_3-R_1\to R_1 \\ \\ \\ -16R_3+R_4\to R_4 \end{array}\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & z & \\ \hline 0 & 1 & 1 & 1 & 0 & -0.75 & 0 & 4500 \\ 0 & -2 & \boxed{3} & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & -1 & 0 & 6000 \\ \hline 0 & -2 & -4 & 0 & 0 & 16 & 1 & -96{,}000 \end{array}$$

All of the variables are now nonnegative so choose the pivot by locating the most negative number in the bottom row and forming the quotients. Pivot on the 3 in row 2, column 3.

$$\begin{array}{r} \\ -R_2+3R_1\to R_1 \\ \\ -R_2+3R_3\to R_3 \\ 4R_2+3R_4\to R_4 \end{array}\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & z & \\ \hline 0 & \boxed{5} & 0 & 3 & -1 & -2.25 & 0 & 13{,}500 \\ 0 & -2 & 3 & 0 & 1 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & -1 & -3 & 0 & 18{,}000 \\ \hline 0 & -14 & 0 & 0 & 4 & 48 & 3 & -288{,}000 \end{array}$$

Pivot on the 5 in row 1, column 2.

$$\begin{array}{r} \\ \\ 2R_1+5R_2\to R_2 \\ -R_1+R_3\to R_3 \\ 14R_1+5R_4\to R_4 \end{array}\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & z & \\ \hline 0 & 5 & 0 & 3 & -1 & -2.25 & 0 & 13{,}500 \\ 0 & 0 & 15 & 6 & 3 & -4.5 & 0 & 27{,}000 \\ 3 & 0 & 0 & -3 & 0 & -0.75 & 0 & 4500 \\ \hline 0 & 0 & 0 & 42 & 6 & 208.5 & 15 & -1{,}251{,}000 \end{array}$$

Create a 1 in the columns corresponding to y_1, y_2, y_3, and z.

$$\begin{array}{r} \\ \frac{1}{5}R_1\to R_1 \\ \frac{1}{15}R_2\to R_2 \\ \frac{1}{3}R_3\to R_3 \\ \frac{1}{15}R_4\to R_4 \end{array}\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & z & \\ \hline 0 & 1 & 0 & 0.6 & -0.2 & -0.45 & 0 & 2700 \\ 0 & 0 & 1 & 0.4 & 0.2 & -0.3 & 0 & 1800 \\ 1 & 0 & 0 & -1 & 0 & -0.25 & 0 & 1500 \\ \hline 0 & 0 & 0 & 2.8 & 0.4 & 13.9 & 1 & -83{,}400 \end{array}$$

Here, $y_1 = 1500$, $y_2 = 2700$, $y_3 = 1800$, and $z = -w = 83{,}400$. Therefore, use 1500 lb of bluegrass, 2700 lb of rye, and 1800 lb of Bermuda for a minimum cost of \$834.

29. Let x_1 = the number of pounds of bluegrass seed,

x_2 = the number of pounds of rye seed,

and x_3 = the number of pounds of Bermuda seed.

If each batch must contain at least 25% bluegrass seed, then

$$y_1 \geq 0.25(y_1 + y_2 + y_3)$$
$$0.75y_1 - 0.25y_2 - 0.25y_3 \geq 0.$$

And if the amount of Bermuda must be no more than $\frac{2}{3}$ the amount of rye, then

$$y_3 \leq \frac{2}{3}y_2$$
$$-2y_2 + 3y_3 = 0.$$

Using these forms for our constraints, we can now state the problem as follows.

Minimize $w = 16y_1 + 14y_2 + 12y_3$

subject to:
$$0.75y_1 - 0.25y_2 - 0.25y_3 \geq 0$$
$$-\ 2y_2 + 3y_3 \leq 0$$
$$y_1 + y_2 + y_3 = 6000$$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$

The initial simplex tableau is

y_1	y_2	y_3	s_1	s_2	a	z	
0.75	−0.25	−0.25	−1	0	0	0	0
0	−2	3	0	1	0	0	0
[1]	1	1	0	0	1	0	6000
16	14	12	0	0	0	1	0

First eliminate the artificial variable a. Pivot on the 1 in row 3, column 1.

	y_1	y_2	y_3	s_1	s_2	a	z	
$0.75R_3 - R_1 \to R_1$	0	1	1	1	0	0.75	0	4500
	0	−2	3	0	1	0	0	0
	1	1	1	0	0	1	0	6000
$-16R_3 + R_4 \to R_4$	0	−2	−4	0	0	−16	1	−96,000

Since $a = 0$, we can drop the a column.

y_1	y_2	y_3	s_1	s_2	z	
0	1	1	1	0	0	4500
0	−2	[3]	0	1	0	0
1	1	1	0	0	0	6000
0	−2	−4	0	0	1	−96,000

Pivot on the 3 in row 2, column 3.

	y_1	y_2	y_3	s_1	s_2	z	
$-R_2 + 3R_1 \to R_1$	0	[5]	0	3	−1	0	13,500
	0	−2	3	0	1	0	0
$-R_2 + 3R_3 \to R_3$	3	5	0	0	−1	0	18,000
$4R_2 + 3R_4 \to R_4$	0	−14	0	0	4	3	−288,000

Pivot on the 5 in row 1, column 2.

	y_1	y_2	y_3	s_1	s_2	z	
	0	5	0	3	−1	0	13,500
$2R_1 + 5R_2 \to R_2$	0	0	15	6	3	0	27,000
$-R_1 + R_3 \to R_3$	3	0	0	−3	0	0	4500
$14R_1 + 5R_4 \to R_4$	0	0	0	42	6	15	−1,251,000

Create a 1 in the columns corresponding to y_1, y_2, y_3, and z.

	y_1	y_2	y_3	s_1	s_2	z	
$\frac{1}{5}R_1 \to R_1$	0	1	0	0.6	−0.2	0	2700
$\frac{1}{15}R_2 \to R_2$	0	0	1	0.4	0.2	0	1800
$\frac{1}{3}R_3 \to R_3$	1	0	0	−1	0	0	1500
$\frac{1}{15}R_4 \to R_4$	0	0	0	2.8	0.4	1	−83,400

Here, $y_1 = 1500$, $y_2 = 2700$, $y_3 = 1800$, and $z = -w = 83,400$. Therefore, use 1500 lb of bluegrass, 2700 lb of rye, and 1800 lb of Bermuda for a minimum cost of \$834.

30. Let y_1 = the number of gallons of ingredient 1,

y_2 = the number of gallons of ingredient 2,

y_3 = the number of gallons of ingredient 3,

y_4 = the number of gallons of ingredient 4,

y_5 = the number of gallons of ingredient 5,

and y_6 = the number of gallons of water.

Note that 10%(15,000) = 1500, and 0.01(15,000) = 150.

The problem becomes:

Minimize

$$w = 0.48y_1 + 0.32y_2 + 0.53y_3 + 0.28y_4 + 0.43y_5 + 0.04y_6$$

subject to:

$$0.28y_1 + 0.19y_2 + 0.43y_3 + 0.57y_4 + 0.22y_5 \leq 1500$$
$$y_3 + y_4 \geq 150$$
$$y_2 + y_5 \geq 150$$
$$y_1 + y_4 \geq 150$$
$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,000$$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0, y_5 \geq 0,$ $y_6 \geq 0.$

This exercise should be solved by graphing calculator or computer methods. The answer is to use 0 gal of ingredient 1, 150 gal of 2, 0 gal of 3, 150 gal of 4, 0 gal of 5, and 14,700 gal of water for a minimum cost of $678.

31. Let x_1 = the amount of chemical I,

x_2 = the amount of chemical II,

and x_3 = the amount of chemical III.

Minimize $w = 1.09x_1 + 0.87x_2 + 0.65x_3$

subject to:

$$x_1 + x_2 + x_3 \geq 750$$
$$0.09x_1 + 0.04x_2 + 0.03x_3 \geq 30$$
$$3x_2 = 4x_3$$

with $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$

We follow the suggestion in the note in the text to reduce the number of variables by using the fact that $x_3 = 0.75x_2$ to express our constraints as follows:

Minimize $w = 1.09x_1 + 1.3575x_2$

subject to

$$x_1 + 1.75x_2 \geq 750$$
$$0.09x_1 + 0.0625x_2 \geq 30$$

We maximize $z = -w$ and after multiplying the second constraint through by 100, our initial tableau is the following:

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline 1 & 1.75 & -1 & 0 & 0 & 750 \\ \boxed{9} & 6.25 & 0 & -100 & 0 & 3000 \\ \hline 1.09 & 1.3575 & 0 & 0 & 1 & 0 \end{array}$$

Since s_1 is negative, we look for a pivot in the first column, and choose 9 because it has the smallest ratio with the corresponding entry in the last column.

$$\begin{array}{r} -R_2 + 9R_1 \to R_1 \\ \\ -1.09R_2 + 9R_3 \to R_3 \end{array} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline 0 & \boxed{9.5} & -9 & 100 & 0 & 3750 \\ 9 & 6.25 & 0 & -100 & 0 & 3000 \\ \hline 0 & 5.405 & 0 & 109 & 9 & -3270 \end{array}$$

s_1 is still negative so we pivot on the 9.5 in column 2.

$$\begin{array}{r} \\ -6.25R_1 + 9.5R_2 \to R_2 \\ -5.405R_1 + 9.5R_3 \to R_3 \end{array} \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline 0 & 9.5 & -9 & 100 & 0 & 3750 \\ 85.5 & 0 & 56.25 & -1575 & 0 & 5062.5 \\ \hline 0 & 0 & 48.645 & 495 & 85.5 & -51{,}333.75 \end{array}$$

This tableau yields the following solution.

$$x_1 = \frac{5062.5}{85.5} = 59.21,\ x_2 = \frac{3750}{9.5} = 394.74,$$
$$x_3 = \frac{3750}{9.5} \cdot \frac{3}{4} = 296.05$$
$$\text{Minimum} = -\left(\frac{-51{,}333.75}{85.5}\right) = 600.39$$

So use 59.21 kg of chemical I, 394.74 kg of chemical II, and 296.05 kg of chemical III, for a minimum cost of $600.39.

32. Let y_1 = the number of ounces of ingredient I,

y_2 = the number of ounces of ingredient II,

and y_3 = the number of ounces of ingredient III.

Expressing the problem in cents, the problem is:

Minimize $w = 30y_1 + 9y_2 + 27y_3$

subject to

$$y_1 + y_2 + y_3 \geq 10$$
$$y_1 + y_2 + y_3 \leq 15$$
$$y_1 \geq \frac{1}{4}y_2$$
$$y_3 \geq y_1$$

with $y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0.$

Rewrite the last two inequalities so that the problem becomes:

Minimize $w = 30y_1 + 9y_2 + 27y_3$

subject to:

$$y_1 + y_2 + y_3 \geq 10$$
$$y_1 + y_2 + y_3 \leq 15$$
$$-4y_1 + y_2 \leq 0$$
$$y_1 - y_3 \leq 0$$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$

We maximize $z = -w$ and have the following initial tableau.

y_1	y_2	y_3	s_1	s_2	s_3	s_4	z	
1	1	1	−1	0	0	0	0	10
1	1	1	0	1	0	0	0	15
−4	1	0	0	0	1	0	0	0
[1]	0	−1	0	0	0	1	0	0
30	9	27	0	0	0	0	1	0

Because the solution is not feasible ($s_1 = -10$), pivot on the 1 in row 4, column 1.

	y_1	y_2	y_3	s_1	s_2	s_3	s_4	z	
$-R_4 + R_1 \to R_1$	0	1	2	−1	0	0	−1	0	10
$-R_4 + R_2 \to R_2$	0	1	2	0	1	0	−1	0	15
$4R_4 + R_3 \to R_3$	0	[1]	−4	0	0	1	4	0	0
	1	0	−1	0	0	0	1	0	0
$-30R_4 + R_5 \to R_5$	0	9	57	0	0	0	−30	1	0

Because the solution is still not feasible ($s_1 = -10$), pivot on the 1 in row 3, column 2.

	y_1	y_2	y_3	s_1	s_2	s_3	s_4	z	
$-R_3 + R_1 \to R_1$	0	1	[6]	−1	0	−1	−5	0	10
$-R_3 + R_2 \to R_2$	0	1	6	0	1	−1	−5	0	15
	0	1	−4	0	0	1	4	0	0
	1	0	−1	0	0	0	1	0	0
$-9R_3 + R_5 \to R_5$	0	0	93	0	0	−9	−66	1	0

Because the solution is still not feasible ($s_1 = -10$), pivot on the 6 in row 1, column 3.

	y_1	y_2	y_3	s_1	s_2	s_3	s_4	z	
	0	0	6	−1	0	−1	−5	0	10
$-R_1 + R_2 \to R_2$	0	0	0	1	1	0	0	0	5
$2R_1 + 3R_3 \to R_3$	0	3	0	−2	0	1	2	0	20
$R_1 + 6R_4 \to R_4$	6	0	0	−1	0	−1	1	0	10
$-31R_3 + 2R_5 \to R_5$	0	0	0	31	0	13	23	2	−310

Create a 1 in the columns corresponding to y_1, y_2, y_2, and z.

	y_1	y_2	y_3	s_1	s_2	s_3	s_4	z	
	0	0	1	$-\frac{1}{6}$	0	$-\frac{1}{6}$	$-\frac{5}{6}$	0	$\frac{5}{3}$
$\frac{1}{6}R_2 \to R_2$	0	0	0	1	1	0	0	0	5
$\frac{1}{3}R_3 \to R_3$	0	1	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{20}{3}$
$\frac{1}{6}R_4 \to R_4$	1	0	0	$-\frac{1}{6}$	0	$-\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{5}{3}$
$\frac{1}{2}R_5 \to R_5$	0	0	0	$\frac{31}{2}$	0	$\frac{13}{2}$	$\frac{23}{2}$	1	−155

Here $y_1 = \frac{5}{3}$, $y_2 = \frac{20}{3}$, $y_3 = \frac{5}{3}$, and $w = -z = 155$.

Therefore, the additive should consist of $\frac{5}{3}$ oz of ingredient I, $\frac{20}{3}$ oz of ingredient II, and $\frac{5}{3}$ oz of ingredient III, for a minimum cost of 155¢/gal, or $1.55/gal. The amount of additive that should be used per gallon of gasoline is $\frac{5}{3} + \frac{20}{3} + \frac{5}{3} = 10$ oz.

33. **(a)** Let x_1 = the number of hours spent doing calisthenics,

x_2 = the number of hours spent swimming,

and x_3 = the number of hours spent playing the drums.

The problem can be stated as follows.

Maximize $z = 388x_1 + 518x_2 + 345x_3$

subject to:

$$x_1 + x_2 + x_3 \le 10$$
$$x_1 - 2x_2 + x_3 \ge 0$$
$$x_3 \le 4$$
$$x_3 \ge 1.$$

with $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$

Write the initial simplex tableau.

x_1	x_2	x_3	s_1	s_2	s_3	s_4	z	
1	1	1	1	0	0	0	0	10
[1]	−2	1	0	−1	0	0	0	0
0	0	1	0	0	1	0	0	4
0	0	1	0	0	0	−1	0	1
−388	−518	−345	0	0	0	0	1	0

Since s_2 is negative, we pivot on the 1 in row 2 of column 1.

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	z	
$-R_2 + R_1 \to R_1$	0	3	0	1	1	0	0	0	10
	1	−2	1	0	−1	0	0	0	0
	0	0	1	0	0	1	0	0	4
	0	0	[1]	0	0	0	−1	0	1
$388R_2 + R_5 \to R_5$	0	−1294	43	0	−388	0	0	1	0

Since s_4 is negative, we pivot on the 1 in row 4 of column 3.

$$\begin{array}{r} \\ \\ -R_4 + R_2 \to R_2 \\ -R_4 + R_3 \to R_3 \\ \\ -43R_4 + R_5 \to R_5 \end{array}
\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z & \\ 0 & \boxed{3} & 0 & 1 & 1 & 0 & 0 & 0 & 10 \\ 1 & -2 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ \hline 0 & -1294 & 0 & 0 & -388 & 0 & 43 & 1 & -43 \end{array}$$

Now we work on the column with the most negative indicator and pivot on 3 in column 2.

$$\begin{array}{r} \\ \\ 2R_1 + 3R_2 \to R_2 \\ \\ \\ 1294R_1 + 3R_5 \to R_5 \end{array}
\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & z & \\ 0 & 3 & 0 & 1 & 1 & 0 & 0 & 0 & 10 \\ 3 & 0 & 0 & 2 & -1 & 0 & 3 & 0 & 17 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1294 & 130 & 0 & 129 & 3 & 12{,}811 \end{array}$$

This tableau gives the solution: Joe should do 17/3 hours of calisthenics, 10/3 hours of swimming, and 1 hour of playing the drums, for a maximum calorie expenditure of 12,811/3 or $4270\frac{1}{3}$ calories.

Chapter 4 Review Exercises

1. True
2. False
3. True
4. False
5. False
6. True
7. True
8. False
9. False
10. True
11. False
12. True
13. False
14. True
15. The simplex method should be used for problems with more than two variables or problems with two variables and many constants.
16. If a surplus variable cannot be made nonnegative, then the inequality which represents one of the constraints can never exist. This means that no solution is possible.
17. **(a)** Maximize $z = 2x_1 + 7x_2$

subject to:
$$\begin{aligned} 4x_1 + 6x_2 &\le 60 \\ 3x_1 + x_2 &\le 18 \\ 2x_1 + 5x_2 &\le 20 \\ x_1 + x_2 &\le 15 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0.$

Adding slack variables s_1, s_2, s_3, and s_4, we obtain the following equations.

$$\begin{aligned} 4x_1 + 6x_2 + s_1 &= 60 \\ 3x_1 + x_2 + s_2 &= 18 \\ 2x_1 + 5x_2 + s_3 &= 20 \\ x_1 + x_2 + s_4 &= 15. \end{aligned}$$

(b) The initial simplex tableau is as follows.

$$\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & z & \\ 4 & 6 & 1 & 0 & 0 & 0 & 0 & 60 \\ 3 & 1 & 0 & 1 & 0 & 0 & 0 & 18 \\ 2 & 5 & 0 & 0 & 1 & 0 & 0 & 20 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 15 \\ \hline -2 & -7 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

18. Maximize $z = 25x_1 + 30x_2$

subject to:
$$\begin{aligned} 3x_1 + 5x_2 &\le 47 \\ x_1 + x_2 &\le 25 \\ 5x_1 + 2x_2 &\le 35 \\ 2x_1 + x_2 &\le 30 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0.$

(a) Add $s_1, s_2, s_3,$ and s_4 as slack variables to obtain

$$\begin{aligned} 3x_1 + 5x_2 + s_1 &= 47 \\ x_1 + x_2 + s_2 &= 25 \\ 5x_1 + 2x_2 + s_3 &= 35 \\ 2x_1 + x_2 + s_4 &= 30. \end{aligned}$$

(b) The initial tableau is

$$\begin{array}{c} \\ \\ \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & z & \\ \hline 3 & 5 & 1 & 0 & 0 & 0 & 0 & 47 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 25 \\ 5 & 2 & 0 & 0 & 1 & 0 & 0 & 35 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 30 \\ \hline -25 & -30 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

19. Maximize $z = 5x_1 + 8x_2 + 6x_3$

subject to:
$$\begin{aligned} x_1 + x_2 + x_3 &\le 90 \\ 2x_1 + 5x_2 + x_3 &\le 120 \\ x_1 + 3x_2 &\ge 80 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

(a) Adding the slack variables s_1 and s_2 and subtracting the surplus variable s_3, we obtain the following equations:

$$\begin{aligned} x_1 + x_2 + x_3 + s_1 &= 90 \\ 2x_1 + 5x_2 + x_3 + s_2 &= 120 \\ x_1 + 3x_2 - s_3 &= 80. \end{aligned}$$

(b) The initial tableau is

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 90 \\ 2 & 5 & 1 & 0 & 1 & 0 & 0 & 120 \\ 1 & 3 & 0 & 0 & 0 & -1 & 0 & 80 \\ \hline -5 & -8 & -6 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

20. Maximize $z = 4x_1 + 6x_2 + 8x_3$

subject to:
$$\begin{aligned} x_1 + x_2 + 2x_3 &\ge 200 \\ 8x_1 + 6x_3 &\le 400 \\ 3x_1 + 5x_2 + x_3 &\le 300 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

(a) Introduce s_1 as a surplus variable and s_2 and s_3 as slack variables to obtain the following equations.

$$\begin{aligned} x_1 + x_2 + 2x_3 - s_1 &= 200 \\ 8x_1 + 6x_3 + s_2 &= 400 \\ 3x_1 + 5x_2 + x_3 + s_3 &= 300. \end{aligned}$$

(b) The initial tableau is as follows.

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline 1 & 1 & 2 & -1 & 0 & 0 & 0 & 200 \\ 8 & 0 & 6 & 0 & 1 & 0 & 0 & 400 \\ 3 & 5 & 1 & 0 & 0 & 1 & 0 & 300 \\ \hline -4 & -6 & -8 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

21.
$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline 4 & 5 & 2 & 1 & 0 & 0 & 18 \\ 2 & 8 & \boxed{6} & 0 & 1 & 0 & 24 \\ \hline -5 & -3 & -6 & 0 & 0 & 1 & 0 \end{array}.$$

The most negative entry in the last row is -6, and the smaller of the two quotients is $\frac{24}{6} = 4.$ Hence, the 6 in row 2, column 3, is the first pivot. Performing row transformations leads to the following tableau.

$$\begin{array}{r} \\ -R_2 + 3R_1 \to R_1 \\ \\ R_2 + R_3 \to R_3 \end{array}\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline \boxed{10} & 7 & 0 & 3 & -1 & 0 & 30 \\ 2 & 8 & 6 & 0 & 1 & 0 & 24 \\ \hline -3 & 5 & 0 & 0 & 1 & 1 & 24 \end{array}.$$

Pivot on the 10 in row 1, column 1.

$$\begin{array}{r} \\ \\ -R_1 + 5R_2 \to R_2 \\ 3R_1 + 10R_3 \to R_3 \end{array}\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline 10 & 7 & 0 & 3 & -1 & 0 & 30 \\ 0 & 33 & 30 & -3 & 6 & 0 & 90 \\ \hline 0 & 71 & 0 & 9 & 7 & 10 & 330 \end{array}.$$

Create a 1 in the columns corresponding to $x_1, x_3,$ and $z.$

$$\begin{array}{r} \\ \frac{1}{10}R_1 \to R_1 \\ \frac{1}{30}R_2 \to R_2 \\ \frac{1}{10}R_3 \to R_3 \end{array}\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \hline 1 & \frac{7}{10} & 0 & \frac{3}{10} & -\frac{1}{10} & 0 & 3 \\ 0 & \frac{11}{10} & 1 & -\frac{1}{10} & \frac{1}{5} & 0 & 3 \\ \hline 0 & \frac{71}{10} & 0 & \frac{9}{10} & \frac{7}{10} & 10 & 33 \end{array}.$$

The maximum value is 33 when $x_1 = 3, x_2 = 0,$ $x_3 = 3, s_1 = 0,$ and $s_2 = 0.$

22.
$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline 2 & \boxed{7} & 1 & 0 & 0 & 14 \\ 2 & 3 & 0 & 1 & 0 & 10 \\ \hline -2 & -4 & 0 & 0 & 1 & 0 \end{array}$$

The most negative indicator is in the second column. The smaller quotient is $\frac{14}{7} = 2$. Pivot on the 7 in row 1, column 2.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ \hline & 2 & 7 & 1 & 0 & 0 & 14 \\ -3R_1 + 7R_2 \to R_2 & \boxed{8} & 0 & -3 & 7 & 0 & 28 \\ \hline 4R_1 + 7R_3 \to R_3 & -6 & 0 & 4 & 0 & 7 & 56 \end{array}$$

Pivot on the 8 in row 2, column 1.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ \hline -R_2 + 4R_1 \to R_1 & 0 & 28 & 7 & -7 & 0 & 28 \\ & 8 & 0 & -3 & 7 & 0 & 28 \\ \hline 3R_2 + 4R_3 \to R_3 & 0 & 0 & 7 & 21 & 28 & 308 \end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, and z.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ \hline \frac{1}{28}R_1 \to R_1 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & 0 & 1 \\ \frac{1}{8}R_2 \to R_2 & 1 & 0 & -\frac{3}{8} & \frac{7}{8} & 0 & \frac{7}{2} \\ \hline \frac{1}{28}R_3 \to R_3 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 1 & 11 \end{array}$$

The maximum value is 11 when $x_1 = \frac{7}{2}, x_2 = 1, s_1 = 0,$ and $s_2 = 0$.

23.

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline 1 & 2 & 2 & 1 & 0 & 0 & 0 & 50 \\ \boxed{3} & 1 & 0 & 0 & 1 & 0 & 0 & 20 \\ 1 & 0 & 2 & 0 & 0 & -1 & 0 & 15 \\ \hline -5 & -3 & -2 & 0 & 0 & 0 & 1 & 0 \end{array}$$

The initial basic solution is not feasible since $s_3 = -15$. In the third row where the negative coefficient appears, the nonnegative entry that appears farthest to the left is the 1 in the first column. In the first column, the smallest nonnegative quotient is $\frac{20}{3}$. Pivot on the 3 in row 2, column 1.

$$\begin{array}{r|ccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline -R_2 + 3R_1 \to R_1 & 0 & 5 & 6 & 3 & -1 & 0 & 0 & 130 \\ & 3 & 1 & 0 & 0 & 1 & 0 & 0 & 20 \\ -R_2 + 3R_3 \to R_3 & 0 & -1 & \boxed{6} & 0 & -1 & -3 & 0 & 25 \\ \hline 5R_2 + 3R_4 \to R_4 & 0 & -4 & -6 & 0 & 5 & 0 & 3 & 100 \end{array}$$

Continue by pivoting on each boxed entry.

$$\begin{array}{r|ccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline -R_3 + R_2 \to R_1 & 0 & \boxed{6} & 0 & 3 & 0 & 3 & 0 & 105 \\ & 3 & 1 & 0 & 0 & 1 & 0 & 0 & 20 \\ & 0 & -1 & 6 & 0 & -1 & -3 & 0 & 25 \\ \hline R_3 + R_4 \to R_4 & 0 & -5 & 0 & 0 & 4 & -3 & 3 & 125 \end{array}$$

The basic solution is now feasible, but there are negative indicators.

Continue pivoting.

$$\begin{array}{r|ccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline & 0 & 6 & 0 & 3 & 0 & \boxed{3} & 0 & 105 \\ -R_1 + 6R_2 \to R_2 & 18 & 0 & 0 & -3 & 6 & -3 & 0 & 15 \\ R_1 + 6R_3 \to R_3 & 0 & 0 & 36 & 3 & 0 & -15 & 0 & 255 \\ \hline 5R_1 + 6R_4 \to R_4 & 0 & 0 & 0 & 15 & 24 & -3 & 18 & 1275 \end{array}$$

$$\begin{array}{r|ccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline & 0 & 6 & 0 & 3 & 0 & 3 & 0 & 105 \\ R_1 + R_2 \to R_2 & 18 & 6 & 0 & 0 & 6 & 0 & 0 & 120 \\ 5R_1 + R_3 \to R_3 & 0 & 30 & 36 & 18 & 0 & 0 & 0 & 780 \\ \hline R_1 + R_4 \to R_4 & 0 & 6 & 0 & 18 & 24 & 0 & 18 & 1380 \end{array}$$

Create a 1 in the columns corresponding to x_1, x_3, s_3, and z.

$$\begin{array}{r|ccccccc|c} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline \frac{1}{3}R_1 \to R_1 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & 35 \\ \frac{1}{18}R_2 \to R_2 & 1 & .33 & 0 & 0 & .33 & 0 & 0 & 6.67 \\ \frac{1}{36}R_3 \to R_3 & 0 & .83 & 1 & .5 & 0 & 0 & 0 & 21.67 \\ \hline \frac{1}{18}R_4 \to R_4 & 0 & .33 & 0 & 1 & 1.33 & 0 & 1 & 76.67 \end{array}$$

The maximum value is about 76.67 when $x_1 \approx 6.67$, $x_2 = 0$, $x_3 \approx 21.67$, $s_1 = 0$, $s_2 = 0$, and $s_3 = 35$.

24.

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ \hline 3 & \boxed{6} & -1 & 0 & 0 & 0 & 28 \\ 1 & 1 & 0 & 1 & 0 & 0 & 12 \\ 2 & 1 & 0 & 0 & 1 & 0 & 16 \\ \hline -1 & -2 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 6 in row 1, column 2.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ \hline & 3 & 6 & -1 & 0 & 0 & 0 & 28 \\ -R_1 + 6R_2 \to R_2 & 3 & 0 & \boxed{1} & 6 & 0 & 0 & 44 \\ -R_1 + 6R_3 \to R_3 & 9 & 0 & 1 & 0 & 6 & 0 & 68 \\ \hline R_1 + 3R_4 \to R_4 & 0 & 0 & -1 & 0 & 0 & 3 & 28 \end{array}$$

Pivot on the 1 in row 2, column 3.

$$\begin{array}{r} \\ R_2 + R_1 \to R_1 \\ \\ -R_2 + R_3 \to R_3 \\ R_2 + R_4 \to R_4 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{cccccc|c} 6 & 6 & 0 & 6 & 0 & 0 & 72 \\ 3 & 0 & 1 & 6 & 0 & 0 & 44 \\ 6 & 0 & 0 & -6 & 6 & 0 & 24 \\ \hline 3 & 0 & 0 & 6 & 0 & 3 & 72 \end{array}\right] \end{array}$$

$$\begin{array}{r} \\ \frac{1}{6}R_1 \to R_1 \\ \\ \frac{1}{6}R_3 \to R_3 \\ \frac{1}{3}R_4 \to R_4 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 1 & 0 & 0 & 12 \\ 3 & 0 & 1 & 6 & 0 & 0 & 44 \\ 1 & 0 & 0 & -1 & 1 & 0 & 4 \\ \hline 1 & 0 & 0 & 2 & 0 & 1 & 24 \end{array}\right] \end{array}$$

The maximum is 24 when $x_1 = 0, x_2 = 12,$ $s_1 = 44,$ $s_2 = 0,$ and $s_3 = 4.$

25. Minimize $w = 10y_1 + 15y_2$

subject to: $y_1 + y_2 \ge 17$
$5y_1 + 8y_2 \ge 42$

with $y_1 \ge 0, y_2 \ge 0.$

Using the dual method:

To form the dual, write the augmented matrix for the given problem.

$$\left[\begin{array}{cc|c} 1 & 1 & 17 \\ 5 & 8 & 42 \\ \hline 10 & 15 & 0 \end{array}\right]$$

Form the transpose of this matrix.

$$\left[\begin{array}{cc|c} 1 & 5 & 10 \\ 1 & 8 & 15 \\ \hline 17 & 42 & 0 \end{array}\right]$$

Write the dual problem.

Maximize $z = 17x_1 + 42x_2$

subject to: $x_1 + 5x_2 \le 10$
$x_1 + 8x_2 \le 15$

with $x_1 \ge 0, x_2 \ge 0.$

The initial simplex tableau is as follows.

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 5 & 1 & 0 & 0 & 10 \\ 1 & \boxed{8} & 0 & 1 & 0 & 15 \\ \hline -17 & -42 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 8 in row 2 column 2.

$$\begin{array}{r} -5R_2 + 8R_1 \to R_1 \\ \\ 21R_2 + 4R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} \boxed{3} & 0 & 8 & -5 & 0 & 5 \\ 1 & 8 & 0 & 1 & 0 & 15 \\ \hline -47 & 0 & 0 & 21 & 4 & 315 \end{array}\right] \end{array}$$

Pivot on the 3 in row 1, column 1.

$$\begin{array}{r} \\ -R_1 + 3R_2 \to R_2 \\ 47R_1 + 3R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 3 & 0 & 8 & -5 & 0 & 5 \\ 0 & 24 & -8 & \boxed{8} & 0 & 40 \\ \hline 0 & 0 & 376 & -172 & 12 & 1180 \end{array}\right] \end{array}$$

Pivot on the 8 in row 2, column 4.

$$\begin{array}{r} 5R_2 + 8R_1 \to R_1 \\ \\ 43R_2 + 2R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 24 & 120 & 24 & 0 & 0 & 240 \\ 0 & 24 & -8 & 8 & 0 & 40 \\ \hline 0 & 1032 & 408 & 0 & 24 & 4080 \end{array}\right] \end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, and z.

$$\begin{array}{r} \frac{1}{24}R_1 \to R_1 \\ \frac{1}{8}R_2 \to R_2 \\ \frac{1}{24}R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 5 & 1 & 0 & 0 & 10 \\ 0 & 3 & -1 & 1 & 0 & 5 \\ \hline 0 & 43 & 17 & 0 & 1 & 170 \end{array}\right] \end{array}$$

The minimum value is 170 when $y_1 = 17$ and $y_2 = 0.$

Using the method of 4.4:

Change the objective function to

Maximize $z = -w = -10y_1 - 15y_2.$

The constraints are not changed.

The initial simplex tableau is as follows.

$$\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} \boxed{1} & 1 & -1 & 0 & 0 & 17 \\ 5 & 8 & 0 & -1 & 0 & 42 \\ \hline 10 & 15 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The solution is not feasible since $s_1 = -17$ and $s_2 = -42.$ Pivot on the 1 in row 1, column 1.

$$\begin{array}{r} \\ -5R_1 + 5R_2 \to R_2 \\ -10R_1 + R_3 \to R_3 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 17 \\ 0 & 3 & 5 & -1 & 0 & -43 \\ \hline 0 & 5 & 10 & 0 & 1 & -170 \end{array}\right] \end{array}$$

Thus when
$y = 17, z = -170$ so the minimum is 170.

26. Minimize $w = 22y_1 + 44y_2 + 33y_3$

subject to:
$$\begin{aligned} y_1 + 2y_2 + y_3 &\geq 3 \\ y_1 + y_3 &\geq 3 \\ 3y_1 + 2y_2 + 2y_3 &\geq 8 \end{aligned}$$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$

Using the dual method:

To form the dual, write the augmented matrix for the given problem.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & 2 & 8 \\ \hline 22 & 44 & 33 & 0 \end{array}\right]$$

Form the transpose of the matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 22 \\ 2 & 0 & 2 & 44 \\ 1 & 1 & 2 & 33 \\ \hline 3 & 3 & 8 & 0 \end{array}\right]$$

Write the dual problem.

Maximize $z = 3x_1 + 3x_2 + 8x_3$

subject to:
$$\begin{aligned} x_1 + x_2 + 3x_3 &\leq 22 \\ 2x_1 + 2x_3 &\leq 44 \\ x_1 + x_2 + 2x_3 &\leq 33 \end{aligned}$$

with $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

The initial simplex tableau is as follows.

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 1 & \boxed{3} & 1 & 0 & 0 & 0 & 22 \\ 2 & 0 & 2 & 0 & 1 & 0 & 0 & 44 \\ 1 & 1 & 2 & 0 & 0 & 1 & 0 & 33 \\ \hline -3 & -3 & -8 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivot on the 3 in row 1, column 3.

$$\begin{array}{r} \\ -2R_1 + 3R_2 \to R_2 \\ -2R_1 + 3R_3 \to R_3 \\ 8R_1 + 3R_4 \to R_4 \end{array} \left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 1 & \boxed{1} & 3 & 1 & 0 & 0 & 0 & 22 \\ 4 & -2 & 0 & -2 & 3 & 0 & 0 & 88 \\ 1 & 1 & 0 & -2 & 0 & 3 & 0 & 55 \\ \hline -1 & -1 & 0 & 8 & 0 & 0 & 3 & 176 \end{array}\right]$$

There is a choice of pivot columns; choose column 2. Pivot on the 1 in row 1, column 2.

$$\begin{array}{r} \\ 2R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \\ R_1 + R_4 \to R_4 \end{array} \left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 1 & 1 & 3 & 1 & 0 & 0 & 0 & 22 \\ 6 & 0 & 6 & 0 & 3 & 0 & 0 & 132 \\ 0 & 0 & -3 & -3 & 0 & 3 & 0 & 33 \\ \hline 0 & 0 & 3 & 9 & 0 & 0 & 3 & 198 \end{array}\right]$$

Create a 1 in the columns corresponding to s_2, s_3, and z.

$$\begin{array}{r} \\ \frac{1}{3}R_2 \to R_2 \\ \frac{1}{3}R_3 \to R_3 \\ \frac{1}{3}R_4 \to R_4 \end{array} \left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 1 & 1 & 3 & 1 & 0 & 0 & 0 & 22 \\ 2 & 0 & 2 & 0 & 1 & 0 & 0 & 44 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 & 11 \\ \hline 0 & 0 & 1 & 3 & 0 & 0 & 1 & 66 \end{array}\right]$$

The minimum value is 66 when $y_1 = 3, y_2 = 0,$ and $y_3 = 0.$

Using the method of 4.4:

Change the objective function to

Maximize $z = -w = -22y_1 - 44y_2 - 33y_3.$

The constraints are not changed.
The initial simplex tableau is as follows.

$$\begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} \boxed{1} & 2 & 1 & -1 & 0 & 0 & 0 & 3 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & 3 \\ 3 & 2 & 2 & 0 & 0 & -1 & 0 & 8 \\ \hline 22 & 44 & 33 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The solution is not feasible since $s_1 = -3$, $s_2 = -3$, and $s_3 = -8$. Pivot on the 1 in row 1, column 1.

$$\begin{array}{r} \\ -R_1 + R_2 \to R_2 \\ 3R_1 - R_3 \to R_3 \\ -22R_1 + R_4 \to R_4 \end{array} \left[\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & z & \\ 1 & 2 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & -2 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 4 & 1 & -3 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 11 & 22 & 0 & 0 & 1 & -66 \end{array}\right]$$

The solution is feasible and there are no negative indicators so the solution is optimal. Create a 1 in the column corresponding to s_2.

$$\begin{array}{r} \\ \\ -R_2 \to R_2 \\ \\ \\ \end{array} \left[\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & z & \\ 1 & 2 & 1 & -1 & 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & -3 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 11 & 22 & 0 & 0 & 1 & -66 \end{array}\right]$$

Since $z = -w = -66,$ the minimum value is 66 when $y_1 = 3, y_2 = 0,$ and $y_3 = 0.$

27. Minimize $w = 7y_1 + 2y_2 + 3y_3$

subject to:
$$\begin{aligned} y_1 + y_2 + 2y_3 &\geq 48 \\ y_1 + y_2 \qquad &\geq 12 \\ y_3 &\geq 10 \\ 3y_1 \qquad + y_3 &\geq 30 \end{aligned}$$

with $y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0.$

Using the dual method:

To form the dual, write the augmented matrix for the given problem.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 48 \\ 1 & 1 & 0 & 12 \\ 0 & 0 & 1 & 10 \\ 3 & 0 & 1 & 30 \\ \hline 7 & 2 & 3 & 0 \end{array}\right]$$

Form the transpose of this matrix.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 7 \\ 1 & 1 & 0 & 0 & 2 \\ 2 & 0 & 1 & 1 & 3 \\ \hline 48 & 12 & 10 & 30 & 0 \end{array}\right]$$

Write the dual problem.

Maximize $z = 48x_1 + 12x_2 + 10x_3 + 30x_4$

subject to:
$$\begin{aligned} x_1 + x_2 \qquad + 3x_4 &\leq 7 \\ x_1 + x_2 \qquad\qquad &\leq 2 \\ 2x_1 + \quad x_3 + \ x_4 &\leq 3 \end{aligned}$$

with $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0.$

The initial simplex tableau is as follows.

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ 1 & 1 & 0 & 3 & 1 & 0 & 0 & 0 & 7 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ \boxed{2} & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 3 \\ \hline -48 & -12 & -10 & -30 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 2 in row 3, column 1.

$$\begin{array}{r} \\ -R_3 + 2R_1 \to R_1 \\ -R_3 + 2R_2 \to R_2 \\ \\ 24R_3 + R_4 \to R_4 \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ 0 & 2 & -1 & 5 & 2 & 0 & -1 & 0 & 11 \\ 0 & \boxed{2} & -1 & -1 & 0 & 2 & -1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 3 \\ \hline 0 & -12 & 14 & -6 & 0 & 0 & 24 & 1 & 72 \end{array}$$

Pivot on the 2 in row 2, column 2.

$$\begin{array}{r} \\ -R_2 + R_1 \to R_1 \\ \\ \\ 6R_2 + R_4 \to R_4 \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 0 & \boxed{6} & 2 & -2 & 0 & 0 & 10 \\ 0 & 2 & -1 & -1 & 0 & 2 & -1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 3 \\ \hline 0 & 0 & 8 & -12 & 0 & 12 & 18 & 1 & 78 \end{array}$$

Pivot on the 6 in row 1, column 4.

$$\begin{array}{r} \\ \\ R_1 + 6R_2 \to R_2 \\ -R_1 + 6R_3 \to R_3 \\ 2R_1 + R_4 \to R_4 \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 0 & 6 & 2 & -2 & 0 & 0 & 10 \\ 0 & 12 & -6 & 0 & 2 & 10 & -6 & 0 & 16 \\ 12 & 0 & 6 & 0 & -2 & 2 & 6 & 0 & 8 \\ \hline 0 & 0 & 8 & 0 & 4 & 8 & 18 & 1 & 98 \end{array}$$

Create a 1 in the columns corresponding to x_1, x_2, and x_4.

$$\begin{array}{r} \\ \frac{1}{6}R_1 \to R_1 \\ \frac{1}{12}R_2 \to R_2 \\ \frac{1}{12}R_3 \to R_3 \\ \\ \end{array}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & z & \\ 0 & 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & \frac{5}{3} \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{6} & \frac{5}{6} & -\frac{1}{2} & 0 & \frac{4}{3} \\ 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{6} & \frac{1}{6} & \frac{1}{2} & 0 & \frac{2}{3} \\ \hline 0 & 0 & 8 & 0 & 4 & 8 & 18 & 1 & 98 \end{array}$$

The minimum value is 98 when $y_1 = 4,\ y_2 = 8,$ and $y_3 = 18.$

Using the method of 4.4:

Change the objective function to

Maximize $z = -w = -7y_1 - 2y_2 - 3y_3.$

The constraints are not changed.

The initial simplex tableau is as follows.

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & s_4 & z & \\ 1 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & 48 \\ 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 10 \\ \boxed{3} & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 30 \\ \hline 7 & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

The solution is not feasible since $s_1 = -48$, $s_2 = -12$, $s_3 = -10$, and $s_4 = -30$. Pivot on the 3 in row 4, column 1.

$$\begin{array}{r} \\ -R_4 + 3R_1 \to R_1 \\ -R_4 + 3R_2 \to R_2 \\ \\ \\ -7R_4 + 3R_5 \to R_5 \end{array}\begin{array}{cccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & s_4 & z & \\ 0 & \boxed{3} & 5 & -3 & 0 & 0 & 1 & 0 & 114 \\ 0 & 3 & -1 & 0 & -3 & 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 10 \\ 3 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 30 \\ \hline 0 & 6 & 2 & 0 & 0 & 0 & 7 & 3 & -210 \end{array}$$

The solution is still not feasible since $s_1 = -38$, $s_2 = -2$, and $s_3 = -10$. Pivot on the 3 in row 1, column 2.

$$\begin{array}{r} \\ \\ R_1 - R_2 \to R_2 \\ \\ \\ -2R_1 + R_5 \to R_5 \end{array}\begin{array}{cccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & s_4 & z & \\ 0 & 3 & 5 & -3 & 0 & 0 & 1 & 0 & 114 \\ 0 & 0 & 6 & -3 & 3 & 0 & 0 & 0 & 108 \\ 0 & 0 & \boxed{1} & 0 & 0 & -1 & 0 & 0 & 10 \\ 3 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 30 \\ \hline 0 & 0 & -8 & 6 & 0 & 0 & 5 & 3 & -438 \end{array}$$

Again, the solution is not feasible since $s_3 = -10$ and $s_4 = -30$. Pivot on the 1 in row 3, column 3.

$$\begin{array}{r} \\ -5R_3 + R_1 \to R_1 \\ -6R_3 + R_2 \to R_2 \\ \\ -R_3 + R_4 \to R_4 \\ 8R_3 + R_5 \to R_5 \end{array}\begin{array}{cccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & s_4 & z & \\ 0 & 3 & 0 & -3 & 0 & 5 & 1 & 0 & 64 \\ 0 & 0 & 0 & -3 & 3 & \boxed{6} & 0 & 0 & 48 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 10 \\ 3 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 20 \\ \hline 0 & 0 & 0 & 6 & 0 & -8 & 5 & 3 & -358 \end{array}$$

The solution is feasible because all variables are nonnegative. But it is still not optimal. Pivot on the 6 in row 2, column 6.

$$\begin{array}{r} \\ -5R_2 + 6R_1 \to R_1 \\ \\ R_2 + 6R_3 \to R_3 \\ -R_2 + 6R_4 \to R_4 \\ 4R_2 + 3R_5 \to R_5 \end{array}\begin{array}{cccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & s_4 & z & \\ 0 & 18 & 0 & -3 & -15 & 0 & 6 & 0 & 144 \\ 0 & 0 & 0 & -3 & 3 & 6 & 0 & 0 & 48 \\ 0 & 0 & 6 & -3 & 3 & 0 & 0 & 0 & 108 \\ 18 & 0 & 0 & 3 & -3 & 0 & -6 & 0 & 72 \\ \hline 0 & 0 & 0 & 6 & 12 & 0 & 15 & 9 & -882 \end{array}$$

Create a 1 in the columns corresponding to y_1, y_2, y_3, s_3 and z.

$$\begin{array}{r} \\ \frac{1}{18}R_1 \to R_1 \\ \frac{1}{6}R_2 \to R_2 \\ \frac{1}{6}R_3 \to R_3 \\ \frac{1}{18}R_4 \to R_4 \\ \frac{1}{9}R_5 \to R_5 \end{array}\begin{array}{cccccccc|c} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & s_4 & z & \\ 0 & 1 & 0 & -\frac{1}{6} & -\frac{5}{6} & 0 & \frac{1}{3} & 0 & 8 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 18 \\ 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{6} & 0 & -\frac{1}{3} & 0 & 4 \\ \hline 0 & 0 & 0 & \frac{2}{3} & \frac{4}{3} & 0 & \frac{5}{3} & 1 & -98 \end{array}$$

Since $z = -w = -98$, the minimum value is 98 when $y_1 = 4$, $y_2 = 8$, and $y_3 = 18$.

28. Minimize $w = 3y_1 + 4y_2 + y_3 + 2y_4$

subject to:
$$4y_1 + 6y_2 + 3y_3 + 8y_4 \ge 19$$
$$13y_1 + 7y_2 + 2y_3 + 6y_4 \ge 16$$

with $y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0,\ y_4 \ge 0.$

Using the dual method:

To form the dual, write the augmented matrix for the given problem.

$$\left[\begin{array}{cccc|c} 4 & 6 & 3 & 8 & 19 \\ 13 & 7 & 2 & 6 & 16 \\ \hline 3 & 4 & 1 & 2 & 0 \end{array}\right]$$

Form the transpose of this matrix.

$$\left[\begin{array}{cc|c} 4 & 13 & 3 \\ 6 & 7 & 4 \\ 3 & 2 & 1 \\ 8 & 6 & 2 \\ \hline 19 & 16 & 0 \end{array}\right]$$

Write the dual problem.

Maximize $z = 19x_1 + 16x_2$

subject to:
$$4x_1 + 13x_2 \le 3$$
$$6x_1 + 7x_2 \le 4$$
$$3x_1 + 2x_2 \le 1$$
$$8x_1 + 6x_2 \le 2$$

with $x_1 \ge 0,\ x_2 \ge 0.$

The initial simplex tableau is as follows.

$$\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & z & \\ \hline 4 & 13 & 1 & 0 & 0 & 0 & 0 & 3 \\ 6 & 7 & 0 & 1 & 0 & 0 & 0 & 4 \\ 3 & 2 & 0 & 0 & 1 & 0 & 0 & 1 \\ \boxed{8} & 6 & 0 & 0 & 0 & 1 & 0 & 2 \\ \hline -19 & -16 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 8 in row 4 of column 1.

$$\begin{array}{r} \\ -R_4 + 2R_1 \to R_1 \\ -3R_4 + 4R_2 \to R_2 \\ -3R_4 + 8R_3 \to R_3 \\ \\ 19R_4 + 8R_5 \to R_5 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & z & \\ 0 & \boxed{20} & 2 & 0 & 0 & -1 & 0 & 4 \\ 0 & 10 & 0 & 4 & 0 & -3 & 0 & 10 \\ 0 & -2 & 0 & 0 & 8 & -3 & 0 & 2 \\ 8 & 6 & 0 & 0 & 0 & 1 & 0 & 2 \\ \hline 0 & -14 & 0 & 0 & 0 & 19 & 8 & 38 \end{array}$$

Pivot on the 20 in row 1 of column 2.

$$\begin{array}{r} \\ \\ -R_1 + 2R_2 \to R_2 \\ R_1 + 10R_3 \to R_3 \\ -3R_1 + 10R_4 \to R_4 \\ 7R_1 + 10R_5 \to R_5 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & z & \\ 0 & 20 & 2 & 0 & 0 & -1 & 0 & 4 \\ 0 & 0 & -2 & 8 & 0 & -5 & 0 & 16 \\ 0 & 0 & 2 & 0 & 80 & -31 & 0 & 24 \\ 80 & 0 & -6 & 0 & 0 & 13 & 0 & 8 \\ \hline 0 & 0 & 14 & 0 & 0 & 183 & 80 & 408 \end{array}$$

Get a 1 in the last row of the z column.

$$\begin{array}{r} \\ \\ \\ \\ \\ R_5/80 \to R_5 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & z & \\ 0 & 20 & 2 & 0 & 0 & -1 & 0 & 4 \\ 0 & 0 & -2 & 8 & 0 & -5 & 0 & 16 \\ 0 & 0 & 2 & 0 & 80 & -31 & 0 & 24 \\ 80 & 0 & -6 & 0 & 0 & 13 & 0 & 8 \\ \hline 0 & 0 & \frac{7}{40} & 0 & 0 & \frac{183}{80} & 1 & \frac{51}{10} \end{array}$$

Now we can read the solution: The minimum is 51/10 when $y_1 = 7/40$ and $y_4 = 183/80$.

Using the method of 4.4:

Change the objective function to

Maximize $z = -w = -3y_1 - 4y_2 - y_3 - 2y_4.$

The constraints are not changed.

The initial simplex tableau is as follows.

$$\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & z & \\ 4 & 6 & 3 & 8 & -1 & 0 & 0 & 19 \\ \boxed{13} & 7 & 2 & 6 & 0 & -1 & 0 & 16 \\ \hline 3 & 4 & 1 & 2 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 13 in row 2 of column 1.

$$\begin{array}{r} \\ -4R_2 + 13R_1 \to R_1 \\ \\ -3R_2 + 13R_3 \to R_3 \end{array}\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & z & \\ 0 & 50 & 31 & \boxed{80} & -13 & 4 & 0 & 183 \\ 13 & 7 & 2 & 6 & 0 & -1 & 0 & 16 \\ \hline 0 & 31 & 7 & 8 & 0 & 3 & 13 & -48 \end{array}$$

Now pivot on the 80 in row 1 of column 4. (We take advantage of the fact mentioned in Section 4 that any positive entry in the row containing the variable we want to eliminate can be used to identify a pivot column.)

$$\begin{array}{r} \\ \\ -3R_1 + 40R_2 \to R_2 \\ -R_1 + 10R_3 \to R_3 \end{array}\begin{array}{ccccccc|c} y_1 & y_2 & y_3 & y_4 & s_1 & s_2 & z & \\ 0 & 50 & 31 & 80 & -13 & 4 & 0 & 183 \\ 520 & 130 & -13 & 0 & 39 & -52 & 0 & 91 \\ \hline 0 & 260 & 39 & 0 & 13 & 26 & 130 & -663 \end{array}$$

We now find $y_1 = \dfrac{91}{520} = \dfrac{7}{40}$, $y_4 = \dfrac{183}{80}$ and a minimum of $-\left(\dfrac{-663}{130}\right) = \dfrac{51}{10}$.

29.

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 5 & 10 & 1 & 0 & 0 & 120 \\ \boxed{10} & 15 & 0 & -1 & 0 & 200 \\ \hline -20 & -30 & 0 & 0 & 1 & 0 \end{array}$$

The initial tableau is not feasible. Pivot on the 10 in row 2, column 1.

$$\begin{array}{r} \\ -R_2 + 2R_1 \to R_1 \\ \\ 2R_2 + R_3 \to R_3 \end{array}\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 0 & 5 & 2 & \boxed{1} & 0 & 40 \\ 10 & 15 & 0 & -1 & 0 & 200 \\ \hline 0 & 0 & 0 & -2 & 1 & 400 \end{array}$$

The basic solution is feasible, but there are negative indicators. Pivot on the 1 in row 1, column 4.

$$\begin{array}{r} \\ \\ R_2 + 2R_1 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array}\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 0 & 5 & 2 & 1 & 0 & 40 \\ 10 & 20 & 2 & 0 & 0 & 240 \\ \hline 0 & 10 & 4 & 0 & 1 & 480 \end{array}$$

Create a one in the column corresponding to x_1.

$$\begin{array}{r} \\ \\ \frac{1}{10}R_2 \to R_2 \\ \\ \end{array}\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ 0 & 5 & 2 & 1 & 0 & 40 \\ 1 & 2 & \frac{1}{5} & 0 & 0 & 24 \\ \hline 0 & 10 & 4 & 0 & 1 & 480 \end{array}$$

The maximum value is $z = 480$ when $x_1 = 24$ and $x_2 = 0$.

30. Minimize $w = 4y_1 + 2y_2$

subject to:
$$\begin{aligned} y_1 + 3y_2 &\ge 6 \\ 2y_1 + 8y_2 &\le 21 \\ y_1 \ge 0,\ y_2 &\ge 0 \end{aligned}$$

Let $z = -w = -4y_1 - 2y_2$ and maximize z.

Introduce the surplus variable s_1 and the slack variable s_2. The initial tableau is as follows.

$$\begin{array}{ccccc|c} y_1 & y_2 & s_1 & s_2 & z & \\ \hline \boxed{1} & 3 & -1 & 0 & 0 & 6 \\ 2 & 8 & 0 & 1 & 0 & 21 \\ \hline 4 & 2 & 0 & 0 & 1 & 0 \end{array}$$

The initial basic solution is not feasible since $s_1 = -6$. Pivot on the 1 in row 1, column 1.

$$\begin{array}{r|ccccc|c} & y_1 & y_2 & s_1 & s_2 & z & \\ \hline & 1 & \boxed{3} & -1 & 0 & 0 & 6 \\ -2R_1 + R_2 \to R_2 & 0 & 2 & 2 & 1 & 0 & 9 \\ \hline -4R_1 + R_3 \to R_3 & 0 & -10 & 4 & 0 & 1 & -24 \end{array}$$

Pivot on the 3 in row 1, column 2.

$$\begin{array}{r|ccccc|c} & y_1 & y_2 & s_1 & s_2 & z & \\ \hline & 1 & 3 & -1 & 0 & 0 & 6 \\ -2R_1 + 3R_2 \to R_2 & -2 & 0 & 8 & 3 & 0 & 15 \\ \hline 10R_1 + 3R_3 \to R_3 & 10 & 0 & 2 & 0 & 3 & -12 \end{array}$$

Create a 1 in the columns corresponding to y_2, s_2, and z.

$$\begin{array}{r|ccccc|c} & y_1 & y_2 & s_1 & s_2 & z & \\ \hline \frac{1}{3}R_1 \to R_1 & \frac{1}{3} & 1 & -\frac{1}{3} & 0 & 0 & 2 \\ \frac{1}{3}R_2 \to R_2 & -\frac{2}{3} & 0 & \frac{8}{3} & 1 & 0 & 5 \\ \hline \frac{1}{3}R_3 \to R_3 & \frac{10}{3} & 0 & \frac{2}{3} & 0 & 1 & -4 \end{array}$$

The maximum value of $z = -w$ is -4. Therefore, the minimum value of w is 4 when $y_1 = 0$ and $y_2 = 2$.

31. Maximize $z = 10x_1 + 12x_2$

subject to:
$$\begin{aligned} 2x_1 + 2x_2 &= 17 \\ 2x_1 + 5x_2 &\ge 22 \\ 4x_1 + 3x_2 &\le 28 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0$.

Introduce artificial variable a, surplus variable s_1, and slack variable s_2. The initial simplex tableau as follows.

$$\begin{array}{cccccc|c} x_1 & x_2 & a & s_1 & s_2 & z & \\ \hline \boxed{2} & 2 & 1 & 0 & 0 & 0 & 17 \\ 2 & 5 & 0 & -1 & 0 & 0 & 22 \\ 4 & 3 & 0 & 0 & 1 & 0 & 28 \\ \hline -10 & -12 & 0 & 0 & 0 & 1 & 0 \end{array}$$

First, eliminate the artificial variable a. Pivot on the 2 in row 1, column 1.

$$\begin{array}{r|cccccc|c} & x_1 & x_2 & a & s_1 & s_2 & z & \\ \hline & 2 & 2 & 1 & 0 & 0 & 0 & 17 \\ -R_1 + R_2 \to R_2 & 0 & 3 & -1 & -1 & 0 & 0 & 5 \\ 2R_1 - R_3 \to R_3 & 0 & 1 & 2 & 0 & -1 & 0 & 6 \\ \hline 5R_1 + R_4 \to R_4 & 0 & -2 & 5 & 0 & 0 & 1 & 85 \end{array}$$

Now $a = 0$, so we can drop the a column.

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & z & \\ \hline 2 & 2 & 0 & 0 & 0 & 17 \\ 0 & \boxed{3} & -1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 & 0 & 6 \\ \hline 0 & -2 & 0 & 0 & 1 & 85 \end{array}$$

Because $s_1 = -5$, we choose the 3 in row 2, column 2, as the next pivot.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ \hline -2R_2 + 3R_1 \to R_1 & 6 & 0 & 2 & 0 & 0 & 41 \\ & 0 & 3 & -1 & 0 & 0 & 5 \\ -R_2 + 3R_3 \to R_3 & 0 & 0 & \boxed{1} & -3 & 0 & 13 \\ \hline 2R_2 + 3R_4 \to R_4 & 0 & 0 & -2 & 0 & 3 & 265 \end{array}$$

The solution is still not feasible since $s_2 = -\frac{13}{3}$. Pivot on the 1 in row 3, column 3.

$$\begin{array}{r|ccccc|c} & x_1 & x_2 & s_1 & s_2 & z & \\ \hline -2R_3 + R_1 \to R_1 & 6 & 0 & 0 & \boxed{6} & 0 & 15 \\ R_3 + R_2 \to R_2 & 0 & 3 & 0 & -3 & 0 & 18 \\ & 0 & 0 & 1 & -3 & 0 & 13 \\ \hline 2R_3 + R_4 \to R_4 & 0 & 0 & 0 & -6 & 3 & 291 \end{array}$$

The solution is now feasible but is not yet optimal. Pivot on the 6 in row 1, column 4.

$$\begin{array}{r} \\ \\ R_1+2R_2\to R_2 \\ R_1+2R_3\to R_3 \\ R_1+R_4\to R_4 \end{array}\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 6&0&0&6&0&15\\ 6&6&0&0&0&51\\ 6&0&2&0&0&41\\ \hline 6&0&0&0&3&306 \end{array}\right]\end{array}$$

Create a 1 in the columns corresponding to x_2, s_1, s_2, and z.

$$\begin{array}{r} \\ \frac{1}{6}R_1\to R_1 \\ \frac{1}{6}R_2\to R_2 \\ \frac{1}{2}R_3\to R_3 \\ \frac{1}{3}R_4\to R_4 \end{array}\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 1&0&0&1&0&\frac{5}{2}\\ 1&1&0&0&0&\frac{17}{2}\\ 3&0&1&0&0&\frac{41}{2}\\ \hline 2&0&0&0&1&102 \end{array}\right]\end{array}$$

The maximum is 102 when $x_1 = 0$ and $x_2 = \frac{17}{2}$.

32. Minimize $w = 24y_1 + 30y_2 + 36y_3$

subject to:
$$\begin{aligned} 5y_1 + 10y_2 + 15y_3 &\ge 1200 \\ y_1 + y_2 + y_3 &\le 50 \\ y_1 \ge 0, y_2 \ge 0, y_3 &\ge 0 \end{aligned}$$

Let $z = -w = -24y_1 - 30y_2 - 36y_3$ and maximize z. Introduce surplus variable s_1 and slack variable s_2. The initial tableau is as follows.

$$\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 5&10&15&-1&0&0&1200\\ \boxed{1}&1&1&0&1&0&50\\ \hline 24&30&36&0&0&1&0 \end{array}\right]\end{array}$$

The initial basic solution is not feasible since $s_1 = -1200$.

Pivot on the 1 in row 2, column 1.

$$\begin{array}{r} \\ -5R_2+R_1\to R_1 \\ \\ -24R_2+R_3\to R_3 \end{array}\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 0&5&10&-1&-5&0&950\\ 1&\boxed{1}&1&0&1&0&50\\ \hline 0&6&12&0&-24&1&-1200 \end{array}\right]\end{array}$$

Pivot on the 1 in row 2, column 2.

$$\begin{array}{r} \\ -5R_2+R_1\to R_1 \\ \\ -6R_2+R_3\to R_3 \end{array}\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} -5&0&5&-1&-10&0&700\\ 1&1&\boxed{1}&0&1&0&50\\ \hline -6&0&6&0&-30&1&-1500 \end{array}\right]\end{array}$$

Pivot on the 1 in row 2, column 3.

$$\begin{array}{r} \\ -5R_2+R_1\to R_1 \\ \\ -6R_2+R_3\to R_3 \end{array}\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & s_1 & s_3 & z \end{array} \\ \left[\begin{array}{cccccc|c} -10&-5&0&-1&-15&0&450\\ 1&1&1&0&1&0&50\\ \hline -12&-6&0&0&-36&1&-1800 \end{array}\right]\end{array}$$

Now $s_1 = -450$ is not a feasible solution, but it is not possible to choose a pivot point. Therefore there is no solution.

33. Any maximizing or minimizing problems can be solved using slack, surplus, and artificial variables. Slack variables are used in problems involving "$\le$" constraints. Surplus variables are used in problems involving "$\ge$" constraints. Artificial variables are used in problems involving "$=$" constraints.

34. A dual can be used to solve any standard minimization problem.

35.
$$\left[\begin{array}{cccccc|c} 4&2&3&1&0&0&9\\ 5&4&1&0&1&0&10\\ \hline -6&-7&-5&0&0&1&0 \end{array}\right]$$

(a) The 1 in column 4 and the 1 in column 5 indicate that the constraints involve $\le$. The problem being solved with this tableau is:

Maximize $z = 6x_1 + 7x_2 + 5x_3$

subject to:
$$\begin{aligned} 4x_1 + 2x_2 + 3x_3 &\le 9 \\ 5x_1 + 4x_2 + x_3 &\le 10 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$.

(b) If the 1 in row 1, column 4 was -1 rather than 1, then the first constraint would have a surplus variable rather than a slack variable, which means the first constraint would be $4x_1 + 2x_2 + 3x_3 \ge 9$ instead of $4x_1 + 2x_2 + 3x_3 \le 9$.

(c)
$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 3&0&5&2&-1&0&8\\ 11&10&0&-1&3&0&21\\ \hline 47&0&0&13&11&10&227 \end{array}\right]\end{array}$$

From this tableau, the solution is $x_1 = 0$, $x_2 = \frac{21}{10} = 2.1$, $x_3 = \frac{8}{5} = 1.6$, and $z = \frac{227}{10} = 22.7$.

(d) The dual of the original problem is as follows:

Minimize $w = 9y_1 + 10y_2$

subject to:
$$\begin{aligned} 4y_1 + 5y_2 &\ge 6 \\ 2y_1 + 4y_2 &\ge 7 \\ 3y_1 + y_2 &\ge 5 \end{aligned}$$

with $y_1 \ge 0,\ y_2 \ge 0.$

(e) From the tableau in part (c), the solution of the dual in part (d) is $y_1 = \frac{13}{10} = 1.3$, $y_2 = \frac{11}{10} = 1.1$, and $w = \frac{227}{10} = 22.7$.

36. (a) Find matrices A, B, C, and X such that the problem

Maximize $z = 3x_1 + 2x_2 + x_3$

subject to:
$$\begin{aligned} 2x_1 + x_2 + x_3 &\le 150 \\ 2x_1 + 2x_2 + 8x_3 &\le 200 \\ 2x_1 + 3x_2 + x_3 &\le 320 \end{aligned}$$

with $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$

can be written as

Maximize CX

subject to: $AX \le B$

with $X \ge O.$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 8 \\ 2 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 150 \\ 200 \\ 320 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) To write the dual, write the augmented matrix for the given problem.

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 150 \\ 2 & 2 & 8 & 200 \\ 2 & 3 & 1 & 320 \\ \hline 3 & 2 & 1 & 0 \end{array}\right]$$

Now form the transpose.

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 3 \\ 1 & 2 & 3 & 2 \\ 1 & 8 & 1 & 1 \\ \hline 150 & 200 & 320 & 0 \end{array}\right]$$

The dual is now stated as:

Minimize $w = 150y_1 + 200y_2 + 320y_3$

subject to:
$$\begin{aligned} 2y_1 + 2y_2 + 2y_3 &\ge 3 \\ y_1 + 2y_2 + 3y_3 &\ge 2 \\ y_1 + 8y_2 + y_3 &\ge 1 \end{aligned}$$

with $y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0.$

This can be stated as:

For $Y = \begin{bmatrix} y_1 & y_2 & y_2 \end{bmatrix}$,

Minimize YB

subject to: $YA \ge C$

with $Y \ge O.$

37. (a) Let x_1 = the number of cake plates,

x_2 = the number of bread plates,

and x_3 = the number of dinner plates.

(b) The objective function to maximize is $z = 15x_1 + 12x_2 + 5x_3$.

(c) The constraints are
$$\begin{aligned} 15x_1 + 10x_2 + 8x_3 &\le 1500 \\ 5x_1 + 4x_2 + 4x_3 &\le 2700 \\ 6x_1 + 5x_2 + 5x_3 &\le 1200. \end{aligned}$$

38. (a) Let x_1 = the amount invested in oil leases;

x_2 = the amount invested in stocks;

and x_3 = the amount invested in bonds.

(b) We want to maximize
$$z = 0.15x_1 + 0.09x_2 + 0.05x_3.$$

(c) The constraints are
$$\begin{aligned} x_1 + x_2 + x_3 &\le 50{,}000 \\ x_1 + x_2 \quad &\le 15{,}000 \\ x_1 \quad + x_3 &\le 25{,}000. \end{aligned}$$

39. (a) Let x_1 = number of gallons of Fruity wine

and x_2 = number of gallons of Crystal wine.

(b) The profit function is
$$z = 12x_1 + 15x_2.$$

(c) The ingredients available are the limitations; the constraints are

$$\begin{aligned} 2x_1 + \ \ x_2 &\le 110 \\ 2x_1 + 3x_2 &\le 125 \\ 2x_1 + \ \ x_2 &\le \ \ 90. \end{aligned}$$

40. **(a)** Let y_1 = the number of kilograms of canned whole tomatoes produced

and y_2 = the number of kilograms of tomato sauce produced.

(b) The minimum cost function is

$$w = 4y_1 + 3.25y_2.$$

(c) The constraints are

$$\begin{aligned} y_1 + \ \ y_2 &\le 3{,}000{,}000 \\ y_1 \qquad &\ge \ \ 800{,}000 \\ y_2 &\ge \ \ \ 80{,}000 \\ 6y_1 + 3y_2 &\ge 6{,}600{,}000. \end{aligned}$$

41. Maximize $z = 15x_1 + 12x_2 + 5x_3$

subject to:
$$\begin{aligned} 15x_1 + 10x_2 + 8x_3 &\le 1500 \\ 5x_1 + \ \ 4x_2 + 4x_3 &\le 2700 \\ 6x_1 + \ \ 5x_2 + 5x_3 &\le 1200 \end{aligned}$$

with $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$

The initial tableau is as follows.

$$\begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline \boxed{15} & 10 & 8 & 1 & 0 & 0 & 0 & 1500 \\ 5 & 4 & 4 & 0 & 1 & 0 & 0 & 2700 \\ 6 & 5 & 5 & 0 & 0 & 1 & 0 & 1200 \\ \hline -15 & -12 & -5 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

Pivot on the 15 in row 1, column 1.

$$\begin{array}{r} \\ \\ -R_1 + 3R_2 \to R_2 \\ -2R_1 + 5R_3 \to R_3 \\ R_1 + R_4 \to R_4 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline 15 & \boxed{10} & 8 & 1 & 0 & 0 & 0 & 1500 \\ 0 & 2 & 4 & -1 & 3 & 0 & 0 & 6600 \\ 0 & 5 & 9 & -2 & 0 & 5 & 0 & 3000 \\ \hline 0 & -2 & 3 & 1 & 0 & 0 & 1 & 1500 \end{array}$$

Pivot on the 10 in row 1, column 2.

$$\begin{array}{r} \\ \\ -R_1 + 5R_2 \to R_2 \\ -R_1 + 2R_3 \to R_3 \\ R_1 + 5R_4 \to R_4 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline 15 & 10 & 8 & 1 & 0 & 0 & 0 & 1500 \\ -15 & 0 & 12 & -6 & 15 & 0 & 0 & 31{,}500 \\ -15 & 0 & 10 & -5 & 0 & 10 & 0 & 4500 \\ \hline 15 & 0 & 23 & 6 & 0 & 0 & 5 & 9000 \end{array}$$

Create a 1 in the columns corresponding to x_2, s_2, s_3, and z.

$$\begin{array}{r} \\ \frac{1}{10}R_1 \to R_1 \\ \frac{1}{15}R_2 \to R_2 \\ \frac{1}{10}R_3 \to R_3 \\ \frac{1}{5}R_4 \to R_4 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline \frac{3}{2} & 1 & \frac{4}{5} & \frac{1}{10} & 0 & 0 & 0 & 150 \\ -1 & 0 & \frac{4}{5} & -\frac{2}{5} & 1 & 0 & 0 & 2100 \\ -\frac{3}{2} & 0 & 1 & -\frac{1}{2} & 0 & 1 & 0 & 450 \\ \hline 3 & 0 & \frac{23}{5} & \frac{6}{5} & 0 & 0 & 1 & 1800 \end{array}$$

The maximum profit of \$1800 when no cake plates, 150 bread plates, and no dinner places are produced.

42. Based on the information given in Exercise 38, the initial tableau is as follows.

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 50{,}000 \\ \boxed{1} & 1 & 0 & 0 & 1 & 0 & 0 & 15{,}000 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 25{,}000 \\ \hline -0.15 & -0.09 & -0.05 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Continue by pivoting on each indicated entry.

$$\begin{array}{r} \\ -R_2 + R_1 \to R_1 \\ \\ -R_2 + R_3 \to R_3 \\ 0.15R_2 + R_4 \to R_4 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline 0 & 0 & 1 & 1 & -1 & 0 & 0 & 35{,}000 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 15{,}000 \\ 0 & -1 & \boxed{1} & 0 & -1 & 1 & 0 & 10{,}000 \\ \hline 0 & 0.06 & -0.05 & 0 & 0.15 & 0 & 1 & 2250 \end{array}$$

$$\begin{array}{r} \\ -R_3 + R_1 \to R_1 \\ \\ \\ 0.05R_3 + R_4 \to R_4 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline 0 & 1 & 0 & 1 & 0 & -1 & 0 & 25{,}000 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 15{,}000 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 10{,}000 \\ \hline 0 & 0.01 & 0 & 0 & 0.1 & 0.05 & 1 & 2750 \end{array}$$

The maximum value is $z = 2750$ when $x_1 = 15{,}000,\ x_2 = 0,$ and $x_3 = 10{,}000.$
He should invest \$15,000 in oil leases and \$10,000 in stock for a maximum return of \$2750.

43. Based on Exercise 39, the initial tableau is

$$\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ \hline 2 & 1 & 1 & 0 & 0 & 0 & 110 \\ 2 & \boxed{3} & 0 & 1 & 0 & 0 & 125 \\ 2 & 1 & 0 & 0 & 1 & 0 & 90 \\ \hline -12 & -15 & 0 & 0 & 0 & 1 & 0 \end{array}.$$

Locating the first pivot in the usual way, it is found to be the 3 in row 2, column 2. After row transformations, we get the next tableau.

	x_1	x_2	s_1	s_2	s_3	z	
$-R_2 + 3R_1 \to R_1$	4	0	3	−1	0	0	205
	2	3	0	1	0	0	125
$-R_2 + 3R_3 \to R_3$	[4]	0	0	−1	3	0	145
$5R_2 + R_4 \to R_4$	−2	0	0	5	0	1	625

Pivot on the 4 in row 3, column 1.

	x_1	x_2	s_1	s_2	s_3	z	
$-R_3 + R_1 \to R_1$	0	0	3	0	−3	0	60
$-R_3 + 2R_2 \to R_2$	0	6	0	3	−3	0	105
	4	0	0	−1	3	0	145
$R_3 + 2R_4 \to R_4$	0	0	0	9	3	2	1395

	x_1	x_2	s_1	s_2	s_3	z	
$\frac{1}{3}R_1 \to R_1$	0	0	1	0	−1	0	20
$\frac{1}{6}R_2 \to R_2$	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{35}{2}$
$\frac{1}{4}R_3 \to R_3$	1	0	0	$-\frac{1}{4}$	$\frac{3}{4}$	0	$\frac{145}{4}$
$\frac{1}{2}R_4 \to R_4$	0	0	0	$\frac{9}{2}$	$\frac{3}{2}$	1	$\frac{1395}{2}$

The final tableau gives the solution $x_1 = \frac{145}{4}$, $x_2 = \frac{35}{2}$, and $z = \frac{1395}{2} = 697.5$. 36.25 gal of Fruity wine and 17.5 gal of Crystal wine should be produced for a maximum profit of $697.50.

44. Based on Exercise 40, the initial tableau is as follows.

y_1	y_2	s_1	s_2	s_3	s_4	z	
1	1	1	0	0	0	0	3,000,000
[1]	0	0	−1	0	0	0	800,000
0	1	0	0	−1	0	0	80,000
6	3	0	0	0	−1	0	6,600,000
4	3.25	0	0	0	0	1	0

Pivot on the 1 in row 2, column 1 since the basic solution is not feasible.

	y_1	y_2	s_1	s_2	s_3	s_4	z	
$-R_2 + R_1 \to R_1$	0	1	1	1	0	0	0	2,200,000
	1	0	0	−1	0	0	0	800,000
	0	[1]	0	0	−1	0	0	80,000
$-6R_2 + R_4 \to R_4$	0	3	0	6	0	−1	0	1,800,000
$-4R_2 + R_5 \to R_5$	0	3.25	0	4	0	0	1	−3,200,000

Pivot on the 1 in row 3, column 2 since the basic solution is not feasible.

	y_1	y_2	s_1	s_2	s_3	s_4	z	
$-R_3 + R_1 \to R_1$	0	0	1	1	1	0	0	1,400,000
	1	0	0	−1	0	0	0	800,000
	0	1	0	0	−1	0	0	80,000
$-3R_3 + R_4 \to R_4$	0	0	0	[6]	3	−1	0	1,560,000
$-3.25R_3 + R_5 \to R_5$	0	0	0	4	3.25	0	1	−3,460,000

Pivot on the 6 in row 4, column 4 since basic solution is not feasible.

	y_1	y_2	s_1	s_2	s_3	s_4	z	
$-R_4 + 6R_1 \to R_1$	0	0	6	0	3	1	0	6,840,000
$R_4 + 6R_2 \to R_2$	6	0	0	0	3	−1	0	6,360,000
	0	1	0	0	−1	0	0	80,000
	0	0	0	6	3	−1	0	1,560,000
$2R_4 + 3R_5 \to R_5$	0	0	0	0	−3.75	−2	−3	13,500,000

Create a 1 in the columns corresponding to y_1, y_2, s_1, s_2, and z.

	y_1	y_2	s_1	s_2	s_3	s_4	z	
$\frac{1}{6}R_1 \to R_1$	0	0	1	0	$\frac{1}{2}$	$\frac{1}{6}$	0	1,140,000
$\frac{1}{6}R_2 \to R_2$	1	0	0	0	$\frac{1}{2}$	$-\frac{1}{6}$	0	1,060,000
	0	1	0	0	−1	0	0	80,000
$\frac{1}{6}R_4 \to R_4$	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{6}$	0	260,000
$-\frac{1}{3}R_5 \to R_5$	0	0	0	0	1.25	$\frac{2}{3}$	1	−4,500,000

The final tableau gives the solution $y_2 = 1{,}060{,}000$, $y_2 = 80{,}000$, and $z = 4{,}500{,}000$. Use 1,060,000 kg of whole tomatoes and 80,000 kg for sauce for a minimum cost of $4,500,000.

45. **(a)** Let $y_1 =$ the number of cases of corn,

$y_2 =$ the number of cases of beans

and $y_3 =$ the number of cases of carrots.

Minimize $w = 10y_1 + 15y_2 + 25y_3$

subject to:

$$y_1 + y_2 + y_3 \geq 1000$$
$$y_1 \geq 2y_2$$
$$y_3 \geq 340$$

with $y_1 \geq 0, y_2 \geq 0.$

The second constraint can be rewritten as $y_1 - 2y_2 \geq 0$. Change this to a maximization problem by letting $z = -w = -10y_1 - 15y_2 - 25y_3$. Now maximize $z = -10y_1 - 15y_2 - 25y_3$ subject to the constraints above. Begin by inserting surplus variables to set up the first tableau.

$$\begin{array}{c} \\ \end{array}\begin{bmatrix} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & z & \\ \boxed{1} & 1 & 1 & -1 & 0 & 0 & 0 & 1000 \\ 1 & -2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 340 \\ \hline 10 & 15 & 25 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Multiply row 2 by -1 so that s_2 is positive.

$$\begin{array}{r} \\ \\ -R_2 \to R_2 \\ \\ \\ \end{array}\begin{bmatrix} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & z & \\ \boxed{1} & 1 & 1 & -1 & 0 & 0 & 0 & 1000 \\ -1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 340 \\ \hline 10 & 15 & 25 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Pivot on the 1 in row 1, column 1.

$$\begin{array}{r} \\ \\ R_1 + R_2 \to R_2 \\ \\ -10R_1 + R_4 \to R_4 \end{array}\begin{bmatrix} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & z & \\ 1 & 1 & 1 & -1 & 0 & 0 & 0 & 1000 \\ 0 & 3 & 1 & -1 & 1 & 0 & 0 & 1000 \\ 0 & 0 & \boxed{1} & 0 & 0 & -1 & 0 & 340 \\ \hline 0 & 5 & 15 & 10 & 0 & 0 & 1 & -10{,}000 \end{bmatrix}$$

Pivot on the 1 in row 3, column 3.

$$\begin{array}{r} \\ -R_3 + R_1 \to R_1 \\ -R_3 + R_2 \to R_2 \\ \\ -15R_3 + R_4 \to R_4 \end{array}\begin{bmatrix} y_1 & y_2 & y_3 & s_1 & s_2 & s_3 & z & \\ 1 & 1 & 0 & -1 & 0 & 1 & 0 & 660 \\ 0 & 3 & 0 & -1 & 1 & 1 & 0 & 660 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 340 \\ \hline 0 & 5 & 0 & 10 & 0 & 15 & 1 & -15{,}100 \end{bmatrix}$$

The maximum value of z is $-15{,}100$ when $y_1 = 660$, $y_2 = 0$, and $y_3 = 340$. Hence the minimum value of w is 15,100 when $y_1 = 660$, $y_2 = 0$, and $y_3 = 340$. Produce 660 cases of corn and 340 cases of carrots for a minimum cost of $15,100.

(b) The dual problem is as follows.

Maximize $\quad z = 1000x_1 + 340x_3$

subject to:
$$\begin{aligned} x_1 + x_2 &\le 10 \\ x_1 - 2x_2 &\le 15 \\ x_1 + x_3 &\le 25 \end{aligned}$$

with $\quad x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

The initial simplex tableau is as follows.

$$\begin{bmatrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \boxed{1} & 1 & 0 & 1 & 0 & 0 & 0 & 10 \\ 1 & -2 & 0 & 0 & 1 & 0 & 0 & 15 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 25 \\ \hline -1000 & 0 & -340 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Pivot on the 1 in row 1, column 1.

$$\begin{array}{r} \\ \\ -R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \\ 1000R_1 + R_4 \to R_4 \end{array}\begin{bmatrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 10 \\ 0 & -3 & 0 & -1 & 1 & 0 & 0 & 5 \\ 0 & -1 & \boxed{1} & -1 & 0 & 1 & 0 & 15 \\ \hline 0 & 1000 & -340 & 1000 & 0 & 0 & 1 & 10{,}000 \end{bmatrix}$$

Pivot on the 1 in row 3, column 3.

$$\begin{array}{r} \\ \\ \\ \\ 340R_3 + R_4 \to R_4 \end{array}\begin{bmatrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 10 \\ 0 & -3 & 0 & -1 & 1 & 0 & 0 & 5 \\ 0 & -1 & 1 & -1 & 0 & 1 & 0 & 15 \\ \hline 0 & 660 & 0 & 660 & 0 & 340 & 1 & 15{,}100 \end{bmatrix}$$

The minimum value of w is 15,100 when $y_1 = 660$, $y_2 = 0$, and $y_3 = 340$, that is, 660 cases of corn, 0 cases of beans, and 340 cases of carrots should be produced to minimize costs, and the minimum cost is $15,100.

(c) The final tableau for the dual solution shows that the shadow cost of acreage (x_1) is $10 acre, so increasing the number of acres planted by 100 will increase the minimum cost by ($10)(100) or $1000, so the new minimum will be $15,100 + $1000 = $16,100.

46. Let y_1 = the number of packages of Sun Hill and y_2 = the number of packages of Bear Valley.

The problem is:

Minimize $\quad w = 3y_1 + 2y_2$

subject to:
$$\begin{aligned} 10y_1 + 2y_2 &\ge 20 \\ 4y_1 + 4y_2 &\ge 24 \\ 2y_1 + 8y_2 &\ge 24 \end{aligned}$$

with $\quad y_1 \ge 0, y_2 \ge 0.$

(a) Change this to a maximization problem by letting $z = -w = -3y_1 - 2y_2$. Now maximize $z = -3y_1 - 2y_2$ subject to the constraints above. Begin by inserting surplus variables to set up the first tableau.

The initial simplex tableau is as follows.

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccc|c} y_1 & y_2 & s_1 & s_2 & s_3 & z & \\ \boxed{10} & 2 & -1 & 0 & 0 & 0 & 20 \\ 4 & 4 & 0 & -1 & 0 & 0 & 24 \\ 2 & 8 & 0 & 0 & -1 & 0 & 24 \\ \hline 3 & 2 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 10 in row 1, column 1.

$$\begin{array}{r} \\ \\ -4\text{R}_1 + 10\text{R}_2 \to \text{R}_2 \\ -\text{R}_1 + 5\text{R}_3 \to \text{R}_3 \\ -3\text{R}_1 + 10\text{R}_4 \to \text{R}_4 \end{array}\begin{array}{cccccc|c} y_1 & y_2 & s_1 & s_2 & s_3 & z & \\ 10 & 2 & -1 & 0 & 0 & 0 & 20 \\ 0 & 32 & 4 & -10 & 0 & 0 & 160 \\ 0 & \boxed{38} & 1 & 0 & -5 & 0 & 100 \\ \hline 0 & 14 & 3 & 0 & 0 & 10 & -60 \end{array}$$

Pivot on the 38 in row 3, column 2.

$$\begin{array}{r} \\ -\text{R}_3 + 19\text{R}_1 \to \text{R}_1 \\ -16\text{R}_3 + 19\text{R}_2 \to \text{R}_2 \\ \\ -7\text{R}_3 + 19\text{R}_4 \to \text{R}_4 \end{array}\begin{array}{cccccc|c} y_1 & y_2 & s_1 & s_2 & s_3 & z & \\ 190 & 0 & -20 & 0 & 5 & 0 & 280 \\ 0 & 0 & \boxed{60} & -190 & 80 & 0 & 1440 \\ 0 & 38 & 1 & 0 & -5 & 0 & 100 \\ \hline 0 & 0 & 50 & 0 & 35 & 190 & -1840 \end{array}$$

Pivot on the 60 in row 2, column 3.

$$\begin{array}{r} \\ \text{R}_2 + 3\text{R}_1 \to \text{R}_1 \\ \\ -\text{R}_2 + 60\text{R}_3 \to \text{R}_3 \\ -5\text{R}_2 + 6\text{R}_4 \to \text{R}_4 \end{array}\begin{array}{cccccc|c} y_1 & y_2 & s_1 & s_2 & s_3 & z & \\ 570 & 0 & 0 & -190 & 95 & 0 & 2280 \\ 0 & 0 & 60 & -190 & \boxed{80} & 0 & 1440 \\ 0 & 2280 & 0 & 190 & -380 & 0 & 4560 \\ \hline 0 & 0 & 0 & 950 & -190 & 1440 & -18,240 \end{array}$$

Pivot on the 80 in row 2, column 5.

$$\begin{array}{r} \\ -19\text{R}_2 + 16\text{R}_1 \to \text{R}_1 \\ \\ 19\text{R}_2 + 4\text{R}_3 \to \text{R}_3 \\ 19\text{R}_2 + 8\text{R}_4 \to \text{R}_4 \end{array}\begin{array}{cccccc|c} y_1 & y_2 & s_1 & s_2 & s_3 & z & \\ 9120 & 0 & -1140 & 570 & 0 & 0 & 9120 \\ 0 & 0 & 60 & -190 & 80 & 0 & 1440 \\ 0 & 9120 & 1440 & -2850 & 0 & 0 & 45,600 \\ \hline 0 & 0 & 1440 & 3990 & 0 & 9120 & -118,560 \end{array}$$

Create a 1 in the columns corresponding to y_1, y_2, and z.

$$\begin{array}{r} \\ \frac{1}{9120}\text{R}_1 \to \text{R}_1 \\ \\ \frac{1}{9120}\text{R}_3 \to \text{R}_3 \\ \frac{1}{9120}\text{R}_4 \to \text{R}_4 \end{array}\begin{array}{cccccc|c} y_1 & y_2 & s_1 & s_2 & s_3 & z & \\ 1 & 0 & -\frac{1}{8} & \frac{1}{16} & 0 & 0 & 1 \\ 0 & 0 & 60 & -190 & 80 & 0 & 1440 \\ 0 & 1 & \frac{1}{8} & -\frac{5}{16} & 0 & 0 & 5 \\ \hline 0 & 0 & \frac{1}{8} & \frac{7}{16} & 0 & 1 & -13 \end{array}$$

The maximum value of z is -13 when $y_1 = 1$ and $y_2 = 5$. Hence the minimum value of w is 13 when $y_1 = 1$ and $y_2 = 5$. Thus the minimum cost is \$13 for 1 package of Sun Hill and 5 packages of Bear Valley.

(b) The dual problem is as follows.

Maximize $z = 20x_1 + 24x_2 + 24x_3$

subject to: $10x_1 + 4x_2 + 2x_3 \le 3$
$2x_1 + 4x_2 + 8x_3 \le 2$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$

The initial simplex tableau is as follows.

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 10 & 4 & 2 & 1 & 0 & 0 & 3 \\ 2 & \boxed{4} & 8 & 0 & 1 & 0 & 2 \\ \hline -20 & -24 & -24 & 0 & 0 & 1 & 0 \end{array}$$

Pivot about the 4 in row 2, column 2.

$$\begin{array}{r} \\ -\text{R}_2 + \text{R}_1 \to \text{R}_1 \\ \\ 6\text{R}_2 + \text{R}_3 \to \text{R}_3 \end{array}\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ \boxed{8} & 0 & -6 & 1 & -1 & 0 & 1 \\ 2 & 4 & 8 & 0 & 1 & 0 & 2 \\ \hline -8 & 0 & 24 & 0 & 6 & 1 & 12 \end{array}$$

Pivot about the 8 in row 1, column 1.

$$\begin{array}{r} \\ \\ -\text{R}_1 + 4\text{R}_2 \to \text{R}_2 \\ \text{R}_1 + \text{R}_3 \to \text{R}_3 \end{array}\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 8 & 0 & -6 & 1 & -1 & 0 & 1 \\ 0 & 16 & 38 & -1 & 5 & 0 & 7 \\ 0 & 0 & 18 & 1 & 5 & 1 & 13 \end{array}$$

$$\begin{array}{r} \\ \frac{1}{8}\text{R}_1 \to \text{R}_1 \\ \frac{1}{16}\text{R}_2 \to \text{R}_2 \\ \\ \end{array}\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 1 & 0 & -\frac{3}{4} & \frac{1}{8} & -\frac{1}{8} & 0 & \frac{1}{8} \\ 0 & 1 & \frac{19}{8} & -\frac{1}{16} & \frac{5}{16} & 0 & \frac{7}{16} \\ \hline 0 & 0 & 18 & 1 & 5 & 1 & 13 \end{array}$$

The minimum value of w is 13 when $y_1 = 1$ and $y_2 = 5$, that is, you should buy 1 package of Sun Hill and 5 packages of Bear Valley for a minimum cost of \$13.

(c) The shadow cost for peanuts is based on the variable x_1 and can be read from the final simplex tableau: $\frac{1}{8}$. To get 8 more ounces of peanuts will cost an additional $8\left(\frac{1}{8}\right) = \$1$, so the total cost will then be \$14.

47. **(a)** Let $x_1 =$ the number of hours doing tai chi,

$x_2 =$ the number of hours riding a unicycle, and x_3 = the number of hours fencing.

If Ginger wants the total time doing tai chi to be at least twice as long as she rides a unicycle, then

$$x_1 \geq 2x_2$$
$$\text{or} \quad -x_1 + 2x_2 \leq 0.$$

The problem can be stated as follows.

$$\text{Maximize} \quad z = 236x_1 + 295x_2 + 354x_3$$

$$\begin{aligned} \text{subject to:} \quad x_1 + x_2 + x_3 &\leq 10 \\ x_3 &\leq 2 \\ -x_1 + 2x_2 &\leq 0 \end{aligned}$$

$$\text{with} \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

The initial simplex tableau is as follows.

$$\begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ 0 & 0 & \boxed{1} & 0 & 1 & 0 & 0 & 2 \\ -1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline -236 & -295 & -354 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Pivot on the 1 in row 2, column 3.

$$\begin{array}{r} \\ -R_2 + R_1 \to R_1 \\ \\ \\ 354R_2 + R_4 \to R_4 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 1 & 1 & 0 & 1 & -1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 \\ -1 & \boxed{2} & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline -236 & -295 & 0 & 0 & 354 & 0 & 1 & 708 \end{array}$$

Pivot on the 2 in row 3, column 2.

$$\begin{array}{r} \\ -R_3 + 2R_1 \to R_1 \\ \\ \\ 295R_3 + 2R_4 \to R_4 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \boxed{3} & 0 & 0 & 2 & -2 & -1 & 0 & 16 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 \\ -1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline -767 & 0 & 0 & 0 & 708 & 295 & 2 & 1416 \end{array}$$

Pivot on the 3 in row 1, column 1.

$$\begin{array}{r} \\ \\ \\ R_1 + 3R_3 \to R_3 \\ 767R_1 + 3R_4 \to R_4 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 3 & 0 & 0 & 2 & -2 & -1 & 0 & 16 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 \\ 0 & 6 & 0 & 2 & -2 & 2 & 0 & 16 \\ \hline 0 & 0 & 0 & 1534 & 590 & 118 & 6 & 16{,}520 \end{array}$$

Create a 1 in the columns corresponding to $x_1, x_2,$ and z.

$$\begin{array}{r} \\ \frac{1}{3}R_1 \to R_1 \\ \\ \frac{1}{6}R_3 \to R_3 \\ \frac{1}{6}R_4 \to R_4 \end{array}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 1 & 0 & 0 & \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} & 0 & \frac{16}{3} \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & \frac{8}{3} \\ \hline 0 & 0 & 0 & \frac{767}{3} & \frac{295}{3} & \frac{59}{3} & 1 & \frac{8260}{3} \end{array}$$

Ginger will burn a maximum of $2753\frac{1}{3}$ calories if she does $\frac{16}{3}$ hours of tai chi, $\frac{8}{3}$ hours riding a unicycle, and 2 hours fencing.

(b) Since fencing burns the most calories, she should do as much fencing as possible, which is 2 hours. This leaves 8 hours to divide between tai chi and the unicycle. The unicycle burns more calories, so she wants as much of unicycle as possible subject to the tai chi getting at least twice as much time as the unicycle. This requires devoting $\frac{1}{3}$ of the remaining 8 hours to the unicycle and $\frac{2}{3}$ of the 8 hours to tai chi. So the times are: $\frac{16}{3}$ hours of tai chi, $\frac{8}{3}$ hours of unicycle, and 2 hours of fencing.

Extended Application: Using Integer Programming in the Stock-Cutting Problem

1. **(a)** With Plan A you will need to buy 8 timbers: one cut will give you the two 4-ft lengths, two more cuts will give you two each of the 3-ft and 5-ft lengths, the two 3-ft pieces will come out of another full length, leaving 2 ft over, and all four 6-ft lengths will come out of 8-ft pieces. Your total waste amounts to 10 ft.

(b) There's no advantage to Plan B; you still need 8 pieces of lumber: two cuts will give you four 4-ft lengths, two more cuts will give you two each of the 3-ft and 5-ft lengths, the two 5-ft lengths will come out of 8-ft pieces as will each of the two 6-ft lengths, leaving a total waste of 10 ft.

(c) If the original timbers were 9 ft in length, you could cut 6 timbers and cut the lengths for either Plan A or Plan B with no waste.

2. Four patterns not in the minimizer's list are, for example, 14|14|33|33|, 31|33|33|, 33|33|33|, and 14|17|31|33|.

3. The patterns are 31|33|36, 17|17|33|33, 14|17|33|36, and 14|14|36|36.

4. $\dfrac{808}{35,600} \approx 2.3\%$

5. A leftover piece less than 14 inches wide can't be used for any standard width, but a leftover piece of 22 inches, for example, could be cut to make either the 14-inch or the 17-inch standard width. So it might be better to reserve the choice about how to cut this leftover until more orders come in.

6. The highest value is 34: choose weights 2, 2.5 and 4.5.

7. Minimize

$$w = y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7$$

subject to:

$$\begin{aligned}
2y_1 + y_2 &\geq 100 \\
y_3 + y_4 &\geq 123 \\
2y_5 + y_6 &\geq 239 \\
y_6 &\geq 121 \\
2y_1 + y_2 + 2y_3 + y_4 + y_5 + y_6 &\geq 444 \\
y_2 + y_4 + 2y_7 &\geq 87
\end{aligned}$$

with

$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0, y_5 \geq 0, y_6 \geq 0, y_7 \geq 0.$$

8. Minimum is 355.5 when

$y_1 = 50,\ y_2 = 0,\ y_3 = 41,\ y_4 = 82,$
$y_5 = 59,\ y_6 = 121,$ and $y_7 = 2.5,$ or

$y_1 = 50,\ y_2 = 0,\ y_3 = 38.5,\ y_4 = 87,$
$y_5 = 59,\ y_6 = 121,$ and $y_7 = 0.$

Minimum is 356 when

$y_1 = 50,\ y_2 = 0,\ y_3 = 41,\ y_4 = 82,$
$y_5 = 59,\ y_6 = 121,$ and $y_7 = 3,$ or

$y_1 = 50,\ y_2 = 3,\ y_3 = 38,\ y_4 = 85,$
$y_5 = 59,\ y_6 = 121,$ and $y_7 = 0.$